Exercises for the first part

1. Given
$$f_1(z) := \frac{(2z^2-1)(z+1)}{z^3-1}, \qquad f_2(z) = z^2 + \frac{1}{z^2}, \qquad f_3(z) = z^3/(1-z^2)\ .$$

   a) Write the holomorphic map $F_i : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ corresponding to f_i in terms of the homogeneous coordinates $(z_0 : z_1)$.

   b) Compute the degree of F_i and its ramification locus.

   c) Verify Riemann-Hurwitz formula for F_i.

   d) Compute the degree of $F_1 \circ F_2$ and its ramification locus.

2. Let X be the projective plane curve of degree d defined by the homogeneous polynomial $F(x, y, z) = x^d + y^d + z^d$. This curve is called the Fermt curve of degree d. Let $\Pi : X \to \mathbb{P}^1(\mathbb{C})$ be given by $\Pi((x : y : z)) = (x : y)$.

   a) Check that the Fermat curve is smooth and show that Π is a well defined holomorphic map of degree d. Find the ramification and branch loci of Π and use Riemann-Hurwitz formula to compute the genus of the Fermat curve.

   c) Find the ramification and the branch loci of the composition $f \circ \Pi$, with f the holomorphic map associated to the polynomial z^2.

3. a) Let X and Y be complex tori defined by lattices $\bigwedge$ and $\bigwedge'$ respectively, and let $f : X \to Y$ be the holomorphic map induced by the map $F(z) = az + b$ with $a \bigwedge \subset \bigwedge'$. Show that the degree of f is equal to $|detA|$ where $A \in M_2(\mathbb{Z})$ represents $F : \bigwedge \to \bigwedge'$.

   b) Let F be a nontrivial automorphism of X. Show that if F is not a translation, then F has a fixed point.

4. Let $h(x)$ be a polynomial of degree $2g + 1 + e$ (with $e \in \{0, 1\}$) having distinct roots and let $U = \{(x, y) \in \mathbb{C}^2 | y^2 = h(x), \quad x \neq 0\}$. Let $k(z) = z^{2g+2}h(1/z)$ and let $V = \{(z, w) \in \mathbb{C}^2 | w^2 = k(z), \quad z \neq 0\}$. Show that the map $\Phi : U \to V$ defined by $(z, w) = (1/x, y/x^{g+1})$ is an isomorphism of Riemann surfaces.

5. Let X be a compact Riemann surface that is a degree 3 cover of $\mathbb{P}^1(\mathbb{C})$ given by $y^3 = f(x)$ with $f \in \mathbb{C}[x]$ a polynomial of degree 5 or 6 with distinct zeros.

   a) Determine the genus of X and the fiber over $(0 : 1) \in \mathbb{P}^1(\mathbb{C})$.

   b) Generalize to the the case of a compact Riemann surface that is a degree d cover of $\mathbb{P}^1(\mathbb{C})$ given by $y^d = f(x)$ with $f \in \mathbb{C}[x]$ a polynomial of degree nd or $nd - 1$ with distinct zeros.

6. Let X a smooth plane projective curve which is the zero locus of a homogeneous polynomial $F(z_0, z_1, z_2) = z_1^3 z_2 - f_4(z_0, z_2)$ where f_4 has four distinct zeroes in $\mathbb{P}^1(\mathbb{C})$.

   Compute $g(X)$. Generalize to the case $F(z_0, z_1, z_2) = z_1^{d-1} z_2 - f_d(z_0, z_2)$.

7. Let $f : X \to E$ the holomorphic map beteween RS associated to the homomorphism:
$$\rho : \pi_1(E - P, Q) \to S_4, \quad \rho(\alpha) = (123), \quad \rho(\beta) = (234),$$
   where E is a genus 1 RS, $P \in E$ and $\pi_1(E, Q) =< \alpha, \beta | [\alpha, \beta] >$. Compute the genus of X.