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On the Direct Construction of Formal Integrals of a Hamiltonian System Near an Equilibrium Point.

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Sunto. – *In questo lavoro viene data una soluzione parziale ad un problema sollevato da Cherry sulla possibilità di compiere una costruzione formale di integrali del moto per un sistema autonomo hamiltoniano intorno ad un punto di equilibrio.*

1. – We consider the well known problem of constructing n formal integrals for an autonomous hamiltonian system of n degrees of freedom about an equilibrium point [1] - [5] (1). A real hamiltonian $H(q, p)$, $q \in R^n$, $p \in R^n$ is given, such that $H = H_2 + H_3 + \dots$, where H_r is a homogeneous polynomial of degree r in the variables and in particular $H_2 = \frac{1}{2}\sum_{l=1}^{n}(p_l^2 + \omega_l^2 q_l^2)$, $\omega_l > 0$ $(l = 1, \dots, n)$. The problem is then to find a consistent and constructive procedure to determine n independent formal integrals $\varphi^{(l)}$ $(l = 1, \dots, n)$, i.e. power series $\varphi^{(l)} = \varphi_2^{(l)} + \varphi_3^{(l)} + \dots$ for which questions of convergence are ignored, such that $(\varphi^{(l)}, H) = 0$, where $(\cdot, \cdot)$ is the Poisson bracket and the equality has the natural formal sense. One can choose $\varphi_2^{(l)} = \frac{1}{2}(p_l^2 + \omega_l^2 q_l^2)$ so that the integrals $\varphi^{(l)}$ are nonlinear generalizations of the normal modes of system H_2.

This problem is of interest in many fields of mathematical physics and astronomy. Our reason of interest was its connection with the foundations of statistical mechanics [6].

Technically we found available two main ways of constructing the formal integrals. The first one, typically considered by BIRKHOFF [3] and GUSTAVSON [5], consists in performing a sequence of

(1) For a general review see A. D. BRJUNO, *Analytical form of differential equations*, Trans. Moscow Math. Soc., **25** (1971), pp. 131-288.

canonical transformations such that the hamiltonian is brought at any order to a so-called normal form,—i.e. that of being independent of the new coordinates—; the new momenta are then found to be integrals as a byproduct. This procedure, which can be called an indirect one, has a certain disadvantage as it requires to perform at every stage an inversion which can be cumbersome in practice (2). This disadvantage is not shared by the second method, of WHITTAKER [1], CHERRY [2], and CONTOPOULOS [4], in which reference is always made to the original variables and the integrals are built in a direct way.

On the other hand the situation is different for what concerns the available theorems on the possibility of performing the construction, where the first method seems to be in a better position. In particular by this method it has been proven [5] that $n-r$ formal integrals can be constructed if there are r, and only r, independent resonance relations, i.e. relations of the form $\sum_{i=1}^{n} m_i \omega_i = 0$, $m_1, \ldots, m_n$ integers, generalizing the analogous result of BIRKHOFF [3] for the nonresonant case $r=0$. The situation for the direct method is instead the following. Even in the nonresonant case a difficulty appears for the possible presence of certain « critical terms », which will be discussed in section 2. Now, in the words of CHERRY [2*a*], « Whittaker makes no esplicit mention of the difficulty arising from the possible presence of critical terms, the satisfactory treatment of which is vital to establishing the validity of the process ». Cherry himself is « not able to find a direct general proof that the coefficients of the critical terms vanish, though the verification for the first few such terms is given », but he can at least afford that « this fundamental point is established in an indirect manner ». A direct proof of the possibility of the direct construction has been given only by Contopoulos [4*a*]. An analogous but somehow more complicated situation obtains also for the resonant case [2*b*], [4*b*]. However, even in the nonresonant case, the proof given by Contopoulos is more involved than that of Birkhoff and Gustavson and in addition it does not have the generality of the latter, a point which will be discussed below. The situation is still worse for the resonant case.

It then seemed to be worthwhile to look for a proof of the result of Contopoulos which would be comparable in simplicity to that of Birkhoff and Gustavson, and indeed we were able to find one, for

(2) See however A. DEPRIT, *Canonical transformations depending on a small parameter*, Celestial Mech., **1** (1969), pp. 12-30.

the nonresonant case and for a large class of hamiltonians including those considered by Contopoulos. Such a proof constitutes the content of the present note.

2. – The equation $(\varphi^{(l)}, H) = 0$, $\varphi_2^{(l)} = \frac{1}{2}(p_l^2 + \omega_l^2 q_l^2)$ is equivalent to the infinite system of equations

$$\sum_{s=0}^{r-2} (\varphi_{r-s}^{(l)}, H_{2+s}) = 0 \qquad r = 2, 3, \ldots \tag{1}$$

It is convenient to introduce the linear space S the elements of which are the real polynomials of any order in the variables q, p, and to define there the linear operator D by $Df = (f, H_2)$, $f \in S$. The linear subspaces of S, $R(D)$ and $N(D)$, the range and the nucleus of D, are defined as usual: $R(D)$ is the image of S by D and $N(D)$ is the inverse image of the null vector 0, $N(D) = \{f \in S : Df = 0\}$. The system of equations (1) can then be written in the form $D\varphi_2^{(l)} = 0$, which is already solved, and

$$\begin{aligned} &D\varphi_r^{(l)} = \psi_r^{(l)}, \\ &\psi_r^{(l)} = \sum_{s=1}^{r-2} (H_{2+s}, \varphi_{r-s}^{(l)}) \end{aligned} \qquad r = 3, 4, \ldots \tag{2}$$

so that at any order r the unknown $\varphi_r^{(l)}$ is put into evidence and the right hand side $\psi_r^{(l)}$ is known.

It is evident that eq. (2) cannot be satisfied if $\psi_r^{(l)}$ has a (non null) component in $N(D)$; the (non null) vectors of $N(D)$ are then the « critical terms » referred to above, « the satisfactory treatment of which is vital to establishing the validity of the process ». In other words, more explicitly, the problem considered here is that of proving that $\psi_r^{(l)}$ belongs to $R(D)$. The explicit construction of the solution is then standard.

The characterization of the nucleus $N(D)$, which corresponds to finding the general solution of the first equation of system (1) is a classical problem ([3]): $N(D)$ coincides with the subspace of vectors f of S which, as functions of q, p, depend only on $p_l^2 + \omega_l^2 q_l^2$ $(l = 1, \ldots, n)$ i.e. are functions of $\varphi_2^{(l)}$. This, by the way, justifies the

([3]) See for example H. POINCARÉ, *Les Méthodes Nouvelles de la Mécanique Céleste*, Paris, 1892, tome I, sec. 82, p. 236; or E. T. WHITTAKER, ref. [1*b*], page 382; or T. M. CHERRY, ref. [2*a*], page 326.

choice made for the first term of the series $\varphi^{(l)}$ $(l = 1, \ldots, n)$. In addition, one easily proves [4] $R(D) \cup N(D) = S$, $R(D) \cap N(D) = 0$ so that every vector f of S can be uniquely decomposed as $f = f' + f''$, $f' \in R(D)$, $f'' \in N(D)$.

Let us now define a vector $f \in S$ as being even if the corresponding polynomial $f(q, p)$ is even in the momenta p, i.e. $f(q, p) = f(q, -p)$ and analogously as being odd if it is $f(q, p) = -f(q, -p)$; the subspaces of S, S_+ and S_-, of the even and the odd vectors are then defined and every vector f of S can be uniquely decomposed as $f = f_+ + f_-$, $f_+ \in S_+$, $f_- \in S_-$, so that $S = S_+ \cup S_-$, $S_+ \cap S_- = 0$. The following properties are easily proven:

a) The Poisson bracket of vectors of the same parity is odd; the Poisson bracket of vectors of different parity is even;

b) In the nonresonant case it is $N(D) \subset S_+$, i.e. the nucleus $N(D)$ is constitued of even functions;

c) The Poisson bracket of vectors of the same parity belongs to $R(D)$, the range of D.

Indeed the proof of *a*) is immediate; *b*) is a conseguence of the characterization given above for $N(D)$, and *c*) follows from *a*) and *b*).

Assume now that H is even, $H = H_+$, and for the moment restrict the search for solutions of eq. (1) to even vectors $\varphi^{(l)} = \varphi_+^{(l)}$. Then, if the quantities $\varphi_s^{(l)}$, $(s = 1, \ldots, r-1)$ have been determined and are even, which is true for $r = 3$, by property *c*) one has $\psi_r^{(l)} \in R(D)$; the solution $\varphi_r^{(l)}$ of eq. (2) is then uniquely defined in the range $R(D)$ and is obviously even, so that the absence of the unwanted critical terms is proven by induction. The restriction made above that $\varphi^{(l)}$ should be a priori even may be shown to be irrelevant, because $\varphi^{(l)}$ turns out to be necessarily even. Indeed, if one lets $\varphi^{(l)} = \varphi_+^{(l)} + \varphi_-^{(l)}$, then eq. (2) decomposes into two equations for $\varphi_{r+}^{(l)}$ and $\varphi_{r-}^{(l)}$ respectively, which turn out to be separated if $H_- = 0$; one is then reduced to the previous situation and the uniqueness of the solution $\varphi_{r-}^{(l)} = 0$, $r \geq 2$, i.e. $\varphi_-^{(l)} = 0$, immediately follows by property *b*).

We have then proved the following

THEOREM. – *Assume the real hamiltonian $H(q, p)$, $q, p \in R^n$, to be a formal series $H = H_2 + H_3 + \ldots$ with $H_2 = \sum_{i=1}^{n} \frac{1}{2}(p_i^2 + \omega_i^2 q_i^2)$ and H_r*

(4) This property is obviously true of any linear operator that can be diagonalized, as is the case for D. See for example ref. [5*a*], page 676, or ref. [5*b*], page 10.

a homogeneous polynomial of degree r in the variables, and let it satisfy the conditions of being:

i) *even in the momenta, i.e.* $H(q, p) = H(q, -p)$,

ii) *nonresonant, i.e.* $\sum_{i=1}^{n} m_i \omega_i = 0$, $m_1, \ldots, m_n$ *integers, implies* $m_1 = \ldots = m_n = 0$.

Then there exist n formal even integrals $\varphi^{(l)} = \varphi_2^{(l)} + \varphi_3^{(l)} + \ldots$ *with* $\varphi_2^{(l)} = \frac{1}{2}(p_l^2 + \omega_l^2 q_l^2)$ $(l = 1, \ldots, n)$, *the components of which in the range* $R(D)$ *are uniquely defined* (5).

By $\varphi^{(l)}$ we will designate precisely those unique formal integrals with $\varphi_2^{(l)} = \frac{1}{2}(p_l^2 + \omega_l^2 q_l^2)$ which have null component in the nucleus $N(D)$, apart from $\varphi_2^{(l)}$, and we could call them the « fundamental integrals ».

We remark that the condition that H be even is satisfied for a system of material points interacting through a potential independent of the velocities, for which it is $H(q, p) = T(q, p) + V(q)$, T being the kinetic energy and V the potential energy, which is indeed the case considered by Contopoulos. Our proof is then more general than his, but still less general than the indirect one of Birkhoff, which is independent of the parity of H. The generalization of our result to the case of any H, a problem which was overlooked by Whittaker, Cherry and Contopoulos, would constitute, in our opinion, a relevant step towards the generalization of the direct method to the resonant case.

3. – Let us close by pointing out some properties of the formal integrals.

i) The fundamental integrals $\varphi^{(l)}$ are in involution, i.e. one has $(\varphi^{(l)}, \varphi^{(k)}) = 0$ $(l, k = 1, \ldots, n)$. The dynamical system characterized by the given hamiltonian is then formally integrable (6).

ii) Any formal integral φ (in particular the hamiltonian H) can be expressed as a power series in the fundamental

(5) More generally, one could prove that given $\bar{\varphi} \in N(D)$, there exists a corresponding even formal integral φ, the component of which in the range $R(D)$ is uniquely defined.

(6) See ref. [1*b*], page 323 and V. I. ARNOL'D - A. AVEZ, *Problèmes ergodiques de la Mécanique Classique*, Paris, 1967.

integrals. Thus n is the maximum number of independent formal integrals (7).

iii) One can determine n integrals $\tilde{\varphi}^{(l)}$ $(l = 1, \ldots, n)$, which are still in involution, such that $H = \sum_{l=1}^{n} \tilde{\varphi}^{(l)}$.

All these properties are obviously true at order 2 and can be proved at any order by induction [7]. Cherry proves i) indirectly and ii) just up to order 6. Property iii) is immediate.

(7) If the origin had been an ordinary point of the Hamilton equations, instead of a point of equilibrium, there would have been $2n - 1$ formal integrals developable about it. See T. M. CHERRY, Proc. Cambridge Philos. Soc., **22** (1924), pp. 273-285.

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