IL NUOVO CIMENTO VOL. 72 B, N. 2 11 Dicembre 1982

An Extension of the Poincaré-Fermi Theorem on the Nonexistence of Invariant Manifolds in Nearly Integrable Hamiltonian Systems (*).

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(ricevuto il 7 Luglio 1982)

Summary. — For an autonomous nearly integrable Hamiltonian system of n degrees of freedom with $n > 1$ it was shown by Poincaré that, in general, no integrals of motion exist which are independent of the Hamiltonian. This result was generalized by Fermi, who showed that in general not even single invariant $(2n-1)$-dimensional manifolds exist, apart from constant-energy surfaces. On the other hand, the Kolmogorov-Arnold-Moser theorem guarantees the existence of n-dimensional invariant tori. In this paper we discuss the possible existence of invariant manifolds of intermediate dimensions and conclude that, apart from very well-defined exceptions (namely, manifolds of the so-called resonant type and $(n+1)$-dimensional families of n tori with mutually proportional frequencies), in general such invariant manifolds do not exist.

(*) In 1923 two important papers in analytical dynamics by E. FERMI (at that time 22 years old) appeared in this journal. We are glad to publish now this interesting paper—a direct sequel of those papers—and with it to pay a tribute to the scope, the depth and the fertility of Fermi's scientific work. The Editor.

1. – Introduction.

Consider an integrable Hamiltonian system with n degrees of freedom, namely a system that, in suitable action-angle variables I, φ, is described by a φ-independent Hamiltonian:

$$H(I, \varphi) = H^0(I) , \tag{1}$$

where $I = (I_1, \ldots, I_n) \in B \subset \mathbf{R}^n$ and $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbf{T}^n$, B being a finite open ball of $\mathbf{R}^n$ and $\mathbf{T}^n$ the n-torus; here $\varphi \in \mathbf{T}^n$ simply means that any function of φ is supposed to have period 2π in $\varphi_1, \ldots, \varphi_n$. The equations of motion $\dot{I} = 0$, $\dot{\varphi} = \omega(I)$, where

$$\omega = (\omega_1, \ldots, \omega_n) = \left(\frac{\partial H^0}{\partial I_1}, \ldots, \frac{\partial H^0}{\partial I_n}\right) = \operatorname{grad} H^0 ,$$

are then trivially integrated, giving $I(t) = I^*$, $\varphi(t) = \varphi^* + \omega(I^*)t$ in correspondence with the initial data I^*, φ^*. The nondegeneracy condition

$$\det \frac{\partial \omega_i}{\partial I_j} = \det \frac{\partial^2 H^0}{\partial I_i \partial I_j} \neq 0 , \tag{2}$$

which ensures that the (angular) frequencies ω can be taken as independent variables in place of the actions I, will also be assumed.

In this case one has obviously n independent integrals of motion, for example the actions $I_1, \ldots, I_n$ themselves. Consequently the phase space $B \times \mathbf{T}^n$ has a very particular structure: each integral generates a foliation into invariant $(2n-1)$-dimensional manifolds and, by intersection of them, one then obtains « finer » foliations into $(n+l)$-dimensional invariant manifolds with $0 \leqslant l \leqslant n-1$. In particular, for $l = 0$ one gets a n-torus of the form $\{I\} \times \mathbf{T}^n$, while for $l > 0$ each of such invariant manifolds has the form $V \times \mathbf{T}^n$, where V is a l-dimensional submanifold of B.

One can say that this class of invariant manifolds is characterized by the existence of constraints on the actions I only, the angles φ remaining free variables. There exist, however, also invariant manifolds defined by constraints on the angles: for example, the $(2n-2)$-dimensional manifold, defined by the pair of equations

$$\sin(m \cdot \varphi) = \text{const} , \tag{3}$$

$$m \cdot \omega(I) = 0 , \tag{4}$$

where $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n$, $m \neq 0$, and the dot denotes the usual scalar product, is trivially checked to be invariant. In general (see the appendix), one can show that, if for an integrable system there exists an invariant manifold defined by constraints on the angles, then on such a manifold « resonance relations » of the type of (4) have to be satisfied. From this fact two consequences can be drawn: 1) if $n + l$ is the dimension of such a manifold, then one has $l < n - 1$; 2) the set of such manifolds has vanishing Lebesgue measure and, in particular, cannot constitute a foliation of phase space.

The natural problem then arises to understand what happens of the invariant manifolds of the integrable system when a perturbation is added to the Hamiltonian, so that, for example, $H = H^0(I) + \mu H^1(I, \varphi)$, μ being a perturbative parameter belonging to a real interval A around the origin. In this connection, one has first the classical negative result of Poincaré (1) according to which, under suitable generality and regularity conditions, no integral of motion (depending smoothly on the perturbative parameter μ) exists which is independent of the Hamiltonian itself. As a consequence, the only foliation of the phase space $B \times \mathbf{T}^n$ into smoothly μ-dependent $(2n - 1)$-dimensional invariant manifolds is the foliation into constant-energy surfaces. However, this result does not preclude the existence of a particular smoothly μ-dependent $(2n - 1)$-dimensional invariant manifold distinct from a constant-energy surface. This possibility was later excluded by FERMI (2-4), under the same conditions of generality and regularity of Poincaré, but with the additional requirement $n > 2$.

Now the necessity of this further assumption of Fermi is easily understood on the basis of the Kolmogorov-Arnold-Moser existence theorem (5). Indeed, according to such a theorem, « many » of the unperturbed n-tori $\{I\} \times \mathbf{T}^n$ give rise to corresponding invariant tori for the perturbed system; more precisely (see the appendix), the Kolmogorov-Arnold-Moser theorem allows for the existence of $(n + 1)$-dimensional invariant manifolds which have the form $V \times \mathbf{T}^n$, V being a very special 1-dimensional submanifold of $B \subset \mathbf{R}^n$; as a consequence, for $n = 2$ actually $(2n - 1)$-dimensional invariant manifolds distinct from constant-energy surfaces can exist. This fact, as was also stressed by CERCIGNANI, may suggest the possibility that nonexistence theorems of the type of Poincaré and Fermi be optimal, in the sense that the manifolds which cannot be excluded by considerations based on similar grounds do actually exist.

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(1) H. POINCARÉ: *Les méthodes nouvelles de la mécanique celeste*, Vol. 1 (Paris, 1892), Chapt. 5. See also E. T. WHITTAKER: *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (New York, 1944), par. 165.

(2) E. FERMI: *Nuovo Cimento*, **26**, 105 (1923).

(3) E. FERMI: *Nuovo Cimento*, **25**, 267 (1923); *Phys. Z.*, **24**, 261 (1923).

(4) W. URBANSKI: *Phys. Z.*, **25**, 47 (1924); E. FERMI: *Phys. Z.*, **25**, 166 (1924).

(5) A. N. KOLMOGOROV: *Dokl. Akad. Nauk SSSR*, **98**, 527 (1954); translated in *Lecture Notes in Physics*, **93**, edited by G. CASATI and J. FORD (Berlin, 1979); V. I. ARNOLD: *Russ. Math. Surv.*, **18**, 9 (1963); **18**, 85 (1963); J. K. MOSER: *Nachr. Akad. Wiss. Gottingen, Math-Phys. Kl.*, 1 (1962).

From this point of view one is then naturally led to try to extend the arguments of Poincaré and Fermi in order to possibly prove the nonexistence of invariant manifolds of dimension $n+l$, with $1<l<n-1$, for any $n>2$. This is indeed the problem studied in the present paper, where we show that (for a generic perturbation) there are no smoothly μ-dependent invariant manifolds of dimension $n+l$ with $1<l<n-1$ which for vanishing perturbation reduce to manifolds of the form $V\times \mathbf{T}^n$ (V being a l-dimensional submanifold of $B\subset \mathbf{R}^n$), *i.e.* reduce to manifolds belonging to the first of the two classes of invariant manifolds described above for the unperturbed system.

In this sense we may then say that our result generalizes the results of Poincaré and Fermi, because the invariant manifolds excluded by them are also in this class; indeed, by remark 1) made above, any $(2n-1)$-dimensional invariant manifold necessarily belongs to this class. On the other hand, the existence of invariant manifolds reducing to manifolds of the second class described above (of resonant type) cannot be excluded in general, as is well known and will be recalled, by examples, in the appendix. Thus it appears that our result fills the gap between the existence theorem of Kolmogorov, Arnold and Moser and the nonexistence theorems of Poincaré and Fermi. For the problem of the existence of d-dimensional invariant manifolds supporting quasi-periodic motions, with $d<n$, see ref. [6].

2. – Statement of the theorem.

To begin with, let us give a precise description of the class of invariant manifolds S_μ that we consider as smooth perturbations of a manifold S_0 of the form $V\times \mathbf{T}^n$ in order to generalize the Poincaré-Fermi theorem. Now first of all we recall that any l-dimensional submanifold V of B can be described locally by conditions of the form

$$F_i^0(I)=0\,, \qquad i=1,\ldots,k, \tag{5}$$

with $k=n-l$, where F_i^0 are smooth functions with the property that

$$\operatorname{rank}\frac{\partial F_i^0}{\partial I_j}=k\,. \tag{6}$$

By possibly restricting the ball B, we can then think of eq. (5) as defined in B. It is also clear that, by thinking of the functions F_i^0 as defined in $B\times \mathbf{T}^n$ by trivial extension, the same equations (5) with property (6) define any $(n+l)$-dimensional submanifold of $B\times \mathbf{T}^n$ of the form $V\times \mathbf{T}^n$.

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(6) J. K. Moser: *Math. Ann.*, **169**, 136 (1967).

We will then consider, in the spirit of the works of Poincaré and Fermi, manifolds S_μ defined by equations of the form

$$F_i(\mu, I, \varphi) = 0\,, \qquad i = 1, \ldots, k, \tag{7}$$

where F_i are smooth functions in $A \times B \times \mathbf{T}^n$ with the property that

$$F_i(0, I, \varphi) = F_i^0(I)\,, \qquad i = 1, \ldots, k\,, \tag{8}$$

together with (6).

A precise statement of our result is then given by the following theorem.

Theorem. Consider an autonomous nearly integrable Hamiltonian system with $n > 2$ degrees of freedom depending on a parameter μ with Hamiltonian $H(\mu, I, \varphi)$, where $\mu \in A \subset \mathbf{R}$, $I \in B \subset \mathbf{R}^n$, $\varphi \in \mathbf{T}^n$, A being a real interval around the origin, B a finite open ball of $\mathbf{R}^n$ and $\mathbf{T}^n$ the n-torus; denote by $H^0(I)$ the corresponding unperturbed Hamiltonian, *i.e.* let

$$H(0, I, \varphi) = H^0(I)\,.$$

Assume that the Hamiltonian satisfies the conditions of regularity, non-degeneracy and genericity defined, respectively, as follows:

a) H be of class C^2 in $A \times B \times \mathbf{T}^n$.

b) $\det \dfrac{\partial^2 H^0}{\partial I_i \partial I_j} \neq 0$ in B.

c) Let $H^1(I, \varphi) = (\partial/\partial\mu) H(\mu, I, \varphi)|_{\mu=0}$ and define the Fourier coefficients h_m ($m \in \mathbf{Z}^n$) by $H^1(I, \varphi) = \sum_{m \in \mathbf{Z}^n} h_m(I) \exp[im \cdot \varphi]$; say two coefficients with non-vanishing indices m, m' are in the same class if $m' = \alpha m$ with $\alpha \in \mathbf{R}$: then the genericity condition is that there is at least a nonvanishing coefficient of H^1 in each class.

Consider now, for $1 < l < n - 1$, a family $\{S_\mu\}$ of $(n + l)$-dimensional differentiable submanifolds S_μ of $B \times \mathbf{T}^n$ depending smoothly on the parameter μ, which for $\mu = 0$ reduce to a manifold of the form $V \times \mathbf{T}^n$, V being a l-dimensional submanifold of $B \subset \mathbf{R}^n$. Precisely with $k = n - l$, $1 < k < n - 1$, suppose that S_μ be defined by

$$F_i(\mu, I, \varphi) = 0\,, \qquad i = 1, \ldots, k,$$

where $F_i(\mu, I, \varphi)$ are functions that

d) a of class C^2 in $A \times B \times \mathbf{T}^n$;

e) a φ-independent for $\mu = 0$, *i.e.* there exist functions $F_i^0(I)$, $i = 1, \ldots, k$, in B with

$$\operatorname{rank} \frac{\partial F_i^0}{\partial I_j} = k\,,$$

defining $V \subset B$ by $F_i^0(I) = 0$, $i = 1, \dots, k$, such that $F_i(0, I, \varphi) = F_i^0(I)$, $i = 1, \dots, k$.

Then it is not possible that, for all $\mu \in A$, S_μ be invariant under the flow generated by the Hamiltonian $H(\mu, I, \varphi)$.

The same holds for $l = 1$ and $l = n - 1$ with the additional hypotheses that

f) in the case $l = 1$ (or $k = n - 1$), on the 1-dimensional manifold $V \subset B$ the frequencies $\omega_1, \dots, \omega_n$ be not mutually proportional (*i.e.* for $I \in V$ one does not have $\omega(I) = \alpha(I)\omega^*$ with a fixed ω^* and a smooth real function $\alpha(I)$);

g) in the case $l = n - 1$ (or $k = 1$), grad F_1^0 be not proportional to $\omega = \operatorname{grad} H^0$ on S_0 (*i.e.* S_0 do not coincide with a constant-energy surface).

Let us now add some comments. The regularity conditions *a*) and *d*) are weaker than the analogous analiticity conditions considered by POINCARÉ and FERMI, while the nondegeneracy and genericity conditions *b*) and *c*) are exactly the same. However, as emphasized by POINCARÉ himself, condition *c*) is only a sufficient one which could be slightly weakened. Finally, a discussion of the significance of condition *g*) in the case $l = n - 1$, which was not required by FERMI, is deferred to the conclusions.

3. – Proof of the theorem.

i) The heart of the proof of the theorem consists in showing that the invariance of S_μ under the flow generated by $H(\mu, I, \varphi)$, for all $\mu \in A$, would require that all the vectors grad F_j^0 ($j = 1, \dots, k$) be proportional to $\omega = \operatorname{grad} H^0$ in an open subset of $V \subset B$. This gives a contradiction if $k > 1$ because the vectors grad F_j^0 are linearly independent by hypothesis *e*), and if $k = 1$ by hypothesis *g*). The proof of the proportionality of grad F_j^0 and ω is obtained by following almost literally a part of Fermi's arguments.

ii) The condition that S_μ be invariant under the flow generated by the given Hamiltonian is expressed by

$$\{H, F_j\} = 0 \quad \text{on} \quad S_\mu\,, \qquad j = 1, \dots, k, \tag{9}$$

where $\{.\,,.\}$ denotes the Poisson bracket. In particular, for $\mu = 0$, eq. (9)

reduces to

$$\{H^0, F_j^0\} = 0 \qquad \text{on} \qquad S_0\,, \qquad\qquad j = 1, \ldots, k, \tag{10}$$

which is evidently satisfied because H^0 and the F_j^0's are φ-independent.

By differentiation of eq. (9) there also follows the « first-order condition »

$$\{H^1, F_j^0\} + \{H^0, F_j^1\} = 0 \qquad \text{on} \qquad S_0\,, \qquad\qquad j = 1, \ldots, k, \tag{11}$$

where

$$F_j^1(I, \varphi) = \frac{\partial}{\partial \mu} F_j(\mu, I, \varphi)\Big|_{\mu=0}\,. \tag{12}$$

Equation (11) looks like the usual first-order condition of the classical perturbation theory, which is obtained from the equation $\{H, F_j\} = 0$ by derivation with respect to μ at $\mu = 0$. In fact, in our casei ts deduction is a little subtler, because one has to take into account the μ-dependent constraint « on S_μ », *i.e.* the conditions $F_j(\mu, I, \varphi) = 0$ $(j = 1, \ldots, k)$. However, in virtue of the φ-independence of H^0 and F_j^0, one has that $\{H, F_j\}$ vanishes identically in $B \times \mathbf{T}^n$ at $\mu = 0$, so that, in the differential

$$\mathrm{d}\{H, F_j\} = \frac{\partial}{\partial \mu}\{H, F_j\}\,\mathrm{d}\mu + \sum_{i=1}^{n}\left(\frac{\partial}{\partial I_i}\{H, F_j\}\,\mathrm{d}I_i + \frac{\partial}{\partial \varphi_i}\{H, F_j\}\,\mathrm{d}\varphi_i\right),$$

all terms but the first one vanish at $\mu = 0$. On the other hand, by condition *e*) μ can be taken (together with $\varphi_1, \ldots, \varphi_n$ and l of the variables I) as a free variable and thus from (9) one gets $(\partial/\partial\mu)\{H, F_j\} = 0$ at $\mu = 0$, *i.e.* eq. (11).

iii) Here we simply write down explicitly eq. (11) in terms of the Fourier coefficients $h_m(I)$ and $f_{jm}(I)$ of $H^1(I, \varphi)$ and $F_j^1(I, \varphi)$, respectively. This is indeed the point at which use is made of the fact that the angles $\varphi_1, \ldots, \varphi_n$ are free co-ordinates on $S_0 = V \times \mathbf{T}^n$, which excludes the possible consideration of manifolds of resonant type. One has thus, for any $j = 1, \ldots, k$, the condition

$$(m\cdot\omega) f_{jm} - (m\cdot \operatorname{grad} F_j^0) h_m = 0\,, \qquad\qquad m \in \mathbf{Z}^n, \tag{13}$$

to be satisfied on V. From (13) one then obtains, on V,

$$m\cdot \operatorname{grad} F_j^0 = 0 \qquad \text{whenever} \qquad m\cdot\omega = 0\,; \tag{14}$$

in fact, the further condition « if $h_m \neq 0$ », that should be added, turns out to be superfluous in virtue of the genericity condition *c*) on the Hamiltonian,

iv) We finally show that, as a consequence of (14), grad F_j^0 is proportional to ω for $j = 1, \ldots, k$, in an open subset U of V. To this end, let us first consider the case in which all of the frequencies $\omega_1, \ldots, \omega_n$ nowhere vanish in V, so that one can there define the functions

$$\beta_{rs}(I) = \frac{\omega_r(I)}{\omega_s(I)} \qquad (r, s = 1, \ldots, n) . \tag{15}$$

We notice, first of all, that in any open subset $U \subset V$ one at least of the functions β_{rs}, say β_{12}, is strictly nonconstant, *i.e.* for any I in U there exists I' arbitrarily close to I such that $\beta_{12}(I') \neq \beta_{12}(I)$. Indeed, because of condition *b*), which ensures that the frequencies ω can be taken as independent variables in a suitable open set $\Omega \subset \mathbf{R}^n$ in place of the actions $I \in B \subset \mathbf{R}^n$, the conditions $\beta_{rs} = c_{rs}$ $(r, s = 1, \ldots, n)$, where the c_{rs} are constants, define a 1-dimensional manifold (precisely a straight line through the origin in the space Ω of the frequencies ω); on the other hand, this manifold cannot coincide with U, in the case $l > 1$ just because of its dimensions, and, in the case $l = 1$, by condition *f*). Consider then the subset $\tilde{U}$, dense in U, where $\beta_{12}(I)$ takes rational values. Now, for any $I \in \tilde{U}$ there exists a $m \in \mathbf{Z}^n$ satisfying the condition $m \cdot \omega = 0$ such that $m_1 \neq 0$, $m_2 \neq 0$, $m_i = 0$ for $i \neq 1, 2$; by (14) one has then, for any j,

$$\frac{\partial F_j^0/\partial I_1}{\omega_1} = \frac{\partial F_j^0/\partial I_2}{\omega_2} \tag{16}$$

in $\tilde{U}$ and thus, by continuity, also in U.

In such a way it is clear that, if all the $n-1$ functions $\beta_{12}(I), \beta_{23}(I), \ldots, \beta_{n-1,n}(I)$ are nonconstant in U, one gets there

$$\frac{\partial F_j^0/\partial I_1}{\omega_1} = \frac{\partial F_j^0/\partial I_2}{\omega_2} = \ldots = \frac{\partial F_j^0/\partial I_n}{\omega_n} ,$$

namely that grad F_j^0 is proportional to ω, for all j, in U. The same conclusion is evidently reached if one considers other suitable sets of $n-1$ functions $\beta_{rs}(I)$, namely any set in which each index appears at least once, for example the set $\beta_{12}, \beta_{13}, \beta_{34}, \ldots, \beta_{n-1,n}$. On the other hand, it is easy to construct a suitable set of $n-1$ nonconstant functions $\beta_{rs}(I)$ if one at least of such functions, say $\beta_{12}(I)$, is nonconstant; indeed, starting from the set $\beta_{12}, \beta_{23}, \ldots, \beta_{n-1,n}$, if, in going from left to right, β_{hk} is the first bad function, then it suffices to replace it by its product with the function immediately on the left, thus obtaining a new set, and so on.

There remains to consider the possibility that some of the frequencies $\omega_1, \ldots, \omega_n$ vanish somewhere in V. To this end, we remark first of all that there exists an open subset $W \subset V$ where p of the frequencies, say $\omega_1, \dots, \omega_p$, never vanish, while $\omega_{p+1}, \dots, \omega_n$ vanish identically. Indeed one at least of the frequencies, say ω_1, does not vanish identically in V and thus never vanishes in a suitable open subset of V. Consider now ω_2: either it vanishes identically there, or one can take a smaller open subset where both ω_1 and ω_2 never vanish, and so on (one easily sees $p > l$). Proceeding then as above to obtain the analog of (16), one gets the parallelism of grad F_j^0 and ω restricted to the first p components. Finally, the complete parallelism is established by remarking that the last $n-p$ components of grad F_j^0 vanish, as the last components of ω, in virtue of (14).

Thus we have come to the conclusion that the invariance of S_μ requires the existence of an open subset $U \subset V$ where grad F_j^0 is proportional to ω for $j = 1, \dots, k$, as announced in (*i*). The theorem is proven.

4. – Concluding remarks.

Geometrically speaking, what we have proven is essentially the following: generically, if a Hamiltonian system admits a smooth invariant manifold on which no resonance relation is identically satisfied, then such a manifold is necessarily the union of purely nonresonant n-tori. But the fact of excluding all resonant tori immediately prevents the possibility of building up smooth manifolds of dimensions greater than $n+1$, which thus, in general, cannot exist.

The reader familiar with the original works of Poincaré and Fermi will recognize that our proof exploits the same ideas and techniques of those authors, making, of course, the necessary extensions. The only difference is in the use we make of the parallelism of grad F_j^0 with ω, for $j = 1, \dots, k$. In fact, for $k > 1$ this already gives a contradiction and leads directly to the conclusion; instead, in the case $k = 1$ considered by FERMI, from the proportionality of grad F_1^0 and $\omega =$ grad H^0 one can only conclude that S_0 coincides with a constant-energy surface, and this is not a contradiction (unless one introduces an explicit hypothesis such as *g*)), because it does not preclude the existence of a family $\{S_\mu\}$ with S_μ distinct from a constant-energy surface for $\mu \neq 0$. Thus for $k = 1$ the proof of Fermi is a little more complicated and requires considering the functions $F_j(\mu, I, \varphi)$ at all higher perturbative orders to show that S_μ necessarily coincides with a constant-energy surface even for any $\mu \neq 0$. By the way, analyticity conditions on the Hamiltonian and the invariant manifolds are thus required, as was also the case for the theorem of Poincaré.

A reason for reporting here a detailed proof of our theorem and not just giving a short sketch of it by making reference to Fermi's work is due to the fact that, notwithstandig a correct appreciation of Moser (7), in the scientific community there is a widespread opinion that Fermi's work is incorrect. In fact, in the two first versions of Fermi's work (3), the accent was rather put on an application to ergodic theory which was not correct (as was already remarked by URBANSKI (4) and acknowledged by FERMI himself (4)). Instead, the nonexistence of invariant $(2n-1)$-dimensional manifolds, which is the only point considered in the last paper of Fermi on the subject, appears to be correct and is, in fact, generalized here.

The point is the following one. FERMI proved that there cannot exist open invariant sets having smooth manifolds as boundary; however, as pointed out in ref. (4), one cannot exclude the existence of open invariant sets whose boundary is not smooth. In fact, the latter situation turns out to be a common one: for example, the set of invariant tori considered in the Kolmogorov-Arnold-Moser theorem (which is certainly not a smooth $(2n-1)$-dimensional manifold; for a few details see the appendix) is nevertheless the border of its complement, which is open and invariant. This situation, well known today but difficult to be imagined 60 years ago, comes out from the diophantine condition (A.3) defining in the space of frequencies the set of the invariant tori.

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We thank C. CERCIGNANI, C. F. MANARA and J.-M. STRELCYN for useful discussions. We also thank L. CHIERCHIA and G. GALLAVOTTI for communicating us their results on the smooth prime integrals in nearly integrable Hamiltonian systems, use of which is made in the appendix.

APPENDIX

This appendix is devoted first of all to a discussion of possible invariant manifolds S_μ which, for vanishing μ, cannot be decomposed as $S_0 = V \times \boldsymbol{T}^n$ (with the notations of the introduction). Precisely we make the following two points: i) If S_0 is invariant for the flow induced by the Hamiltonian H^0 and is not of the form $V \times \boldsymbol{T}^n$, then it is of resonant type, *i.e.* for any $(I, \varphi) \in S_0$ there exists a nonvanishing $m \in \boldsymbol{Z}^n$ such that $m \cdot \omega(I) = 0$. ii) The hypothesis *e*) of our theorem, namely that $S_0 = V \times \boldsymbol{T}^n$, is necessary, as is shown by counter-examples. Furthermore, in iii) we make some considerations on the invariant manifolds whose existence is guaranteed by the Kolmogorov-Arnold-Moser theorem.

(7) J. K. MOSER: in *Topics in Nonlinear Dynamics*, edited by S. JORNA (New York, N. Y., 1978). See also C. L. SIEGEL: *Vorlesungen über Himmelsmechanik* (Berlin, 1956), p. 201; C. L. SIEGEL and J. K. MOSER: *Lectures on Celestial Mechanics* (Berlin, Heidelberg and New York, N. Y., 1971).

i) Consider a manifold S_0 which is invariant under the flow induced by H^0; if it is not of the form $V\times \mathbf{T}^n$, then obviously the angles $\varphi_1, \ldots, \varphi_n$ are not free co-ordinates on it. Assume now that, if $(I^*, \varphi^*)\in S_0$ and $\omega^* = \omega(I^*)$, then ω^* is nonresonant, *i.e.* there does not exist $m \in \mathbf{Z}^n$, $m\neq 0$, such that $m\cdot\omega^* = 0$. This contradicts the invariance of S_0 under the flow induced by H^0. Indeed, by a well-known result of the ergodic theory, the orbit issuing from (I^*, φ^*), namely $I(t) = I^*$, $\varphi(t) = (\varphi^* + \omega^* t) \bmod 2\pi$, is dense on the torus $\{I^*\}\times\mathbf{T}^n$; such a torus then belongs to S_0, against the requirement that the angles $\varphi_1, \ldots, \varphi_n$ be not all free co-ordinates on S_0.

ii) Let us consider the family of Hamiltonians

$$H(\mu, I, \varphi) = \sum_{j=1}^{n} \tfrac{1}{2} I_j^2 + \mu \sum_{m\in Z^n} h_m \exp[im\cdot\varphi] \tag{A.1}$$

with h_m independent of I and satisfying $h_m = h_{m'}$ if $m = (m_1, m_2, \ldots, m_n)$ and $m' = (-m_1, m_2, \ldots, m_n)$. Then the $(2n-2)$-dimensional manifold S_0 defined by

$$\varphi_1 = 0\,, \qquad I_1 = 0 \qquad (\textit{i.e.} \text{ with } \omega_1(I) = 0) \tag{A.2}$$

is trivially checked to be invariant for all μ. On the other hand, H^0 and S_0 satisfy all the requirements of the theorem, apart from condition *e*), *i.e.* that S_0 be of the form $V\times\mathbf{T}^n$.

iii) It is well known that, under suitable regularity hypotheses, for nearly integrable nondegenerate Hamiltonian systems the Kolmogorov-Arnold-Moser theorem guarantees the existence of invariant n-tori which are perturbations of the tori $I = \text{const}$. More precisely, for any sufficiently small μ there exists a set of invariant tori, each corresponding to a frequency ω satisfying the diophantine condition

$$|m\cdot\omega|^{-1} \leqslant C|m|^n\,, \qquad m\in\mathbf{Z}^n\,, \quad m\neq 0\,, \tag{A.3}$$

with a positive number C, where $|m| = |m_1| + \ldots + |m_n|$. Now, in the space Ω of the frequencies ω, such a set is decomposed into pieces of straight lines; indeed, if ω^* satisfies (A.3), then $\omega = \alpha\omega^*$, α being a real number not too different from 1, also satisfies it. This fact suggests that the set of invariant tori in $B\times\mathbf{T}^n$ be arranged into disjoint sets of invariant $(n+1)$-dimensional manifolds of the form $V\times\mathbf{T}^n$, where V is a 1-dimensional submanifold of B, all tori of each set having mutually proportional frequencies. This is in agreement with a recent result ([8]) of Chierchia and Gallavotti and of Poeschel, according to which, for any sufficiently small μ, there exists a smooth function F defined in an open subset of the phase space $B\times\mathbf{T}^n$ with values in Ω which is invariant for the initial data lying on the invariant tori. If we consider then the 1-dimensional manifold $\{\alpha\omega^*\}$ with α belonging to a suitable real interval and ω^* satisfying (A.3), the preimage $F^{-1}\{\alpha\omega^*\}$ will be an invariant $(n+1)$-dimensional submanifold of $B\times\mathbf{T}^n$.

([8]) L. Chierchia and G. Gallavotti: *Nuovo Cimento B*, **67**, 277 (1982); J. Poeschel: *Differentiable foliation of invariant tori in Hamiltonian systems*, preprint.

● RIASSUNTO

Poincaré mostrò che per, un sistema hamiltoniano autonomo quasi integrabile ad n gradi di libertà, con $n > 1$, in generale non esistono integrali del moto indipendenti dall'hamiltoniana. Fermi generalizzò poi il risultato mostrando che, in generale, non esistono neppure singole varietà invarianti di dimensione $2n-1$, a parte le superfici di energia costante. D'altro canto il teorema di Kolmogorov, Arnold e Moser garantisce l'esistenza di tori invarianti di dimensione n. Nel presente lavoro si discute l'eventuale esistenza di varietà invarianti di dimensione intermedia e si conclude che, a parte ben definite eccezioni (varietà cosiddette risonanti, e famiglie $(n+1)$-dimensionali di n-tori, con frequenze mutuamente proporzionali), in generale tali varietà invarianti non esistono.

Резюме не получено.

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11 Dicembre 1982
Il Nuovo Cimento
Serie 11, Vol. **72** B, pag. 137-148