INVARIANT TORI IN THE SECULAR MOTIONS OF THE THREE–BODY PLANETARY SYSTEMS

UGO LOCATELLI
Dipartimento di Matematica, Via Saldini 50,
20133 — Milano, Italy.

ANTONIO GIORGILLI
Dipartimento di Matematica e Applicazioni, Via degli Arcimboldi 8,
20126 — Milano, Italy.

In memory of Michèle Moons

Abstract. We consider the problem of the applicability of KAM theorem to a realistic problem of three bodies. In the framework of the averaged dynamics over the fast angles for the Sun–Jupiter–Saturn system we can prove the perpetual stability of the orbit. The proof is based on semi–numerical algorithms requiring both explicit algebraic manipulations of series and analytical estimates. The proof is made rigorous by using interval arithmetics in order to control the numerical errors.

1. Introduction and statement of the result

We reconsider the classical problem of stability for the solar system in the light of the KAM theory. After the celebrated works of Kolmogorov[14], Moser[24][25][26] and Arnold[2][3] the relevance of persistence of conditionally periodic motions for a near to integrable Hamiltonian system for the dynamics of our solar system has been emphasized by many authors. However, it has been soon remarked (e.g., by Hénon[11]) that it is not evident that the perturbation due to the mutual interaction of the planets is so small that the KAM theory may be safely applied. The aim of this work is to use both analytical theories and explicit perturbation expansions in order to prove that KAM theorem actually applies at least to an approximate model describing the dynamics of the Sun–Jupiter–Saturn (SJS) system.

Numerical attempts to reveal a possible quasi–periodic behaviour of the planetary orbits have been performed by some authors during the last decades (see, e.g., [28], [33] and [17]). The typical conclusion was that the dynamics of the major planets is rather close to a motion on an invariant torus, while the motion of the internal planets exhibits a non–negligible chaotic component. However, in the recent work of Murray & Holman[27] it has been shown that a very small chaotic behaviour appears also in the orbit of Uranus. Nevertheless, the KAM theory may be expected to represent a quite good approximation of the orbits of Jupiter and Saturn.

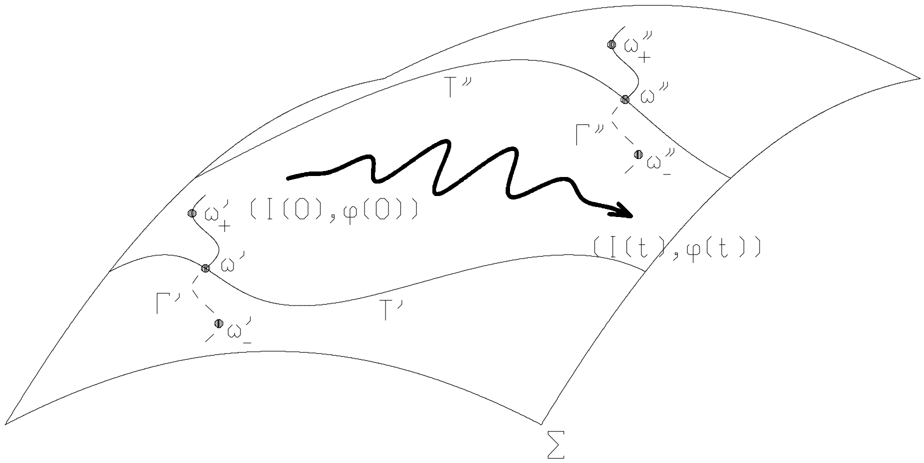

Figure 1. Illustrating the topological confinement of the orbit in the 4D phase space. The continuous curves Γ' and Γ'' represent two sets of 2D invariant tori that intersect transversally an energy surface. An orbit with initial datum in the gap between two tori will be eternally trapped in the same region (see text).

In this spirit, we investigate the stability of the orbits of Jupiter and Saturn in the framework of KAM theory. This involves three main difficulties, namely: (i) the degeneration of frequencies in the Keplerian approximation, (ii) the effectiveness of KAM estimates for a realistic problem, and (iii) the arithmetic properties of the frequencies.

The first problem is a well known one in Celestial Mechanics, and it is usually overcome by averaging over the fast angles. We partially follow the tradition. Starting with the Hamiltonian for the problem of three bodies in an heliocentric reference system we first perform the usual reduction of the first integrals of the momentum and of the angular momentum. Then we expand the Hamiltonian up to order 2 in the masses and we perform an averaging transformation so as to remove the dependence on the fast angles. Forgetting for a moment the behaviour of the secular variables, we approximate the fast motion of Jupiter and Saturn with an orbit on a two dimensional torus. In the neighbourhood of the latter torus we expand the Hamiltonian of the three–body problem in Poincaré variables up to order 6 in eccentricities, thus obtaining a system of two degrees of freedom in the neighbourhood of an elliptic equilibrium point. On this system we perform a Birkhoff normalization up to order 6, truncating the expansions at order 70. This defines the model to be investigated in the framework of KAM theory. Let us refer to this model as "the approximated secular model for the SJS system".

Concerning the effectiveness of the KAM estimates, it is well known that implementing a perturbation algorithm on a computer greatly improves the final estimates. For instance, Celletti & Chierchia[5] proved the stability of orbits for a (planar circular restricted) three–body problem not too far from reality, but different from the present one. We actually consider a model similar to the one in [30], where the behaviour of the orbits has been explored by using the numerical tool of frequency analysis. Previous investigations of this model have been performed in [21]. The present work uses further refinements of the demonstration technique and a more effective software package developed on purpose. This allowed us to also consider the contributions of order 2 in the masses and to avoid artificial reductions of the size of the perturbation.

We finally come to the last possible obstruction to an application of the KAM theory, namely the problem that the frequencies are required to satisfy an irrationality condition that is hardly compatible with the fact that the frequencies are actually known only within some degree of approximation. Hence we make use of a topological confinement, that we describe in some detail. The argument is illustrated by Fig. 1.
Let us write the Hamiltonian in action–angle variables I, φ in the usual form $h(I)+\varepsilon f(I,\varphi)$, where ε is the perturbation parameter. Consider the initial conditions $(I(0),\varphi(0))$ and let us denote by $\bar{E}$ the corresponding value of the energy. After having assumed suitable non–degeneracy conditions on the unperturbed Hamiltonian $h(I)$, we can uniquely identify the invariant tori surviving the perturbation by their angular frequencies. Making reference to the frequency plane (ω_1,ω_2) , we consider two straight lines $\omega_1/\omega_2 = \mathrm{const}'$, const'' of frequencies satisfying a Diophantine condition and we choose two segments Γ' and Γ'' lying on each of the previous lines, respectively. Let us imagine we are able to prove the existence of all tori corresponding to the frequencies belonging to Γ' and Γ'', then the images in the phase space of these two segments are two families of invariant tori depending on a parameter. Let us remark that in Fig. 1 we drew the images of Γ' and Γ'' which correspond only to a fixed value of the angles. Moreover, let us suppose to know two pairs of frequencies ω'_- , $\omega'_+ \in \Gamma'$ and ω''_- , $\omega''_+ \in \Gamma''$, such that $E(\omega'_-), E(\omega''_-) < \bar{E}$ and $E(\omega'_+), E(\omega''_+) > \bar{E}$, where $E(\omega)$ is the energy related to the torus with frequency ω. Since KAM theory ensures us that the function $E(\omega)$ is continuous on the sets Γ' and Γ'', then there are two frequencies $\omega' \in \Gamma'$ and $\omega'' \in \Gamma''$ corresponding to two invariant tori, say T' and T'' respectively, such that they belong to the energy surface Σ related to the level $\bar{E}$. Thus, it is enough to check that the initial data belong to the gap between T' and T'' on the surface Σ in order to assure that the orbit will be trapped there forever.

Applying the procedure above to our model we prove the following

Theorem 1: *The Hamiltonian of the approximated secular model for the SJS system possesses two invariant tori bounding the orbit with the initial data of Jupiter and Saturn.*

The form of the Hamiltonian referred to in the statement is given by truncating at order 70 the expression (19) in sect. 3.1. The initial data in the appropriate canonical coordinates are given in table 4. The frequencies related to the trapping tori belong to the sets defined in formula (51).

Acknowledgements. The advice of Michèle Moons and Jacques Henrard was invaluable in dealing with some of the classical expansions in Celestial Mechanics. Part of the calculations were performed using an algebraic manipulation package developed at the University of Namur. During the preparation of this paper, one of the authors (U.L.) has been supported by the research program of the Fondazione Cariplo on: “Problemi teorici ed applicativi in meccanica celeste ed ingegneria aerospaziale”. U.L. would also like to thank the Dublin Institute for Advanced Studies for their warm hospitality while part of this work was carried out. The project of this study has been discussed together with Michèle, but her illness, followed by her passing, interrupted the collaboration. We wish to dedicate this work to her memory, to remember her sincere friendship and dedication to her fellow

Table 1. Masses, heliocentric position and velocities for Jupiter and Saturn. We adopt the UA as unit of length, the year as time unit and set the gravitational constant $\mathcal{G} = 1$. With these units, the solar mass is equal to $(2\pi)^2$. The data are taken by JPL at the Julian Day 2451220.5.

	Jupiter	Saturn
Mass	$(2\pi)^2/1047.355$	$(2\pi)^2/3498.5$
X	4.9193878348583491	7.6616865939311696
Y	0.56774366109437100	5.1843531385874693
Z	−0.11252196237771750	−0.39486592787023467
$\dot{X}$	−0.35254936932700516	−1.2500060951403738
$\dot{Y}$	2.8698796063436852	1.6866882187249184
$\dot{Z}$	−0.0039904051778461076	0.020339209854943361

collaborators.

2. Reduction of the secular Hamiltonian

In this section we discuss some classical expansions of the Hamiltonian of the problem of three bodies in Poincaré variables. Then we introduce the secular Hamiltonian by eliminating the fast variables.

2.1 Expansion in canonical variables

We start with the expression of the Hamiltonian F of the three–body problem in Poincaré variables, after having performed the reduction of the classical first integrals of the momentum and of the angular momentum (see, e.g. [29] and [16]). The Hamiltonian writes

$$F = -\frac{1}{2}\left(\frac{\mu_1^2\beta_1^3}{\Lambda_1^2} + \frac{\mu_2^2\beta_2^3}{\Lambda_2^2}\right) - \mathcal{G}\frac{m_1 m_2}{\Delta} + T \ . \tag{1}$$

where Λ_1, Λ_2 are the action variables and λ_1, λ_2 the conjugate angles for the two planets, $\mu_j = \mathcal{G}(m_0 + m_j)$ and $\beta_j = \frac{m_j m_0}{m_j + m_0}$ for $j = 1, 2$, m_0 being the mass of the Sun and m_1, m_2 the masses of the planets and $\mathcal{G}$ being the gravitational constant. Moreover, Δ is the distance between the planets and T a term coming from the expression of the kinetic energy in heliocentric coordinates. We recall that the Poincaré variables are

$$\begin{cases} \Lambda_j = \beta_j\sqrt{\mu_j a_j} & \xi_j = \sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\cos\omega_j \\ \lambda_j = l_j + \omega_j & \eta_j = -\sqrt{2\Lambda_j}\sqrt{1-\sqrt{1-e_j^2}}\sin\omega_j \end{cases} \qquad j = 1, 2 \ , \tag{2}$$

with the usual notations a_j, e_j, l_j and ω_j for the semi–major axes, the eccentricities, the mean anomalies and the perihelion arguments, respectively.

Table 2. Fundamental frequencies n^* and g^* (related to the angles λ and ω, respectively) of Jupiter and Saturn as calculated by using Laskar's method.

Jupiter	$n_1^* = 0.52989041594$ rad/year	$g_1^* = -30.06829$ ″/year
Saturn	$n_2^* = 0.213454442910$ rad/year	$g_2^* = -54.04533$ ″/year

As usual we expand the perturbation in F. It is known that the main difficulty is represented by the expansion of the inverse of the mutual distance $1/\Delta$: we essentially followed, with minor changes, the scheme sketched in sect. 3.3 of [31] (see also [20] for more details). We just add a few remarks.

a) We found that the algorithm described in [12] is very effective for the expansion of the true anomaly and the ratio r/a (where r is the distance of a planet from the central star and a is the semi–major axis) as functions of the eccentricities e and of the mean anomaly l.

b) We need the expansion of the expression

$$[1 + \varrho^2 + 2\varrho\cos(\lambda_1 - \lambda_2)]^{-s/2} = \frac{1}{2} b^0_{s/2}(\varrho) + \sum_{j=1}^{+\infty} b^j_{s/2}(-\varrho)\cos(j(\lambda_1 - \lambda_2)) \ , \tag{3}$$

where ϱ represents the ratio a_1/a_2 of the semi–major axes. The Laplace coefficients $b^j_{s/2}(\varrho)$ can be calculated, for instance, according to the algorithm described in [1].

c) For what concerns the so-called complementary term T coming from the kinetic energy, we use the expression

$$T = -\frac{\beta_1 n_1 a_1 \beta_2 n_2 a_2}{m_0\sqrt{1-e_1^2}\sqrt{1-e_2^2}} \Big[(\cos(v_1+\omega_1) + e_1\cos(\omega_1))(\cos(v_2+\omega_2) + e_2\cos(\omega_2))\cos J + (\sin(v_1+\omega_1) + e_1\sin(\omega_1))(\sin(v_2+\omega_2) + e_2\sin(\omega_2)) \Big] \ ,$$

where n_j indicates the mean motion frequency of the j–th planet (for the osculating orbit) and J is the mutual inclination of the two orbital planes. An useful expression for $1 - \cos J$ is reported, e.g., in formula (12) of [31].

All expansions above are quite standard. We come now to the part which is strictly related to the search of an invariant torus according to Kolmogorov's algorithm. We look for fixed values $(\Lambda_1^*, \Lambda_2^*)$ by solving the equation

$$\left.\frac{\partial \langle F \rangle_\lambda}{\partial \Lambda_j}\right|_{\substack{\Lambda=\Lambda^* \\ \xi\,,\,\eta=0}} = n_j^* \ , \qquad j = 1, 2 \ , \tag{4}$$

where the symbol $\langle \cdot \rangle_\lambda$ denotes the average with respect to the fast angles, and n_j^* are the fundamental mean motion frequencies related to the angles λ_j. In order to determine these frequencies we integrate the Newton equations by taking the initial conditions reported in

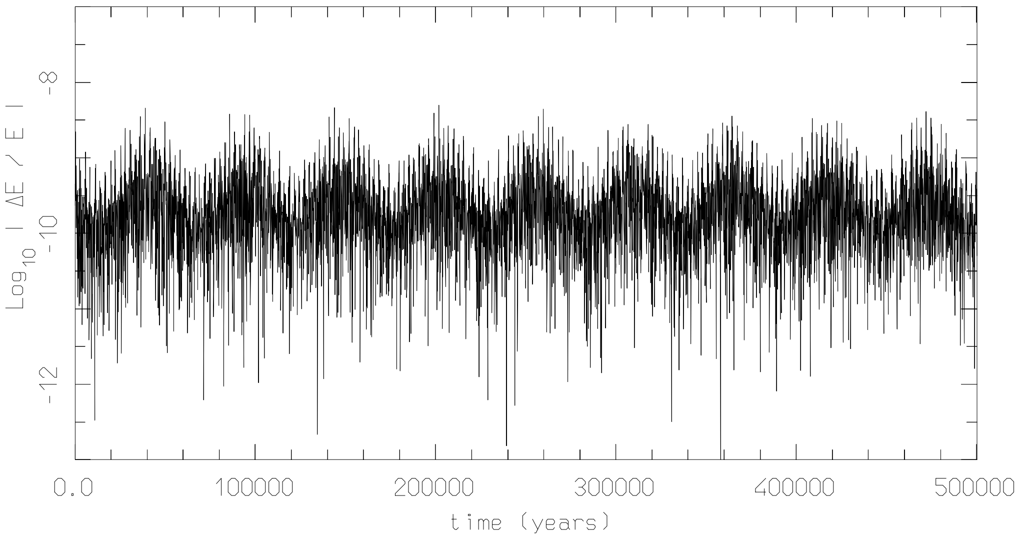

Figure 2. Check of our expansions of Hamiltonian (1).

table 1 and use Laskar's method for the frequency analysis (see [18], [19]). The values so determined are reported in table 2.

After having determined $(\Lambda_1^*, \Lambda_2^*)$ by numerically solving equation (4), we introduce the displacements L by

$$L_j = \Lambda_j - \Lambda_j^* \ , \qquad \forall\, j = 1, 2 \ .$$

Finally, we expand the perturbation in powers of L_1 , L_2 in a neighbourhood of $(0, 0)$.

In our calculations we expand the perturbation as a function of the canonical variables (L, λ, ξ, η) with the following limits: 1) up to degree 2 in L; 2) up to order 7 in eccentricity; 3) up to a trigonometric degree in λ high enough to assure that at least the first 21 terms of the series in (3) are taken into account. The limits above have been chosen by attempting to preserve the value of the energy as much as possible in the domain of the expansions. As a check, we find a number of points in the prescribed neighbourhood by numerically integrating the Newton's equations with the initial data in table 1; then we calculate the difference ΔE between the initial energy and the energy given by our expansion of (1) at different points of the orbit. The results are reported in Fig. 2. For comparison, the value of the energy computed by formula (1) is preserved during the numerical integration within a relative error of $\sim 10^{-11}$.

2.2 The secular Hamiltonian at order two in the masses

In this section we discuss the process of calculation of the secular Hamiltonian via elimination of the fast angles. This is a quite classical procedure that we implement in a way that takes into account the Kolmogorov's algorithm for the construction of an invariant torus. Recall that after the expansions in sect. 2.1 the perturbation has order 1 in the masses, and it is a polynomial function in the variables L , ξ and η and a trigonometric polynomial in the fast angles λ. We remove the dependence on the angles from terms which are independent of and linear in the action variables L, regarding the variables ξ, η as parameters. This will assure that the torus $L = 0$ is invariant up to terms of order two in the masses.

Let us denote with $F_j^{(s)}$ the part of the Hamiltonian F having order s with respect to the masses (i.e., the small parameter $\bar{m}/m_0$, where $\bar{m} = \max\{m_1, m_2\}$) and degree j in the mean motion actions L. Thus we write the Hamiltonian as

$$F = F_1^{(0)} + F_2^{(0)} + \ldots + F_0^{(1)} + F_1^{(1)} + F_2^{(1)} + \ldots \tag{5}$$

where an unessential constant has been neglected. Here the terms $F_j^{(0)}$ come from the unperturbed Keplerian part of the Hamiltonian, while $F_j^{(1)}$ come from the perturbation. We now take into account the first order correction to the frequencies by considering $F_1^{(0)} + \langle F_1^{(1)}|_{\xi\,,\,\eta=0}\rangle_\lambda$ as the linear unperturbed Hamiltonian, and replacing $F_1^{(1)}$ with $F_1^{(1)} - \langle F_1^{(1)}|_{\xi\,,\,\eta=0}\rangle_\lambda$. With a minor abuse we denote again the latter quantities by $F_1^{(0)}$ and $F_1^{(1)}$, respectively. Thanks to (4), the latter replacements imply $F_1^{(0)} = n^* \cdot L$. In view of D'Alembert rules the dependency of the new $F_1^{(1)}$ on the secular variables satisfies $\langle F_1^{(1)}\rangle_\lambda = \mathcal{O}(\bar{m}/m_0) \times \mathcal{O}(e^2)$. We also assume $\mathcal{O}(e^2) \simeq \mathcal{O}(\bar{m}/m_0)$, which is true for the SJS system.

We remove the angle–dependent terms in $F_0^{(1)}$ via a canonical transformation with generating function $X_{\mathcal{M}}^{(1)}$ determined by the equation

$$n^* \cdot \frac{\partial X_{\mathcal{M}}^{(1)}}{\partial \lambda} + F_0^{(1)} - \langle F_0^{(1)}\rangle_\lambda = 0\ . \tag{6}$$

Here, $\langle F_0^{(1)}\rangle_\lambda$ is the average on the angles λ and it depends only on ξ, η. Using the formalism of Lie series (see, e.g., [10] and [7]) the transformed Hamiltonian is

$$\bar{F} = \exp \mathcal{L}_{X_{\mathcal{M}}^{(1)}} F = \sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{L}^j_{X_{\mathcal{M}}^{(1)}} F\ , \tag{7}$$

where, as usual, the symbol $\mathcal{L}_f \cdot$ indicates the Poisson bracket $\{f, \cdot\}$. We write the first terms that will be useful for the discussion which follows:

$$\begin{gathered} \bar{F}_0^{(1)} = \langle F_0^{(1)}\rangle_\lambda\ , \qquad\qquad \bar{F}_1^{(1)} = \mathcal{L}_{X_{\mathcal{M}}^{(1)}} F_2^{(0)} + F_1^{(1)}\ , \\ \bar{F}_0^{(2)} = \frac{1}{2}\left\{X_{\mathcal{M}}^{(1)}, \mathcal{L}_{X_{\mathcal{M}}^{(1)}} F_2^{(0)}\right\}_{L,\lambda} + \left\{X_{\mathcal{M}}^{(1)}, F_1^{(1)}\right\}_{L,\lambda} + \frac{1}{2}\left\{X_{\mathcal{M}}^{(1)}, F_0^{(1)}\right\}_{\xi,\eta}\ , \end{gathered} \tag{8}$$

where we denoted with $\{\cdot,\cdot\}_{L,\lambda}$ and $\{\cdot,\cdot\}_{\xi,\eta}$ the terms of the Poisson bracket involving only the derivatives with respect the variables (L,λ) and (ξ,η), respectively.

Then we proceed by eliminating the angular dependence in $\bar{F}_1^{(1)}$ via a canonical transformation with generating function $Y_{\mathcal{M}}^{(1)}$ determined by the equation

$$n^* \cdot \frac{\partial Y_{\mathcal{M}}^{(1)}}{\partial \lambda} + \bar{F}_1^{(1)} - \langle \bar{F}_1^{(1)}\rangle_\lambda = 0\ . \tag{9}$$

As a matter of fact, we just remove $\bar{F}_1^{(1)} - \langle \bar{F}_1^{(1)}\rangle_\lambda$ without performing explicitly the transformation. This is justified because up to order two in the masses no terms independent of both L and λ will be generated. This means that all terms generated by this expansion

Table 3. Initial conditions of the secular coordinates $(\xi\,,\,\eta)$ for Jupiter and Saturn.

Jupiter	$\xi_1 =$	0.013902710681323937	$\eta_1 =$	−0.033461832907706032
Saturn	$\xi_2 =$	0.021222294541265476	$\eta_2 =$	0.013936304647086138

will be removed when reducing the system to the secular one in the fast angles, as we shall do in a moment.

After the transformation above the Hamilton's equations for the mean motion variables satisfy

$$\dot{L}_j = \mathcal{O}((\bar{m}/m_0)^2)\ , \qquad \dot{\lambda}_j = \mathcal{O}((\bar{m}/m_0)^2)\ , \qquad \text{for } j=1,2$$

(recall that $\langle \bar{F}_1^{(1)} \rangle_\lambda = \langle F_1^{(1)} \rangle_\lambda = \mathcal{O}((\bar{m}/m_0)^2)$, due to our assumption on the size of the eccentricities).

We now proceed with the reduction to the secular system. This is performed via a truncation of the Hamiltonian by removing all terms depending on the fast variables L and λ. This means that we consider the fast variables frozen on the torus $L=0$, with fast frequencies n^*. The resulting Hamiltonian has the form

$$\mathcal{H}_{Sec} = \bar{F}_0^{(1)} + \langle F_0^{(2)} \rangle_\lambda\ , \tag{10}$$

with $\bar{F}_0^{(1)}$ and $\bar{F}_0^{(2)}$ given by (8). As a matter of fact, this turns out to be an infinite sum of even polynomials in the canonical variables ξ,η of the form

$$\mathcal{H}_{Sec}^{(\mathrm{I})}(\xi_1,\xi_2,\eta_1,\eta_2) = \sum_{s=1}^{3} \sum_{\substack{i_1+i_2+\\ j_1+j_2=2s}} c_{i_1,i_2,j_1,j_2}\, \xi_1^{i_1}\xi_2^{i_2}\eta_1^{j_1}\eta_2^{j_2}\ , \tag{11}$$

that we calculate up to the sixth order in eccentricity (the numerical values of the coefficients c are reported in App. A).

In order to test our model, we compare the fundamental frequencies calculated via Newton's equations for the complete SJS system with those related to the Hamiltonian (11). First, we determine the values of the initial conditions in the variables $(\xi\,,\,\eta)$. We express the initial conditions given in table 1 in orbital elements. Then, by using formula (2) we get the values in the original variables ξ,η, to which the canonical transformations related to the generating functions $X_{\mathcal{M}}^{(1)}$ and $Y_{\mathcal{M}}^{(1)}$ must be applied. By performing the latter transformation at order one in the masses we end up with the table 3.[(#)] By applying Laskar's frequency analysis we determine the fundamental frequencies for the Hamiltonian (11) with the initial conditions of table 3, thus getting

$$\bar{g}_1 = -30.03577993494953\ ''/\text{year}\ , \qquad \bar{g}_2 = -53.33350277285140\ ''/\text{year}\ . \tag{12}$$

The comparison with the secular frequencies calculated for the complete three–body problem (see the values of g_1^* and g_2^* in table 2) shows that the relative errors on the secular

[(#)] Remark that we limit ourselves to consider the correction $\mathcal{O}(\bar{m}/m_0)$ in the change of coordinates. The calculation of the further corrections is demanding and does not significantly modify the resulting frequencies (12).

frequency of Jupiter and Saturn are about 0.1 % and 1.3 %, respectively. As a comparison with other works, Laskar found an agreement of the same order of magnitude between the "real" secular frequencies of Jupiter and Saturn and those calculated from his more complicated model including the 8 main planets (see table 2 in [15]). Therefore, we consider that our model should be a reliable one for testing the applicability of KAM theory to the secular dynamics of the SJS system.

3. Construction of the secular torus

With the calculations of the previous sections we have reduced the Hamiltonian to a form corresponding to a conservative system close to an equilibrium point. Our aim now is to apply the Kolmogorov's normalization algorithm in order to find an invariant torus with given frequencies. This requires a few transformations in order to give the Hamiltonian a form suitable for starting the Kolmogorov's algorithm, namely the form

$$H(p,q) = \omega \cdot p + h_2(p) + f(p,q) + \mathcal{O}(p^3) \ , \tag{13}$$

where p, q are action–angle variables, ω are the frequencies of the unperturbed motion on the torus $p = 0$, $h_2(p)$ is a homogeneous polynomial of degree 2 and $f(p,q)$ is the part of the perturbation which is at most of degree 2 in the actions p. Here $f(p,q)$ is assumed to be small.

3.1 Preliminary transformations

We prepare the Hamiltonian by performing four operations: (i) diagonalization of the quadratic part of the Hamiltonian, (ii) transformation to action–angle variables, (iii) partial Birkhoff's normalization in order to remove the degeneration of the unperturbed Hamiltonian, and (iv) translation on the unperturbed torus with prescribed frequencies.

The diagonalization of the Hamiltonian is a classical well known topic: the quadratic part of the Hamiltonian may be given a diagonal form in case all eigenvalues of the corresponding linear system of differential equations are different; if, moreover, the eigenvalues are pure imaginary then one can find a linear canonical transformation such that the quadratic Hamiltonian takes the form of a system of harmonic oscillators (see, e.g., App. 6 of [4]). In our case the eigenvalues $i\nu_j$ are actually pure imaginary, so that we can give the quadratic Hamiltonian the form

$$\frac{\nu_1}{2}\left(\xi_1'^{\,2} + \eta_1'^{\,2}\right) + \frac{\nu_2}{2}\left(\xi_2'^{\,2} + \eta_2'^{\,2}\right)$$

The linear transformation $(\xi, \eta) \to (\xi', \eta')$ may be be constructed by adapting the procedure in [8], sec. 7.

Action–angle variables are introduced via the canonical transformation

$$\xi_j' = \sqrt{2I_j}\cos(\varphi_j) \ , \qquad \eta_j' = \sqrt{2I_j}\sin(\varphi_j) \ , \qquad j = 1,2 \ . \tag{14}$$

With this change of coordinates the Hamiltonian (11) takes the form

$$
(15)\qquad \begin{aligned} &\mathcal{H}^{(\mathrm{II})}_{Sec}(I_1, I_2, \varphi_1, \varphi_2) = \nu_1 I_1 + \nu_2 I_2 + \\ &\sum_{s=2}^{3} \sum_{i_1+i_2=2s} \sum_{j_1=0}^{i_1} \sum_{j_2=0}^{i_2} c^{(\mathrm{II})}_{i_1,i_2,j_1,j_2} \sqrt{I_1^{i_1} I_2^{i_2}} \cos\left[(i_1 - 2j_1)\varphi_1 + (i_2 - 2j_2)\varphi_2\right] , \end{aligned}
$$

where only cosines occur because our model (11) is invariant with respect to the symmetry $(\xi, \eta) \mapsto (\xi, -\eta)$.

The unperturbed part of the Hamiltonian above is degenerate, being linear in the actions. In order to remove the degeneration we perform a partial Birkhoff's normalization up to a finite order. As a matter of fact, normalization up to terms of the fourth order would be enough for a generic system close to an equilibrium point. However, in our model the 5/2 resonance of the SJS system produces quite big coefficients at order 6 in the ξ, η variables. Hence we push the Birkhoff's normalization up to order 6. The procedure is quite known. Let us rewrite the expansion (15) as

$$
(16)\qquad \mathcal{H}^{(\mathrm{II})}_{Sec}(I_1, I_2, \varphi_1, \varphi_2) = \sum_{s=1}^{3} f^{(\mathrm{II})}_s(I_1, I_2, \varphi_1, \varphi_2) ,
$$

where $f^{(\mathrm{II})}_1 = \nu_1 I_1 + \nu_2 I_2$ and the functions $f^{(\mathrm{II})}_s$ are homogeneous polynomials of degree $2s$ in $I^{1/2}$ and trigonometric polynomials of degree $2s$ in the angles φ, with $1 \le s \le 3$. We first determine a generating function $\mathcal{B}^{(\mathrm{III})}$ by solving the equation

$$
(17)\qquad \nu \cdot \frac{\partial \mathcal{B}^{(\mathrm{III})}}{\partial \varphi} + f^{(\mathrm{II})}_2 - \langle f^{(\mathrm{II})}_2 \rangle = 0 ,
$$

where $\langle \cdot \rangle$ indicates the average over the angles φ_1 and φ_2. With this transformation we get a new Hamiltonian of the form

$$
\mathcal{H}^{(\mathrm{III})}_{Sec} = \exp \mathcal{L}_{\mathcal{B}^{(\mathrm{III})}} \mathcal{H}^{(\mathrm{II})}_{Sec} = \sum_{s \ge 1} f^{(\mathrm{III})}_s ,
$$

with functions $f^{(\mathrm{III})}_s$ of the same type as above. This removes the dependence on the angles in $f^{(\mathrm{II})}_2$, so that $f^{(\mathrm{III})}_2 = \langle f^{(\mathrm{II})}_2 \rangle$. Similarly, we remove the dependence on the angles in $f^{(\mathrm{III})}_3$ by applying a canonical transformation with generating function $\mathcal{B}^{(\mathrm{IV})}$ determined by the equation

$$
(18)\qquad \nu \cdot \frac{\partial \mathcal{B}^{(\mathrm{IV})}}{\partial \varphi} + f^{(\mathrm{III})}_3 - \langle f^{(\mathrm{III})}_3 \rangle = 0 .
$$

This process ends up with a Hamiltonian

$$
(19)\qquad \mathcal{H}^{(\mathrm{IV})}_{Sec} = \exp \mathcal{L}_{\mathcal{B}^{(\mathrm{IV})}} \mathcal{H}^{(\mathrm{III})}_{Sec} = \sum_{s \ge 1} f^{(\mathrm{IV})}_s ,
$$

where $f^{(\mathrm{IV})}_1(I)$, $f^{(\mathrm{IV})}_2(I)$ and $f^{(\mathrm{IV})}_3(I)$ are independent of the angles. All the operations above are performed by truncating the expansions at a reasonably high finite degree $2N$ in $I^{1/2}$, corresponding to order $2N$ in eccentricity. In our case we choose $2N = 70$.

The final step is the translation on the unperturbed torus with prescribed frequencies. To this end we consider the part of the Hamiltonian (19) which is in Birkhoff's normal form. Recall that we are looking for an invariant torus with frequencies ω. First, we determine I^* as the solution of the equation

$$\sum_{s=1}^{3} \frac{\partial f_s^{(\mathrm{IV})}}{\partial I_j}(I) = \omega_j \ , \qquad j = 1, 2 \ ; \tag{20}$$

then we perform a translation of the actions

$$I_j = p_j + I_j^* \ , \qquad \varphi_j = q_j \ , \quad j = 1, 2 \ .$$

This produces a Hamiltonian of the form

$$\mathcal{H}_{Sec}^{(\mathrm{V})} = \sum_{s=1}^{N} \sum_{l=0}^{\infty} f_l^{(\mathrm{V},s)} \ , \tag{21}$$

where

$$f_l^{(\mathrm{V},s)}(p,q) = \sum_{j_1+j_2=l} \frac{1}{j_1!\, j_2!} \frac{\partial^{j_1+j_2} f_s^{(\mathrm{IV})}}{\partial I_1^{j_1} \, \partial I_2^{j_2}} \Bigg|_{\substack{I_1=I_1^* \ , \ I_2=I_2^* \\ \varphi_1=q_1 \ , \ \varphi_2=q_2}} p_1^{j_1} p_2^{j_2} \ . \tag{22}$$

This is essentially enough in order to start the process of Kolmogorov's normalization. However, for the purpose of producing analytical estimates it is convenient to rearrange the Hamiltonian as follows. Let us say that the function $f(p,q)$ is of class $\mathcal{P}_{l,s}$ in case it is a trigonometric polynomial of degree s in the angles q with coefficients that are homogeneous polynomials of degree l in the actions p, and moreover Fourier expansion contains only harmonics $k \cdot q$ with $k_1 + k_2$ an even integer. We write the Hamiltonian as

$$H^{(1)}(p,q) = h_1^{(0)} + \sum_{l=2}^{3} h_l^{(1)}(p) + \sum_{s=2}^{N} \sum_{l=0}^{\infty} f_l^{(1,s)}(p,q) + \mathcal{E}^{(1)} \ , \tag{23}$$

where $h_1^{(0)} = \omega \cdot p$, $h_l^{(1)} \in \mathcal{P}_{l,0}$, $f_l^{(1,s)} \in \mathcal{P}_{l,2s}$ and $\mathcal{E}^{(1)}$ is a constant. This form needs not be unique, so let us add a few details about our reordering scheme. Recalling (15), remark that only even sums $k_1 + k_2$ may occur in the generic term $c\, p_1^{j_1} p_2^{j_2} \cos(k_1 q_1 + k_2 q_2)$ appearing in the expansion of (21). Such a generic term is stored in (23) according to the following rule: *let $l = j_1 + j_2$ and $s = (|k_1| + |k_2|)/2$, then it must be added to $f_l^{(1,s)}$*. The previous general rule has the following exceptions:

a) when $|k_1| + |k_2| = 0$, then
   - a1) if $j_1 + j_2 = 0$ then add the term to $\mathcal{E}^{(1)}$; i.e., $\mathcal{E}^{(1)}$ takes into account all the additive constants in order to later calculate the energy of the invariant torus;
   - a2) if $j_1 + j_2 = 1$ and the term comes from $f_1^{(\mathrm{V},r)}$ with $1 \le r \le 3$, then add it to $h_1^{(0)}$; i.e., the main linear term of the Hamiltonian must be $\omega \cdot p$, according to equation (20);
   - a3) if $j_1 + j_2 = 1$ and the term comes from $f_1^{(\mathrm{V},r)}$ with $r > 3$, then add it to $f_1^{(1,3)}$;

a4) if $l = j_1 + j_2 = 2\,,\,3$ and the term comes from $f_l^{(\mathrm{V},r)}$ with $2 \le r \le 3$, then add it to $h_l^{(1)}$;

a5) if none of the cases a1, a2, a3, a4 occurs then add the term to $f_l^{(1,2)}$, where $l = j_1 + j_2$.

b) when $|k_1| + |k_2| = 2$, then add the term to $f_l^{(1,2)}$, where $l = j_1 + j_2$.

3.2 Kolmogorov's normalization algorithm

We follow [9] with minor variations due to the fact that the quadratic part of the unperturbed Hamiltonian is of order $\mathcal{O}(e^4)$, i.e., it is smaller than the linear part $\omega \cdot p$. This is due to the fact that we are working in the neighborhood of an equilibrium point.

We write the Hamiltonian in Kolmogorov's normal form up to order r as

$$H^{(r)}(p,q) = \omega \cdot p + \sum_{s=1}^{r} \sum_{l=2}^{\infty} h_l^{(s)}(p,q) + \sum_{s=r+1}^{\infty} \sum_{l=0}^{\infty} f_l^{(r,s)}(p,q) + \mathcal{E}^{(r)} \;, \tag{24}$$

where $h_l^{(s)} \in \mathcal{P}_{l,2s}$, $f_l^{(r,s)} \in \mathcal{P}_{l,2s}$, $h_2^{(1)} = \langle h_2^{(1)} \rangle$, $\langle f_1^{(r,r)} \rangle = 0$ and $\mathcal{E}^{(r)}$ is a constant.

We assume that we are given an Hamiltonian $H^{(r-1)}$ in normal form up to order $r-1$ and we show how to calculate $H^{(r)}$. Remark that the Hamiltonian $H^{(1)}$ is already in normal form; hence we start with $r = 2$.

We define the new terms $h_l^{(r)}$ of the normal form as

$$h_l^{(r)} = f_l^{(r-1,r)} \qquad \text{for } l \ge 2 \;. \tag{25}$$

Then we remove the unwanted terms $f_0^{(r-1,r)}(q)$ and $\langle f_1^{(r-1,r+1)} \rangle(p)$ via a canonical transformation with generating function $\chi_1^{(r)}(q) = X^{(r)}(q) + \xi^{(r)} \cdot q$ (being $\xi^{(r)}$ a real vector). To this end we solve with respect to $X^{(r)}(q)$ and $\xi^{(r)}$ the equations

$$\begin{aligned} \omega \cdot \frac{\partial X^{(r)}}{\partial q}(q) + f_0^{(r-1,r)}(q) &= \langle f_0^{(r-1,r)} \rangle \;, \\ \mathcal{C}^{(r)} \xi^{(r)} \cdot p + \langle f_1^{(r-1,r+1)} \rangle(p) &= 0 \;, \end{aligned} \tag{26}$$

where $\frac{1}{2}\mathcal{C}^{(r)} p \cdot p = \sum_{s=1}^{r} \langle h_2^{(s)} \rangle$. An unique solution satisfying $\langle X^{(r)} \rangle = 0$ exists if the frequencies ω are non–resonant up to order $2r$, i.e., $k_1\omega_1 + k_2\omega_2 \ne 0\,,\ \forall\, 0 < |k_1| + |k_2| \le 2r$, $k \in \mathbf{Z}^2$ and if $\mathcal{C}^{(r)}$ is a non–degenerate matrix.

The transformation produces an intermediate Hamiltonian

$$\hat{H}^{(r)}(p,q) = \omega \cdot p + \sum_{s=1}^{r} \sum_{l=2}^{\infty} h_l^{(s)}(p,q) + \hat{f}_1^{(r,r)}(p,q) + \sum_{s=r+1}^{\infty} \sum_{l=0}^{\infty} \hat{f}_l^{(r,s)}(p,q) + \mathcal{E}^{(r)} \;, \tag{27}$$

where $\mathcal{E}^{(r)} = \mathcal{E}^{(r-1)} + \omega \cdot \xi^{(r)} + \langle f_0^{(r-1,r)} \rangle$ is a new constant. The functions $f_l^{(r,s)}$, are recursively defined according to the following general rules, with a couple of exceptions. We initially set $\hat{f}_1^{(r,r)} = f_1^{(r-1,r)}$ and $\hat{f}_l^{(r,s)} = f_l^{(r-1,s)}$ for $l \ge 0$ and $s > r$. Then we perform the replacements [♮]

$$
(28)\quad
\begin{aligned}
&\hat f_{l-j}^{(r,jr+s)} \leftarrow \hat f_{l-j}^{(r,jr+s)} + \frac{1}{j!}\mathcal{L}^{j}_{\chi_1^{(r)}} h_l^{(s)} &&\text{for } 0<s<r\,,\ l\ge 2\,,\ 1\le j\le l\,,\\
&\hat f_{l-j}^{(r,jr+s)} \leftarrow \hat f_{l-j}^{(r,jr+s)} + \frac{1}{j!}\mathcal{L}^{j}_{\chi_1^{(r)}} f_l^{(r-1,s)} &&\text{for } s\ge r\,,\ l\ge 1\,,\ 1\le j\le l\,.
\end{aligned}
$$

The exceptions to the prescriptions above take into account the terms that are cancelled out in view of the second equation in (26). We define

$$
(29)\quad
\begin{aligned}
&\hat f_1^{(r,r+1)} = \mathcal{L}_{X^{(r)}} h_2^{(1)} + f_1^{(r-1,r+1)} - \langle f_1^{(r-1,r+1)}\rangle &&,\\
&\hat f_1^{(r,r+m)} = \mathcal{L}_{X^{(r)}} h_2^{(m)} + \mathcal{L}_{\xi^{(r)}\cdot q}\left(h_2^{(m)} - \langle h_2^{(m)}\rangle\right) + f_1^{(r-1,r+m)} &&\forall\ 1<m\le r\,.
\end{aligned}
$$

It is an easy matter to check that $h_l^{(r)}\in\mathcal{P}_{l,2r}$, $\hat f_1^{(r,r)}\in\mathcal{P}_{1,2r}$, $\hat f_l^{(r,s)}\in\mathcal{P}_{l,2s}$ for $s>r$, $\langle\hat f_1^{(r,r)}\rangle=0$ and $\langle\hat f_1^{(r,r+1)}\rangle=0$.

We finally remove $\hat f_1^{(r,r)}(p,q)$ via a canonical transformation with generating function $\chi_2^{(r)}(p,q)$. To this end we solve the equation

$$
(30)\qquad \omega\cdot\frac{\partial\chi_2^{(r)}}{\partial q}(p,q) + \hat f_1^{(r,r)}(p,q) = 0\ .
$$

A solution exists because $\langle\hat f_1^{(r,r)}\rangle=0$, and the frequencies ω are non–resonant up to order $\le 2r$. The transformation gives the Hamiltonian the normal form (24). The functions $f_l^{(r,s)}$ are determined by initially setting $f_l^{(r,s)}=\hat f_l^{(r,s)}$ and then performing the replacements

$$
(31)\quad
\begin{aligned}
&f_l^{(r,jr+s)} \leftarrow f_l^{(r,jr+s)} + \frac{1}{j!}\mathcal{L}^{j}_{\chi_2^{(r)}} h_l^{(s)} &&\text{for } l\ge 2\,,\ 1\le s\le r\,,\ j\ge 1\,,\\
&f_l^{(r,jr+s)} \leftarrow f_l^{(r,jr+s)} + \frac{1}{j!}\mathcal{L}^{j}_{\chi_2^{(r)}} \hat f_l^{(r,s)} &&\text{for } l\ge 0\,,\ s>r\,,\ j\ge 1\,,\\
&f_1^{(r,(j+1)r)} \leftarrow f_1^{(r,(j+1)r)} + \frac{j}{(j+1)!}\mathcal{L}^{j}_{\chi_2^{(r)}} \hat f_1^{(r,r)} &&\text{for } j\ge 1\,.
\end{aligned}
$$

Again, it is an easy matter to check that $f_l^{(r,s)}\in\mathcal{P}_{l,2s}$ for $s>r$ and $\langle f_1^{(r,r+1)}\rangle=0$. This concludes the normalization step.

3.3 Numerical tests

A first test concerns the effect of the truncation at order $2N=70$ of the Hamiltonian (19). We consider the value of the energy. To this end we calculate the values of the initial conditions in coordinates (I,φ), starting from those in table 3 and taking into account the first three preliminary transformations listed in sect. 3.1; the results are reported in table 4. Then, putting these values of (I,φ) as arguments in the sum of the first 35 terms of the Hamiltonian (19), we determine the corresponding energy $\bar E$. This agrees except for

[♮] The notation $f\leftarrow f+g$ means that f is replaced by $f+g$. This corresponds exactly to what we do in our program.

Table 4. Initial conditions of the coordinates $(I\,,\,\varphi)$ corresponding to that in table 3.

Jupiter	$I_1 = 6.5759961186315956 \cdot 10^{-4}$	$\varphi_1 = 1.6385266516501007$
Saturn	$I_2 = 3.2097485037333470 \cdot 10^{-4}$	$\varphi_2 = 3.1140763924923589$

the last significant digit with the energy calculated via the Hamiltonian (11) for the point in table 3. Therefore, we estimate that the effect of the truncation should be at most of the order of magnitude of the roundoff error.

A second test is concerned with the Kolmogorov's normal form. According to the analytic theory, if the Kolmogorov's algorithm converges to a normal form then it produces an invariant torus carrying quasi–periodic orbits with the prescribed frequencies ω. In the next section we shall prove that the approximated secular model possesses invariant tori with frequencies ω close to the values of $\bar{g}$ in (12), with appropriate non–resonance conditions. For the moment, let us set $\omega = \bar{g}$ and perform only a finite number of steps of Kolmogorov's algorithm, thus getting an approximated invariant torus. In this case the high order resonances between the frequencies are harmless. We proceed as follows. Denote by $\mathcal{T}_r$ the composition of the canonical transformations bringing the Hamiltonian in Kolmogorov's normal form up to order r (i.e., $H^{(r)}$ as in (24)). Taking the initial point $(\xi(0),\eta(0))$, reported in table 3, we calculate $(\tilde{\xi}_r(t),\tilde{\eta}_r(t))$ according to the scheme

$$
\begin{array}{ccc}
(\xi(0),\eta(0)) & \stackrel{\mathcal{T}_r}{\longrightarrow} & (p(0)\equiv 0\,,\,q(0)) \\
 & & \downarrow \\
(\tilde{\xi}_r(t),\tilde{\eta}_r(t)) & \stackrel{\mathcal{T}_r^{-1}}{\longleftarrow} & (p(t)\equiv 0\,,\,q(t)\equiv q(0)+\omega t)
\end{array}
\qquad . \tag{32}
$$

The point $\tilde{\xi}_r(t),\tilde{\eta}_r(t)$ so obtained is compared with the point $\xi(t),\eta(t)$ given by numerical integration of the system (11), with the same initial point. The results for different values of r are reported in Fig. 3.

The improvement of the approximation at increasing order is quite evident by looking at the logarithmic scale on the vertical axis. It is also noticed that in Fig. 3d a slow drift starts to appear. The drift becomes the main effect in Fig. 3e, corresponding to $r = 13$. We ascribe this effect to the error in determining the frequencies via frequency analysis. Indeed, a rough evaluation of the drift rate is $4\,\times 10^{-18}$. On the other hand $\|(\xi(t),\eta(t))\|$ nearly behaves as a constant and its value is $\sim 4.4\times 10^{-2}$. We conclude that the calculation of the frequencies (12) is affected by an uncertainty $\sim 10^{-16}$ rad/year. Our evaluation of the error of the Laskar's method substantially agrees with that given in sect. III.2.2 of [30].

4. Computer–assisted proof of stability

Our aim is to prove that the Hamiltonian (19) truncated at $s = 35$ and obtained as described in sect. 3.1 possesses two invariant tori which confine forever the orbit starting

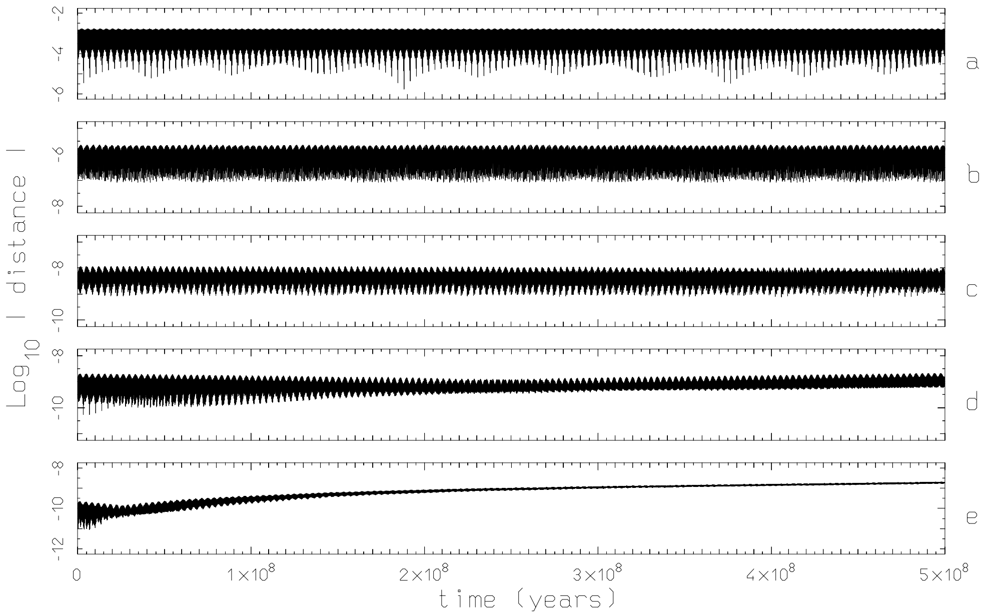

Figure 3. The distance $d(t)$ between the numerically integrated orbit and the approximated motion $(\tilde{\xi}_r(t), \tilde{\eta}_r(t))$ calculated via the scheme (32). The curves a–e refer to the step r of Kolmogorov's algorithm with $r = 1, 5, 9, 11, 13$, respectively. The convergence may be appreciated by looking at the vertical scale. The drift effect in figure e is due to the error in the determination of the frequencies.

from the initial conditions reported in table 4.

Following the scheme in [6] we calculate R' steps of Kolmogorov's algorithm.[(@)] The result improves with increasing R', of course. We found that setting $R' = 33$ is sufficient for our purposes. In order to make the proof rigorous all coefficients in the expansions of sect. 3 have been calculated by using the interval arithmetics (see, e.g., [13] and [32]).

4.1 Iteration of the estimates

Let us first introduce some notations. For $v \in \mathbf{R}^n$ we denote $|v| = \sum_{j=1}^{n} |v_j|$. Let us write the expansion of a generic function $g \in \mathcal{P}_{l,K}$, with multi-index notation, as $g(p,q) = \sum_{|j|=l} \sum_{|k|\le K} c_{jk} p^j \,{}^{\sin}_{\cos}(k \cdot q)$, where the expression ${}^{\sin}_{\cos}$ means that both the contributions with the sines and the cosines may occur. Then we introduce the norm

$$\|g\| = \sum_{|j|=l} \sum_{|k|\le K} |c_{jk}| \ . \tag{33}$$

We look now for recursive estimates for the norms of the functions $h_l^{(s)}$, $f_l^{(r,s)}$ appearing in (24). Precisely, given a positive integer R'' we look for a positive constant E and two

(@) This means that at the r–th step (with $1 \le r \le R'$) of the Kolmogorov's algorithm we calculate at least the explicit expansion of the generating functions $\chi_1^{(r)}$ and $\chi_2^{(r)}$, of the functions $h_l^{(r)}$ such that $2 \le l \le \lfloor (R'+1-r)/r \rfloor + 2$ and of the functions $\hat{f}_l^{(r,s)}$ and $f_l^{(r,s)}$ such that $0 \le l \le \lfloor (R'+1-s)/r \rfloor + 2$ and $r < s \le R'+1$.

finite sequences $\{\varepsilon_r\}_{r=1}^{R''}$ and $\{\zeta_r\}_{r=1}^{R''}$ of positive real numbers such that for $1 \le r \le R''$ we have

$$\begin{aligned} \left\| h_l^{(s)} \right\| &\le \varepsilon_r^{s+1} E\, \zeta_r^{\,l} && \text{for } 1 \le s \le r\,,\ l \ge 2 \quad , \\ \left\| f_l^{(r,s)} \right\| &\le \varepsilon_r^{s+1} E\, \zeta_r^{\,l} && \text{for } s > r\,,\ l \ge 0 \quad , \end{aligned} \tag{34}$$

The iteration of the estimates is performed as follows:

a) Estimate of the functions $\chi_1^{(1)}$, $\chi_2^{(1)}$, $f_2^{(1,2)}$, …, $\chi_1^{(R'')}$, $\chi_2^{(R'')}$, $f_2^{(R'',R''+1)}$ and of all the intermediate functions necessary to evaluate the previous ones.

b) Derivation of the upper bounds (34) on the infinite sequence of terms appearing in the expansion (24), for $1 \le r \le R''$.

4.1.1 Estimates on the truncated Hamiltonians

We start with the *truncated* Hamiltonian (19) after the Birkhoff transformation. This contains a finite number of monomials; hence it is easy to evaluate some upper bounds for

$$\frac{1}{2\pi^2} \sup_{(I_1,I_2)\in\mathcal{D}_{2I_1^*,2I_2^*}} \left| \int_{\varphi\in\mathbf{T}^2} f_s^{(\mathrm{IV})}(I,\varphi) \begin{matrix}\sin\\ \cos\end{matrix}(k\cdot\varphi)\,d\varphi \right| \ , \qquad |k| \le 2s\,,\ 1 \le s \le N\ , \tag{35}$$

where $N = 35$ and the domain of analyticity $\mathcal{D}_{2I_1^*,2I_2^*}$ is defined as

$$\mathcal{D}_{2I_1^*\,,\,2I_2^*} = \{(I_1,I_2):\ 0 < I_1 < 2I_1^*\,,\ 0 < I_2 < 2I_2^*\}\ .$$

Then, we estimate the functions appearing in expansion (21) via Cauchy estimates, thus getting

$$\left\| f_l^{(\mathrm{V},s)} \right\| \le \frac{\left(1 - \frac{\min\{I_1^*,I_2^*\}}{\max\{I_1^*,I_2^*\}}\right)^{-1}}{2\pi^2\left[\min\{I_1^*,I_2^*\}\right]^l} \sum_{|k|\le 2s} \sup_{(I_1,I_2)\in\mathcal{D}_{2I_1^*,2I_2^*}} \left| \int_{\varphi\in\mathbf{T}^2} f_s^{(\mathrm{IV})}(I,\varphi) \begin{matrix}\sin\\ \cos\end{matrix}(k\cdot\varphi)\,d\varphi \right| \ . \tag{36}$$

The functions $h_l^{(1)}$ and $f_l^{(1,s)}$ and constant $\mathcal{E}^{(1)}$ appearing in (23) can be similarly estimated, just taking into account all the prescriptions at the end of sect. 3.1.

The following inequalities are the basic tool for most of the estimates in the algorithm below. Let $g \in \mathcal{P}_{l,2s}$ a generic function appearing in the expansions of the Hamiltonians described in sect. 3.2. Then

$$\begin{aligned} \left\| \frac{1}{j!} \mathcal{L}^j_{\chi_1^{(r)}} g \right\| &\le \binom{l}{j} \left(\max_{j'} \left\{ \left\| \frac{\partial X^{(r)}}{\partial q_{j'}} \right\| \right\} + \max_{j'} \left\{ \left| \xi_{j'}^{(r)} \right| \right\} \right)^j \|g\| \\ \left\| \frac{1}{j!} \mathcal{L}^j_{\chi_2^{(r)}} g \right\| &\le \frac{1}{j!} \prod_{i=1}^{j} \left[l \max_{j'} \left\{ \left\| \frac{\partial \chi_2^{(r)}}{\partial q_{j'}} \right\| \right\} + 2(s + (i-1)r) \max_{j'} \left\{ \left\| \frac{\partial \chi_2^{(r)}}{\partial p_{j'}} \right\| \right\} \right] \|g\| \,. \end{aligned} \tag{37}$$

We now give the formulæ that allow us to estimate an iteration step in the form of a computational algorithm. We emphasize that in our calculation we implemented exactly this scheme in order to calculate recursive estimates.

Start with the functions $h_l^{(s)}$ *and* $f_l^{(r-1,s)}$ *appearing in the Hamiltonian* $H^{(r-1)}$ *in the form (24). Let* $\|h_l^{(s)}\| \le \mathcal{F}_l^{(s,s)}$, $\|f_l^{(r-1,s)}\| \le \mathcal{F}_l^{(r-1,s)}$ *and* $\|\langle f_1^{(r-1,s)}\rangle\| \le \mathcal{F}_{\langle\rangle}^{(r-1,s)}$ *with known* $\mathcal{F}_l^{(s,s)}$, $\mathcal{F}_l^{(r-1,s)}$ *and* $\mathcal{F}_{\langle\rangle}^{(r-1,s)}$. *Let the matrix* $\mathcal{C}^{(r)}$ *in (26) satisfy the non–degeneracy condition* $|\mathcal{C}^{(r)} \cdot v| \ge \varrho^{(r)}|v| \; \forall \; v \in \mathbf{R}^n$, *with some* $\varrho^{(r)}$. *Then:*

(i) the norms of the generating functions are bounded by $\max_{j'}\{\|\partial X^{(r)}/\partial q_{j'}\|\} \le \mathcal{G}_{11}^{(r)}$, $\max_{j'}\{|\xi_{j'}^{(r)}|\} \le \mathcal{G}_{12}^{(r)}$, $\max_{j'}\{\|\partial \chi_2^{(r)}/\partial q_{j'}\| \le \mathcal{G}_{21}^{(r)}$ *and* $\max_{j'}\{\|\partial \chi_2^{(r)}/\partial p_{j'}\|\} \le \mathcal{G}_{22}^{(r)}$ *where*

$$
\begin{aligned}
&\mathcal{G}_{11}^{(r)} \le 2r\Bigg(\min_{\substack{0<|k|\le 2r\\ |k|\ \mathrm{even}}} |k\cdot\omega|\Bigg)^{-1} \mathcal{F}_0^{(r-1,r)} \,, \qquad \mathcal{G}_{12}^{(r)} \le \frac{1}{\varrho^{(r)}}\mathcal{F}_{\langle\rangle}^{(r-1,r+1)} \,, \\
&\mathcal{G}_{21}^{(r)} \le 2r\Bigg(\min_{\substack{0<|k|\le 2r\\ |k|\ \mathrm{even}}} |k\cdot\omega|\Bigg)^{-1} \mathcal{F}_1^{(r-1,r)} \,, \qquad \mathcal{G}_{22}^{(r)} \le \Bigg(\min_{\substack{0<|k|\le 2r\\ |k|\ \mathrm{even}}} |k\cdot\omega|\Bigg)^{-1} \mathcal{F}_1^{(r-1,r)} \;;
\end{aligned}
\tag{38}
$$

(ii) the norms of the functions appearing in expansion (27) of Hamiltonian $\hat{H}^{(r)}$ *are bounded by the inequalities* $\|h_l^{(r)}\| \le \mathcal{F}_l^{(r,r)}$ *for* $l \ge 2$, $\|\hat{f}_1^{(r,r)}\| \le \hat{\mathcal{F}}_1^{(r,r)}$ *and* $\|\hat{f}_l^{(r,s)}\| \le \hat{\mathcal{F}}_l^{(r,s)}$ *for* $l \ge 0$ *and* $s > r$. *Here* $\mathcal{F}_l^{(r,r)} = \mathcal{F}_l^{(r-1,r)}$ *for* $l \ge 2$, *and the constants* $\hat{\mathcal{F}}_l^{(r,s)}$ *are determined by initially setting* $\hat{\mathcal{F}}_l^{(r,s)} = \mathcal{F}_l^{(r-1,s)}$ *for either* $l \ge 0$ *and* $s > r$ *or* $l = 1$ *and* $s = r$, *then performing the replacements*

$$
\begin{aligned}
&\hat{\mathcal{F}}_{l-j}^{(r,jr+s)} \leftarrow \hat{\mathcal{F}}_{l-j}^{(r,jr+s)} + \binom{l}{j}\left(\mathcal{G}_{11}^{(r)} + \mathcal{G}_{12}^{(r)}\right)^j \mathcal{F}_l^{(s,s)} \qquad \text{for } l \ge 2\,,\ 0 < s < r\,,\ 1 \le j \le l\,,\\
&\hat{\mathcal{F}}_{l-j}^{(r,jr+s)} \leftarrow \hat{\mathcal{F}}_{l-j}^{(r,jr+s)} + \binom{l}{j}\left(\mathcal{G}_{11}^{(r)} + \mathcal{G}_{12}^{(r)}\right)^j \mathcal{F}_l^{(r-1,s)} \qquad \text{for } l \ge 1\,,\ s \ge r\,,\ 1 \le j \le l;
\end{aligned}
\tag{39}
$$

(iii) the constant $\mathcal{E}^{(r)}$ *appearing in expansion (27) and (24) satisfies*

$$
\left|\mathcal{E}^{(r)} - \mathcal{E}^{(r-1)}\right| \le \mathcal{G}_{12}^{(r)}\,|\omega| + \mathcal{F}_0^{(r-1,r)} \;;
\tag{40}
$$

(iv) the norms of functions $f_l^{(r,s)}$ *appearing in expansion (24) of Hamiltonian* $H^{(r)}$ *are bounded by* $\|f_l^{(r,s)}\| \le \mathcal{F}_l^{(r,s)}$ *for* $l \ge 0$ *and* $s > r$. *Here the constants* $\mathcal{F}_l^{(r,s)}$ *are determined by initially setting* $\mathcal{F}_l^{(r,s)} = \hat{\mathcal{F}}_l^{(r,s)}$ *for* $l \ge 0$ *and* $s > r$, *and then performing the replacements*

$$
\begin{aligned}
&\mathcal{F}_l^{(r,jr+s)} \leftarrow \mathcal{F}_l^{(r,jr+s)} + \frac{1}{j!}\prod_{i=0}^{j-1}\Big[l\mathcal{G}_{21}^{(r)} + 2((j-1-i)r+s)\mathcal{G}_{22}^{(r)}\Big]\mathcal{F}_l^{(s,s)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } l \ge 2\,,\ 1 \le s \le r\,,\ j \ge 1\,, \\
&\mathcal{F}_l^{(r,jr+s)} \leftarrow \mathcal{F}_l^{(r,jr+s)} + \frac{1}{j!}\prod_{i=0}^{j-1}\Big[l\mathcal{G}_{21}^{(r)} + 2((j-1-i)r+s)\mathcal{G}_{22}^{(r)}\Big]\hat{\mathcal{F}}_l^{(r,s)} \\
&\qquad\qquad\qquad \text{for either } l \ge 0\,,\ s > r\,,\ j \ge 1\,, \quad \text{or } l = 1\,,\ s = r\,,\ j \ge 1\,;
\end{aligned}
\tag{41}
$$

(v) *for the average over the angles of the functions linear in p appearing in expansion (27), the inequality* $\|\langle \hat{f}_1^{(r,s)}\rangle\| \le \hat{\mathcal{F}}_{\langle\rangle}^{(r,s)}$ *holds, where* $\hat{\mathcal{F}}_{\langle\rangle}^{(r,r)} = \hat{\mathcal{F}}_{\langle\rangle}^{(r,r+1)} = 0$ *and*

$$
\hat{\mathcal{F}}_{\langle\rangle}^{(r,r+s)} = \mathcal{F}_{\langle\rangle}^{(r-1,r+s)} + 2\mathcal{G}_{11}^{(r)}\mathcal{F}_2^{(s,s)} \quad \forall\ 1 < s \le r\,, \quad \hat{\mathcal{F}}_{\langle\rangle}^{(r,s)} = \hat{\mathcal{F}}_1^{(r,s)} \quad \forall\ s > 2r\,;
\tag{42}
$$

and, for what concerns the expansion (24), the inequality $\|\langle f_1^{(r,s)}\rangle\| \le \hat{\mathcal{F}}_{\langle\rangle}^{(r,s)}$ *holds, where*

$$
\begin{aligned}
&\mathcal{F}_{\langle\rangle}^{(r,s)} = \hat{\mathcal{F}}_{\langle\rangle}^{(r,s)} \quad \forall\ r < s < 2r\,, \qquad \mathcal{F}_{\langle\rangle}^{(r,2r)} = \hat{\mathcal{F}}_{\langle\rangle}^{(r,2r)} + \big(\mathcal{G}_{21}^{(r)} + 2r\mathcal{G}_{22}^{(r)}\big)\hat{\mathcal{F}}_1^{(r,r)}\,, \\
&\qquad\qquad\qquad\qquad \mathcal{F}_{\langle\rangle}^{(r,s)} = \mathcal{F}_1^{(r,s)} \quad \forall\ s > 2r\,;
\end{aligned}
\tag{43}
$$

(vi) *the matrix* $\mathcal{C}^{(r+1)}$*, defined by the equation* $\frac{1}{2}\mathcal{C}^{(r+1)}p\cdot p = \sum_{s=1}^{r+1}\langle h_2^{(s)}\rangle$*, satisfies the inequality* $|\mathcal{C}^{(r+1)}\cdot v| \ge \varrho^{(r+1)}|v|\ \forall\ v \in \mathbf{R}^n$*, where* $\varrho^{(r+1)}$ *is given by*

$$
\varrho^{(r+1)} = \varrho^{(r)} - 2\mathcal{F}_2^{(r,r+1)}\,.
\tag{44}
$$

The actual implementation goes as follows: at low orders, up to $R' = 33$, we explicitly calculate the functions required by Kolmogorov's algorithm using an algebraic manipulator and we evaluate the sequence of bounds $\mathcal{G}$ and $\mathcal{F}$ according to (33); for higher orders, up to $R'' = 2048$ we use the recursive estimates given by the algorithm above. The worsening effect of the change of method is illustrated in Fig. 4 for the generating function $\chi_2^{(r)}$. We emphasize that the choice of R' and R'' may be delicate: the final result may critically depend on this choice. On one hand, if R' is too small then the recursive estimates will fail to work. On the other hand choosing too high values may critically increase the computational time.

4.1.2 Estimates on the infinite power series expansions

Although the computer implementation of the iterative estimates of the previous section can provide the upper bounds for a very large number of terms appearing in expansion (24), this is not sufficient to ensure the existence of the invariant tori. Indeed, the version of the KAM theorem reported in sect. 4.2 needs some upper limits on all the terms appearing in the infinite power series expansion of the Hamiltonian. Therefore, all the upper bounds

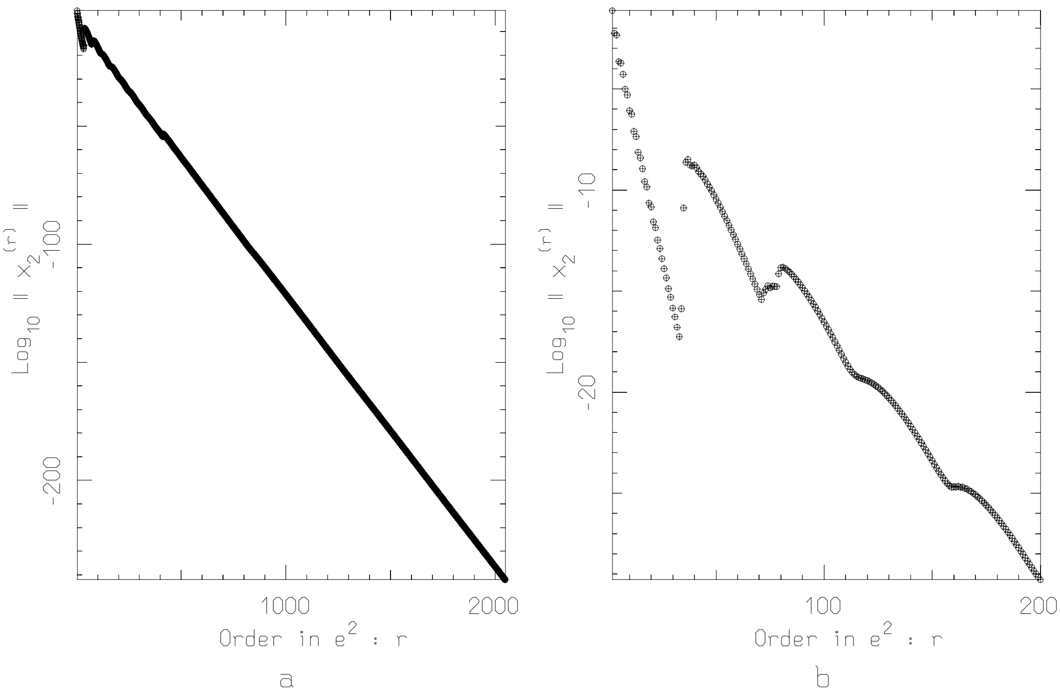

Figure 4. Decrease of the generating functions defined by the Kolmogorov's normalization algorithm. Figure 4a considers the construction of the invariant tori corresponding to the frequencies belonging to the set Γ'' defined in (51). The plotted values are the uniform upper bounds of the norms of $\chi_2^{(r)}$ on the whole set Γ'' . Figure 4a has been enlarged in figure 4b, where we can appreciate the change of the slope occurring when the calculation of the norms is no more made starting by the coefficients of the expansions, but only iterating the estimates (i.e., for $r = 33$).

calculated in the previous section will be now reduced to a positive constant E and two finite sequences $\{\varepsilon_r\}_{r=1}^{R''}$ and $\{\zeta_r\}_{r=1}^{R''}$ satisfying inequalities (34) with $1 \le r \le R''$.

We first look for values E , ε_1 and ζ_1 bounding expansion (23) of $H^{(1)}$. In view of estimate (36) we can take

$$\zeta_1 = \left[\, \min\{I_1^*, I_2^*\}\,\right]^{-1} \ . \tag{45}$$

Next, we evaluate (35) for $f_s^{(\mathrm{IV})}$ in Hamiltonian $\mathcal{H}_{Sec}^{(\mathrm{IV})}$, also recalling the prescriptions at the end of sect. 3.1. This allows us to determine E and ε_1 so that (34) are satisfied for $r = 1$ with ζ_1 given by (45). Finally, the finite sequences $\{\varepsilon_r\}_{r=1}^{R''}$ and $\{\zeta_r\}_{r=1}^{R''}$ are determined by iterating the following algorithm.

Start with three positive numbers E , ε_{r-1} and ζ_{r-1} satisfying inequalities (34) at step $r-1$. Moreover, assume that the bounds $\mathcal{G}_{11}^{(r)}$, $\mathcal{G}_{12}^{(r)}$, $\mathcal{G}_{21}^{(r)}$ and $\mathcal{G}_{22}^{(r)}$ for the generating functions $\chi_1^{(r)}$ and $\chi_2^{(r)}$ be known. Then calculate

$$\hat{\varepsilon}_r = e^{1/r^3} \max\left\{ \varepsilon_{r-1}\, , \, \left[r^2 \left(\mathcal{G}_{11}^{(r)} + \mathcal{G}_{12}^{(r)} \right) \zeta_{r-1} \right]^{1/r} \right\} \ , \qquad \hat{\zeta}_r = e^{1/r^2} \zeta_{r-1} \ . \tag{46}$$

If $r \le R'$ evaluate the new constants ε_r and ζ_r as
(47)

$$\varepsilon_r = \hat{\varepsilon}_r \left(1 + \frac{1}{\hat{\varepsilon}_r^{\,r}} \max\left\{ \mathcal{G}_{21}^{(r)}\,,\; 2r\mathcal{G}_{22}^{(r)} \right\}\right)^{1/r} \,, \qquad \zeta_r = \hat{\zeta}_r \left(1 + \frac{1}{\hat{\varepsilon}_r^{\,r}} \max\left\{ \mathcal{G}_{21}^{(r)}\,,\; 2r\mathcal{G}_{22}^{(r)} \right\}\right) \,,$$

else, if $r > R'$, set

$$\varepsilon_r = e^{1/r^3} \max\left\{ \hat{\varepsilon}_r\,,\; \left[r^2 \max\left\{ \mathcal{G}_{21}^{(r)}\,,\; 2r\mathcal{G}_{22}^{(r)} \right\} \right]^{1/r} \right\} \,, \qquad \zeta_r = e^{1/r^2} \hat{\zeta}_r \,. \tag{48}$$

Recalling the estimates in sect. 4.1.1, one can prove by induction that inequalities (34) are verified by the values of E, ε_r and ζ_r, given by (46)–(48).

The procedure described above can be easily implemented on a computer using the interval arithmetics. This allows us to rigorously prove inequalities (34) for $r = R''$, which will be used to apply the theorem below.

4.2 Statement of the theorem

The following statement is an adaptation of KAM theorem to our context.

Theorem 2: *Consider a n–degrees of freedom real analytic Hamiltonian $H^{(R'')}$ of the form (24). Assume:*

(a) $h_l^{(s)} \in \mathcal{P}_{l,sK}$ and $f_l^{(R'',s)} \in \mathcal{P}_{l,sK}$, where K is a fixed positive integer;

(b) the frequencies ω satisfy the Diophantine non–resonant condition, i.e., there are $\gamma > 0$ and $\tau \ge n-1$ such that $|k \cdot \omega| \ge \gamma r^{-\tau}$ for $0 < |k| \le rK$ and $r \ge 1$;

(c) the main quadratic part in the actions of the Hamiltonian is angle–independent, i.e. $h_2^{(1)} = \langle h_2^{(1)} \rangle$. Moreover, the angular average of the quadratic terms of the Hamiltonian already in normal form satisfies the non–degeneracy condition $|\mathcal{C}^{(R'')} \cdot v| \ge \varrho^{(R'')} |v|$ $\forall\, v \in \mathbf{R}^n$ for some positive $\varrho^{(R'')}$, where the matrix $\mathcal{C}^{(R'')}$ is defined by the relation $\frac{1}{2}\mathcal{C}^{(R'')} p \cdot p = \sum_{s=1}^{R''} \langle h_2^{(s)} \rangle$;

(d) the main perturbing term linear in the actions has zero average over the angles, i.e., $\langle f_1^{(R'',R''+1)} \rangle = 0$;

(e) the bounds (34) on the norms of the functions $h_l^{(s)}$ and $f_l^{(R'',s)}$ hold when $r = R''$;

(f) the inequality $\varepsilon_{R''} < \varepsilon^*(\varepsilon_{R''}, E, \zeta_{R''}, \gamma, \tau, \varrho^{(R'')}, K, R'')$ is satisfied by the small parameter $\varepsilon_{R''}$, ε^* being the only positive root of the equation

$$\frac{(\varepsilon^* a^*)^{R''}}{1 - \varepsilon^* a^*} = \min\left\{ \frac{\varrho^{(R'')}}{4\varepsilon_{R''}^2 E\, \bar{\zeta}^2}\,,\; \frac{e^{-3/R''}}{2\pi^2} \right\} \,,$$

where $\bar{\zeta} = \zeta_{R''} e^{3/R''}$ and

$$a^* = 2^{(2\tau+12)b_{\bar{l}}} \left(Z_1\, Z_2\right)^{2/R''} e^{3/[2(R'')^2]} \,,$$

with $2^{\bar{l}} \le R'' < 2^{\bar{l}+1}$ and where the quantities Z_1, Z_2 and $b_{\bar{l}}$ are defined as

$$Z_1 = \max\left\{ \frac{\varepsilon_{R''} K E\, \bar{\zeta}}{\gamma}\,,\, 1 \right\} \,, \qquad Z_2 = \max\left\{ \frac{2\varepsilon_{R''}^2 E\, \bar{\zeta}^2}{\varrho^{(R'')}}\,,\, 1 \right\} \,, \qquad b_{\bar{l}} = \frac{\bar{l}+2}{2^{\bar{l}-1}} \,.$$

Then the following statement holds true: there exists a canonical transformation $(p,q)=\psi(P,Q)$*, real analytic for* $P\in B_{3/(4\bar\zeta)}(0)$ *and* $Q\in\mathbf{T}^n$ *, which brings the Hamiltonian* $H^{(R'')}$ *to Kolmogorov's normal form*

$$H^{(\infty)}(P,Q)=\omega\cdot P+\sum_{s=1}^{\infty}\sum_{l=2}^{\infty}h_l^{(s)}(P,Q)+\mathcal{E}^{(\infty)}\ . \tag{49}$$

The canonical transformation is near to the identity, i.e. $\psi-\mathbf{I}\sim O(\varepsilon_{R''}^{R''+1})$ *. Moreover, the energy* $\mathcal{E}^{(\infty)}$ *corresponding to the invariant torus* $P=0$ *satisfies the inequality*

$$\left|\mathcal{E}^{(\infty)}-\mathcal{E}^{(R'')}\right|\le\left(\varepsilon_{R''}E+\frac{|\omega|}{\zeta_{R''}}\right)\frac{\left(\varepsilon_{R''}a^*\right)^{R''+1}}{1-\varepsilon_{R''}a^*}\ . \tag{50}$$

The interested reader will find the proof of this version of the KAM theorem in [22].

4.3 *Topological confinement of the orbit*

We follow the discussion in sect. 1.

We look for an approximation of two frequencies ω' and ω'' which fulfill the requirements above. To this purpose, we can make our calculations without using the interval arithmetics on the initial Hamiltonian (11), that is the simplest one. We determine $\bar E$ as the energy of the point ξ,η in table 3; the actual value is the centre of the interval in (52). We take two new initial conditions by multiplying by 1 ± 0.02 both values ξ_1 and η_1 and modifying ξ_2 and η_2 so that the ratio ξ_2/η_2 is kept constant and that the two new initial conditions still lie on an energy level approximatively equal to $\bar E$. Then, by using the frequency analysis, we calculate the angular frequencies, say $\omega'_\sim$ and $\omega''_\sim$, of the orbits starting from the two new initial conditions. We get [(%)]

$$\begin{aligned}\omega'_\sim&=(-0.0001456732145781655\,,\ -0.0002592343751518042)\ ,\\ \omega''_\sim&=(-0.0001455615260122723\,,\ -0.0002579079711284839)\ .\end{aligned}$$

[(%)] The choice above of the two points is quite delicate. By construction, the tori T' and T'' are very close to these points. These tori should be far enough to allow us to claim that they are well separated notwithstanding the shade effect of the interval arithmetics. On the other hand, the argument below requires that the "true" initial point be in the analyticity region of the transformation to Kolmogorov's normal form, which requires the tori to be close enough. We located a sufficiently good pair of initial conditions by trial and error.

We now define the sets Γ' and Γ'' of Diophantine frequencies as

$$\begin{aligned}\Gamma' = \Bigg\{(\omega_1,\omega_2)\,:\,\frac{\omega_1}{\omega_2} &= \frac{5797287 + 2625484\frac{\sqrt{5}-1}{2}}{10316626 + 4672209\frac{\sqrt{5}-1}{2}}\,,\\ &1+\omega_2/0.00025923437515169406\in\left[-5\cdot 10^{-10}\,,\,5\cdot 10^{-10}\right]\Bigg\}\,,\\ \Gamma'' = \Bigg\{(\omega_1,\omega_2)\,:\,\frac{\omega_1}{\omega_2} &= \frac{2899483 + 818219\frac{\sqrt{5}-1}{2}}{5137345 + 1449732\frac{\sqrt{5}-1}{2}}\,,\\ &1+\omega_2/0.0002579079711283471\in\left[-5\cdot 10^{-10}\,,\,5\cdot 10^{-10}\right]\Bigg\}\,.\end{aligned} \tag{51}$$

Our choice is based on some well known relations of the continued fraction expansions. The ratios ω_1/ω_2 have been fixed according to the following two criteria: (1) when $(\omega_1,\omega_2)\in\Gamma'$ (Γ'', resp.), then $\omega_1/\omega_2\approx\omega'_{\sim;\,1}\,/\,\omega'_{\sim;\,2}$ ($\omega_1/\omega_2\approx\omega''_{\sim;\,1}\,/\,\omega''_{\sim;\,2}$, resp.); (2) Γ' and Γ'' are constituted by noble frequencies, which are expected to correspond to the locally most robust tori with respect to the perturbations (see, e.g., [23]). The definition of sets Γ' and Γ'' is parameterized with respect to ω_2 and the range of the values of ω_2 has been chosen in such a way that both $\omega'_{\pm}\in\Gamma'$ and $\omega''_{\pm}\in\Gamma''$, where

$$\omega'_{-;\,2} = -0.00025923437525826712 \qquad \omega'_{+;\,2} = -0.00025923437504512100\,,$$
$$\omega''_{-;\,2} = -0.00025790797123354561 \qquad \omega''_{+;\,2} = -0.00025790797102314870\,.$$

The latter four frequencies have been chosen in such a way that the two values of the energy of the tori corresponding to the frequencies $\omega'_{\pm}$ ($\omega''_{\pm}$, resp.) are $\approx 1\pm 10^{-8}\bar{E}$. In order to actually calculate these frequencies, we use the averages over the angles of the two Hamiltonians in Kolmogorov's normal form expanded around the initial frequencies $\omega'_{\sim}$ and $\omega''_{\sim}$.

Hereafter, all calculation are performed using the interval arithmetics. A straightforward evaluation of the energy level $\bar{E}$ allows us to get

$$\bar{E}\in[-1.6986715765408438\cdot 10^{-7}\,,\,-1.6986715763842084\cdot 10^{-7}]\,; \tag{52}$$

let us recall that $\bar{E}$ is the value corresponding to the initial conditions in table 4 for the Hamiltonian (19) truncated at $s=35$.

We finally apply the computer–assisted proof described in the present section 4, in order to ensure the existence of the invariant tori corresponding to all the frequencies on the curves Γ' and Γ''. Let us remark that we can simultaneously deal with a set of frequencies thanks to the interval arithmetics implemented in our code. As anticipated in sect. 4.1 we perform an explicit calculation of Kolmogorov's algorithm up to order $R'=33$ followed by iteration of the estimates up to the step $R''=2048$.[b] Then, we can check

[b] The procedure above has actually been applied to the curves Γ' and Γ'' in order to check the existence of the families of tori, and then repeated for the extrema of the curves in order to check the conditions on the energy. This makes a total of six times. To give an idea of the required computational power, a single application of the procedure took

that all the considered frequencies satisfy condition *(b)* of theorem 2, with $\gamma = 8 \cdot 10^{-6}$ and $\tau = 1$. We explicitly calculate the values of $\varrho^{(R')}$ and $\mathcal{E}^{(R')}$ and we estimate $\varrho^{(R'')}$ and $\mathcal{E}^{(R'')}$ by using iteratively inequalities (44) and (40); this allows us to evaluate $\varrho^{(R'')} \geq 6$, in all the considered cases. After having iterated the estimates, we can check that hypothesis *(e)* of theorem 2 is always satisfied by the following values of the parameters:

$$\varepsilon^{(R'')} = 0.840 \qquad E = 2 \cdot 10^{-6} \qquad \zeta^{(R'')} = 1.2 \times 10^{4} \ .$$

Furthermore, the algorithm we adopted ensures that hypotheses *(a)*, with $K = 2$, and *(d)* are fulfilled by $H^{(R'')}$. Finally, from a straightforward calculation of the threshold value ε^{*} defined in hypothesis *(f)*, we get

$$\varepsilon^{*} = 0.868 \ .$$

Since $\varepsilon^{(R'')} < \varepsilon^{*}$, our version of KAM theorem applies and ensures that all the considered frequencies correspond to invariant tori. Moreover, we can evaluate the energy $\mathcal{E}^{(\infty)}$ of the invariant tori corresponding to the frequencies $\omega'_{\pm}$ and $\omega''_{\pm}$, by making use of estimate (50). Such a calculation allows us to prove that the energy of both the tori ω'_{+} and ω''_{+} (ω'_{-} and ω''_{-} , resp.) is $> \bar{E}$ ($< \bar{E}$, resp.). Therefore, there is a frequency $\omega' \in \Gamma'$ ($\omega'' \in \Gamma''$) such that the corresponding torus T' (T'') is invariant and its energy level is $\bar{E}$.

In order to complete the proof, we check that the initial point is in the gap between two tori on the same energy surface. Let us refer to the KAM torus T' with frequency ω', and let P', Q' be the canonical coordinates that give the Hamiltonian the Kolmogorov's normal form $H^{(\infty)}$. This Hamiltonian is analytic for $P' \in B_{3/(4\bar{\zeta})}(0)$ and $Q' \in \mathbf{T}^2$. We check that $\partial H^{(\infty)} / \partial P'_2 \,(P', Q') \neq 0$ in the latter domain; this implies that P'_1, Q'_1, Q'_2 may be used as local coordinates on the intersection between the surface of constant energy $H^{(\infty)} = \bar{E}$ and the domain $B_{3/(4\bar{\zeta})}(0) \times \mathbf{T}^2$. Next, we check that the coordinate P'_1 of the initial point and the coordinate P'_1 of any point of the torus T'' are both positive. We have indeed for the initial point $P'_1 \in [10^{-5} \, , \, 2.1 \times 10^{-5}]$, and choosing the point $Q'_1 = Q'_2 = 0$ on the torus T'' we get $P'_1 \in [2.6 \times 10^{-5} \, , \, 3.6 \times 10^{-5}]$. Since a trajectory cannot cross an invariant torus, we conclude that both the orbit and the torus T'' lie on the same side of the energy surface with respect to the torus T'. By reversing the argument, and denoting by P'', Q'' the coordinates of Kolmogorov's normal form in the neighbourhood $B_{3/(4\bar{\zeta})}(0) \times \mathbf{T}^2$ of the KAM torus T'' we first prove that $\partial H^{(\infty)} / \partial P''_2 \,(P'', Q'') \neq 0$, and then check that the initial point has $P''_1 \in [-1.7 \times 10^{-5} \, , \, -1.5 \times 10^{-5}]$, while the point $Q''_1 = Q''_2 = 0$ of the torus T' has $P''_1 \in [-3.2 \times 10^{-5} \, , \, -3 \times 10^{-5}]$. Hence, both the orbit and the torus T' lie on the same side of the energy surface with respect to the torus T''. We conclude that the orbit is eternally trapped between two close KAM tori T', T'', thus assuring a topological confinement. This concludes the proof of theorem 1.

17 h of CPU–time on a 400Mhz Pentium II processor for the explicit calculation of the expansions, and 13 h on an AlphaServer 1000/400 EV5 for the iteration of the estimates.

A. Expansion of the secular Hamiltonian of the SJS system up to order 2 in the masses and 6 in eccentricity

In the following expansion of formula (11), we neglected the additive constant because it has no influence on the equations of motion:

$$
\begin{array}{ll}
\mathcal{H}^{(\mathrm{I})}_{Sec}(\xi_1,\xi_2,\eta_1,\eta_2) = -7.9817980076760655\times 10^{-5}\xi_1^2 & -3.8147366852664142\times 10^{-5}\xi_1\xi_2 \\
-1.0268297004552840\times 10^{-4}\,\xi_2^2 & -7.7243404322933866\times 10^{-5}\,\eta_1^2 \\
-3.3707608458413656\times 10^{-5}\,\eta_1\eta_2 & -9.9999984803151030\times 10^{-5}\,\eta_2^2 \\
-5.6436223750712493\times 10^{-4}\,\xi_1^4 & -1.6416392742006527\times 10^{-3}\,\xi_1^3\xi_2 \\
-6.2973632765649690\times 10^{-3}\,\xi_1^2\xi_2^2 & -2.2899781169560516\times 10^{-3}\,\xi_1^2\eta_1^2 \\
-3.4760626376493233\times 10^{-3}\,\xi_1^2\eta_1\eta_2 & -4.2185659067704027\times 10^{-3}\,\xi_1^2\eta_2^2 \\
-8.3857323933716650\times 10^{-3}\,\xi_1\xi_2^3 & -1.3039354499285056\times 10^{-3}\,\xi_1\xi_2\eta_1^2 \\
-4.7425253805301244\times 10^{-3}\,\xi_1\xi_2\eta_1\eta_2 & -6.8531825574944021\times 10^{-3}\,\xi_1\xi_2\eta_2^2 \\
-6.2286577646952779\times 10^{-3}\,\xi_2^4 & -3.9500271580332256\times 10^{-3}\,\xi_2^2\eta_1^2 \\
-8.3527962865484184\times 10^{-3}\,\xi_2^2\eta_1\eta_2 & -1.1373126532783053\times 10^{-2}\,\xi_2^2\eta_2^2 \\
-1.7268965474147260\times 10^{-3}\,\eta_1^4 & -3.1475592574520577\times 10^{-3}\,\eta_1^3\eta_2 \\
-6.6349860468623900\times 10^{-3}\,\eta_1^2\eta_2^2 & -6.8332272847022573\times 10^{-3}\,\eta_1\eta_2^3 \\
-5.1387074709303473\times 10^{-3}\,\eta_2^4 & -6.0701804574696823\times 10^{-3}\,\xi_1^6 \\
-3.2119532714927534\times 10^{-2}\,\xi_1^5\xi_2 & -2.7549740027179186\times 10^{-1}\,\xi_1^4\xi_2^2 \\
-7.4777792652719661\times 10^{-2}\,\xi_1^4\eta_1^2 & -2.0878989193600397\times 10^{-1}\,\xi_1^4\eta_1\eta_2 \\
-2.2556116819663772\times 10^{-1}\,\xi_1^4\eta_2^2 & -1.2799874137338532\times 10^{0}\,\xi_1^3\xi_2^3 \\
-2.1957803068778765\times 10^{-1}\,\xi_1^3\xi_2\eta_1^2 & -1.0754609127577874\times 10^{0}\,\xi_1^3\xi_2\eta_1\eta_2 \\
-1.2525556241525955\times 10^{0}\,\xi_1^3\xi_2\eta_2^2 & -3.0996153271501390\times 10^{0}\,\xi_1^2\xi_2^4 \\
-6.4320374599537216\times 10^{-1}\,\xi_1^2\xi_2^2\eta_1^2 & -3.9977642968441485\times 10^{0}\,\xi_1^2\xi_2^2\eta_1\eta_2 \\
-4.7878280093164562\times 10^{0}\,\xi_1^2\xi_2^2\eta_2^2 & -1.3017334854810814\times 10^{-1}\,\xi_1^2\eta_1^4 \\
-5.6443027855889856\times 10^{-1}\,\xi_1^2\eta_1^3\eta_2 & -1.6091206979341728\times 10^{0}\,\xi_1^2\eta_1^2\eta_2^2 \\
-2.4623339524089656\times 10^{0}\,\xi_1^2\eta_1\eta_2^3 & -1.6707157626697120\times 10^{0}\,\xi_1^2\eta_2^4 \\
-3.2771819750467199\times 10^{0}\,\xi_1\xi_2^5 & -6.5831703615139023\times 10^{-1}\,\xi_1\xi_2^3\eta_1^2 \\
-6.1109646323834843\times 10^{0}\,\xi_1\xi_2^3\eta_1\eta_2 & -7.3380945672001685\times 10^{0}\,\xi_1\xi_2^3\eta_2^2 \\
-1.8774424112239909\times 10^{-1}\,\xi_1\xi_2\eta_1^4 & -1.0570482388082987\times 10^{0}\,\xi_1\xi_2\eta_1^3\eta_2 \\
-3.6262703950516264\times 10^{0}\,\xi_1\xi_2\eta_1^2\eta_2^2 & -6.1600785219363594\times 10^{0}\,\xi_1\xi_2\eta_1\eta_2^3 \\
-4.0619119716320187\times 10^{0}\,\xi_1\xi_2\eta_2^4 & -1.2962091387801093\times 10^{0}\,\xi_2^6 \\
-6.0253186540222901\times 10^{-1}\,\xi_2^4\eta_1^2 & -4.3130729937971006\times 10^{0}\,\xi_2^4\eta_1\eta_2 \\
-5.0726955487757852\times 10^{0}\,\xi_2^4\eta_2^2 & -3.7694653469144916\times 10^{-1}\,\xi_2^2\eta_1^4 \\
-1.8736854171280113\times 10^{0}\,\xi_2^2\eta_1^3\eta_2 & -5.8658678966225395\times 10^{0}\,\xi_2^2\eta_1^2\eta_2^2 \\
-9.4014451177163672\times 10^{0}\,\xi_2^2\eta_1\eta_2^3 & -6.2640076184402940\times 10^{0}\,\xi_2^2\eta_2^4 \\
-6.1464575048410168\times 10^{-2}\,\eta_1^6 & -3.5591582332419240\times 10^{-1}\,\eta_1^5\eta_2 \\
-1.3743527471116275\times 10^{0}\,\eta_1^4\eta_2^2 & -3.3335872837233516\times 10^{0}\,\eta_1^3\eta_2^3 \\
-5.2949133058695930\times 10^{0}\,\eta_1^2\eta_2^4 & -5.0893561220459285\times 10^{0}\,\eta_1\eta_2^5 \\
-2.4875190197883796\times 10^{0}\,\eta_2^6 &
\end{array}
$$

References

[1] Abu-El-Ata, N. and Chapront, J.: *Développements analytiques de l'inverse de la distance en mécanique céleste*, Astronomy and Astrophysics, **38**, 57–66 (1975).

[2] Arnold, V. I.: *Proof of a theorem of A. N. Kolmogorov on the invariance of quasi–periodic motions under small perturbations of the Hamiltonian*, Usp. Mat. Nauk, **18**, 13 (1963); Russ. Math. Surv., **18**, 9–36 (1963).

[3] Arnold, V. I.: *Small denominators and problems of stability of motion in classical and celestial mechanics*, Usp. Math. Nauk **18** N.6, 91 (1963); Russ. Math. Surv. **18** N.6, 85 (1963).

[4] Arnold, V. I.: *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York (1978).

[5] Celletti, A. and Chierchia, L.: *On the stability of realistic three–body problems*, Comm. Math. Phys., **186**, 2, 413–449 (1997).

[6] Celletti, A., Giorgilli, A. and Locatelli, U.: *Improved Estimates on the Existence of Invariant Tori for Hamiltonian Systems*, Nonlinearity, **13**, 397–412 (2000).

[7] Giorgilli, A.: *Quantitative methods in classical perturbation theory*, proceedings of the Nato ASI school "From Newton to chaos: modern techniques for understanding and coping with chaos in N–body dynamical systems", A.E. Roy e B.D. Steves eds., Plenum Press, New York (1995).

[8] Giorgilli, A., Delshams, A., Fontich, E., Galgani, L. and Simò, C.: *Effective stability for a Hamiltonian system near an elliptic equilibrium point, with an application to the restricted three body problem*, J. Diff. Equations, **77**, 167–198 (1989).

[9] Giorgilli, A. and Locatelli, U., *Kolmogorov theorem and classical perturbation theory*, J. of App. Math. and Phys. (ZAMP), **48**, 220–261 (1997).

[10] Gröbner, W.: *Die Lie–Reihen und Ihre Anwendungen*, Springer Verlag, Berlin (1960); it. transl.: *Le serie di Lie e le loro applicazioni*, Cremonese, Roma (1973).

[11] Henon, M.: *Exploration numérique du problème restreint IV: Masses égales, orbites non périodiques*, Bulletin Astronomique, **3**, N. 1, fasc. 2, 49–66, (1966).

[12] Henrard, J.: *A note on a general algorithm for two–body expansions*, Celestial Mechanics and Dynamical Astronomy, *to appear*.

[13] Koch, H., Schenkel, A. and Wittwer, P.: *Computer-assisted proofs in analysis and programming in logic: a case study*, SIAM Review, **38**, N. 4, 565–604 (1996).

[14] Kolmogorov, A. N.: *Preservation of conditionally periodic movements with small change in the Hamilton function*, Dokl. Akad. Nauk SSSR, **98**, 527–530 (1954).

[15] Laskar, J.: *Secular evolution over 10 million years*, Astronomy and Astrophysics, **198**, 341–362 (1988).

[16] Laskar, J.: *Les variables de Poincaré et le développement de la fonction perturbatrice*, Groupe de travail sur la lecture des Méthodes nouvelles de la Mécanique Céleste, Notes scientifiques et techniques du Bureau des Longitudes **S026** (1989).

[17] Laskar, J.: *Large scale chaos in the solar system*, Astronomy and Astrophysics, **287**, L9–L12 (1994).

[18] Laskar, J.: *Frequency map analysis of an Hamiltonian system*, Workshop on Non–Linear Dynamics in Particle Accelarators, AIP Conf. Proc., **344**, 130–159 (1995).

[19] Laskar, J.: *Introduction to frequency map analysis*, in C. Simò (managing ed.), Proceedings of the NATO ASI school: "Hamiltonian Systems with Three or More Degrees of Freedom", S'Agaro (Spain), June 19–30, 1995, Kluwer, 134–150 (1999).

[20] Laskar, J. and Robutel, P.: *Stability of the Planetary Three–Body Problem – I. Expansion of the Planetary Hamiltonian*, Celestial Mechanics and Dynamical Astronomy, **62**, 193–217 (1995).

[21] Locatelli, U.: *Three–body planetary problem: study of KAM stability for the secular part of the Hamiltonian*, Planetary and Space Science, **46**, 11/12, 1453–1464, (1998).

[22] Locatelli, U.: *Proof of a KAM theorem on the existence of invariant tori near an equilibrium point*, Quaderni del dipartimento di matematica, Università di Milano, (2000).

[23] MacKay, R. S. and Stark, J.: *Locally most robust circles and boundary circles for area–preserving maps*, Nonlinearity, **5**, 867–888, (1992).

[24] Moser, J.: *On invariant curves of area–preserving mappings of an annulus*, Nachr. Akad. Wiss. Gött,. II Math. Phys. Kl 1962, 1–20 (1962).

[25] Moser, J.: *Convergent series expansions for quasi–periodic motions*, Math. Ann. **169**, 136–176 (1967).

[26] Moser, J.: *Stable and random motions in dynamical systems*, Princeton University press, Princeton (1973).

[27] Murray, N. and Holman, M.: *The Origin of Chaos in Outer Solar System*, Science, **283**, Iss. 5409, 1877 (1999).

[28] Nobili, A. M., Milani, A. and Carpino, M.: *Fundamental frequencies and small divisors in the orbits of the outer planets*, Astronomy and Astrophysics, **210**, 313–336 (1989).

[29] Poincaré, H.: *Les méthodes nouvelles de la Mécanique Céleste*, Gauthier–Villars, Paris (1892), reprinted by Blanchard (1987).

[30] Robutel, P.: *Contribution à l'étude de la stabilité du problème planetaire des trois corps*, Ph.D. thesis, Observatoire de Paris (1993).

[31] Robutel, P.: *Stability of the Planetary Three–Body Problem – II. KAM Theory and Existence of Quasiperiodic Motions*, Celestial Mechanics and Dynamical Astronomy, **62**, 219–261 (1995).

[32] Schenkel, A., Wehr, J. and Wittwer, P.: *Computer-assisted proofs for fixed point problems in Sobolev Spaces*, MPEJ, **6**, 3, 1–67 (2000).

[33] Sussman, G. J. and Wisdom, J.: *Chaotic evolution of the solar system*, Science, **241**, 56–62 (1992).