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# Universal Properties in Conservative Dynamical Systems.

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Universal properties have been observed for certain classes of dynamical systems of dissipative type. In the present letter we communicate some new results for systems of nondissipative type.

Let us first briefly recall what is known for dissipative systems. The first and simplest examples are concerned with one-parameter families of endomorphisms of an interval of which a classical example is $\psi_\mu$: $[0, 1] \to [0, 1]$, with $\psi_\mu(x) = 4\mu x(1-x)$. For a large class of these families having a unique maximum increasing with $\mu$ and satisfying certain technical requirements, universal properties were observed in connection with bifurcation phenomena. Precisely, for each family, in correspondence with an initial stable periodic orbit of period $p_0$ for a given $\mu = \mu_{-1}$, one can determine an increasing sequence $\{\mu_k\}_{k=0}^{\infty}$ accumulating to a critical value $\mu_\infty$, with the following properties: for $\mu_{k-1} < \mu < \mu_k$ there exists a stable periodic orbit of period $p_k = p_0 \times 2^k$; at $\mu_k$ this orbit becomes unstable, and each of its $p_k$ points produces by bifurcation a pair of homologous points, on which a new orbit runs of period $p_{k+1} = 2p_k$. This orbit in turn is stable up to $\mu = \mu_{k+1}$.

A first universal phenomenon, observed numerically by FEIGENBAUM (1) and afterwards, independently, by TRESSER and COULLET (2), consists in the fact that the differences $\mu_k - \mu_{k-1}$ decrease asymptotically according to a geometric law, namely

$$\lim_{k\to\infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = \delta > 1 .$$

(1) M. J. FEIGENBAUM: *J. Stat. Phys.*, **19**, 25 (1978); **21**, 6 (1979).
(2) C. TRESSER and P. COULLET: *C. R. Acad. Sci. Ser. A*, **287**, 577 (1978).

The number $\delta$ appears to be the same for all families having the same behaviour around the maximum $\bar{x}$, namely behaving as $c_1 - c_2|x - \bar{x}|^{1+\varepsilon}$ with the same $\varepsilon$, $c_1$ and $c_2$ being constants. For example, for the common case of mappings with a quadratic maximum ($\varepsilon = 1$), one has found $\delta = 4.669201 \ldots$

A second universal asymptotic property is related to the fact that, at each bifurcation, the system possesses the same local structure up to a suitable rescaling of distances. In particular, one can consider the distances between two homologous points of a periodic orbit of period $p_k$. Following a branching along the bifurcations and choosing the parameter in the interval $(\mu_{k-1}, \mu_k)$ with a definite criterion, one thus obtains a sequence of distances $\{d_k\}_{k=1}^{\infty}$. Again these distances decrease asymptotically according to a geometric law, namely

$$\lim_{k\to\infty} \frac{d_{k-1}}{d_k} = \alpha > 1 .$$

The number $\alpha$ has a local character, depending on the particular branching and the values of $\alpha$ should also be universal for families of the same class. In particular, for mappings with a quadratic maximum and following the branching around the maximum one has found $\alpha = 2.502907 \ldots$

Analogous bifurcation properties with the same universal number $\delta$ have been observed for other dissipative systems of higher dimension, namely the Hénon two-dimensional mapping (3) and, among flows, the well-known Lorentz model for the Bénard problem (4) and a five-mode truncation of the Navier-Stokes equation (5).

In these cases, for what concerns the local rescaling parameter $\alpha$, in the choice of the branching there is no immediate analog of the condition that the bifurcation be around the maximum of the mapping $\psi_\mu$. However, following an indication contained in ref. (1), FRANCESCHINI and TEBALDI (5) remarked that an analogue could be given. Precisely, for $k \geq 1$, at $\mu_k$ one has available $p_{k-1}$ pairs of homologous points; one can thus consider the corresponding $p_{k-1}$ distances, and in particular the maximal one $d_k^+$. They found that following with increasing $k$ the maximal distance $d_k^+$ gives a branching and for that branching they got approximately the value 2.44, *i.e.* presumably the same one as for the branching around the maximum in the case of one-dimensional mappings with a quadratic maximum.

A theoretical explanation of these universal properties, at least for the case of endomorphisms of an interval, was first suggested at a heuristic level (1,2) by using techniques of the renormalization group. Rigorous demonstrations were then provided by COLLET, ECKMANN and LANFORD (6) for dimension one and by COLLET, ECKMANN and KOCH (7) for higher dimension. In particular, in sect. **8** of ref. (6) one can find a discussion on the role of the distance $d_k^+$ quoted above.

After having recalled these facts, a problem naturally arises, *i.e.* whether analogous properties, with possibly the same universal numbers, occur also for nondissipative dynamical systems: in particular Hamiltonian systems and symplectic diffeomorphisms of symplectic manifolds, for example area-preserving diffeomorphisms of the plane. Indeed, successive bifurcations analogous to the ones described above were already reported by CONTOPULOS for Hamiltonian systems with two degrees of freedom as energy increases. However, the possibility of different universal numbers with respect

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(3) B. DERRIDA, A. GERVOIS and Y. POMEAU: *J. Phys. A*, **12**, 269 (1979).
(4) V. FRANCESCHINI: preprint.
(5) V. FRANCESCHINI and C. TEBALDI: preprint.
(6) P. COLLET, J.-P. ECKMANN and O. E. LANFORD III: preprint.
(7) P. COLLET, J.-P. ECKMANN and H. KOCH: preprint.

TABLE I.

| $k$ | $p_k$ | $\mu_k$ | $\delta_k$ | $\alpha_k^+$ |
|---|---|---|---|---|
| 0 | 1 | 0.636 619 772 3 | | |
| 1 | 2 | 1.000 000 000 0 | | |
| 2 | 2 | 1.049 438 508 7 | 7.3501 | 5.0971 |
| 3 | 4 | 1.055 226 601 6 | 8.5414 | 2.8165 |
| 4 | 8 | 1.055 891 882 0 | 8.7002 | 4.5529 |
| 5 | 16 | 1.055 968 188 6 | 8.7185 | 3.8963 |
| 6 | 32 | 1.055 976 938 6 | 8.7208 | 4.0491 |
| 7 | 64 | 1.055 977 941 9 | 8.7210 | 4.0104 |
| 8 | 128 | 1.055 978 956 9 | 8.7212 | 4.0201 |

TABLE II.

| $k$ | $p_k$ | $\mu_k$ | $\delta_k$ | $\alpha_k^+$ |
|---|---|---|---|---|
| 0 | 2 | 0.318 309 886 19 | | |
| 1 | 4 | 0.394 083 897 94 | | |
| 2 | 4 | 0.402 795 395 63 | 8.6982 | 6.7481 |
| 3 | 8 | 0.403 802 541 78 | 8.6497 | 3.5347 |
| 4 | 16 | 0.403 918 051 65 | 8.7191 | 4.1489 |
| 5 | 32 | 0.403 931 298 20 | 8.7200 | 3.9866 |
| 6 | 64 | 0.403 932 817 11 | 8.7211 | 4.0259 |
| 7 | 128 | 0.403 932 991 27 | 8.7213 | 4.0162 |

TABLE III

| $k$ | $p_k$ | $\mu_k$ | $\delta_k$ | $\alpha_k^+$ |
|---|---|---|---|---|
| 0 | 1 | 1.405 284 734 6 | | |
| 1 | 2 | 1.524 360 170 8 | | |
| 2 | 4 | 1.541 761 657 8 | 6.8428 | 6.3056 |
| 3 | 8 | 1.543 782 612 1 | 8.6105 | 3.3382 |
| 4 | 16 | 1.544 015 465 8 | 8.6791 | 4.2022 |
| 5 | 32 | 1.544 042 170 9 | 8.7194 | 3.9711 |
| 6 | 64 | 1.544 045 233 3 | 8.7205 | 4.0299 |
| 7 | 128 | 1.544 045 584 4 | 8.7210 | 4.0151 |

to the dissipative cases had been suggested in private conversation with COLLET and ECKMANN, based on an unpublished work by DERRIDA.

Thus we came to consider some specific nondissipative dynamical systems, in the class of the area-preserving diffeomorphisms $\varphi_\mu$ of the torus $\pi^2$ defined by

$$x' = x + y \pmod 1 ,$$

$$y' = y - \mu f(x + y) \pmod 1 ,$$

$f$ being a continuously differentiable function of period one. These mappings have been extensively studied with suitable generalization to higher dimensions [8,9].

The first example refers to the function $f(z) = \sin(2\pi z)$. In this case two different initial stable periodic orbits were considered. One is the origin (0, 0), which is a fixed point ($p_0 = 1$) for any $\mu$; the other is of period $p_0 = 2$ and is formed by the pair of points $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$ for each $\mu$. These orbits produce sequences of bifurcations similar to the ones described above. Only, the difference was noticed that each bifurcation is not necessarily associated with a doubling of the period. Indeed, in each sequence we just once observed that the $p_0 \times 2^{k+1}$ points, produced by bifurcation by the $p_0 \times 2^k$ points of the previously stable periodic orbit, do not form a single orbit of period $p_0 \times 2^{k+1}$, but two separate orbits of period $p_{k+1} = p_0 \times 2^k$ (correspondingly, at the bifurcation the eigenvalues of the linear part of $\varphi_{\mu_k}^{p_k}$ had value 1 and not $-1$). The values of $k$, $p_k$, $\mu_k$, $\delta_k = (\mu_k - \mu_{k-1})/(\mu_{k+1} - \mu_k)$ and $\alpha^+ = d_{k-1}^+/d_k^+$ for these two sequences are reported in table I and table II, respectively. One sees quite clearly that $\delta_k$ and $\alpha_k^+$ approach the limit values $\delta \simeq 8.721$ and $\alpha^+ \simeq 4.02$, respectively, which are quite different from the corresponding values for dissipative systems.

As a second example we considered the same type of mapping with the function $f(z) = -[\cos(\pi z)]^2$. A fixed point for $\mu \geqslant 1$ is given by $(\pi^{-1} \arccos(\mu^{-\frac{1}{2}}), 0)$. The results referring to this orbit, reported in table III, are seen to agree with the previous ones. Similar results were also obtained for another function $f(z)$.

Thus we conclude that bifurcation phenomena analogous to those reported for dissipative systems are also found for nondissipative ones; only, the universal numbers thereby found are different. For what concerns the claim of universality, this is also supported by the comparison with the results available for other models, as we come to discuss now. First, unpublished results by ECKMANN and KOCH for the Hénon two-dimensional mapping (in the conservative case) up to period 16 give, for the analogs of $\delta$ and $\alpha$, the values 8.7210 ... and 4.071 ..., respectively. Moreover, a sequence of bifurcations for a Hamiltonian system with two degrees of freedom has been obtained by CONTOPOULOS and ZIKIDES [10]. From the data available by a private communication, one has four numbers $E_0 = 0.111$, $E_1 = 0.12356$, $E_2 = 0.124833$, $E_3 = 0.12499835$; from these one gets, as successive approximations for $\delta$, the values 9.866 and 8.458, which are in rather good agreement with our estimates.

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(8) C. FROESCHLÉ: *Astron. Astrophys.*, **16**, 172 (1972).
(9) G. BENETTIN, L. GALGANI, A. GIORGILLI and J.-M. STRELCYN: *Meccanica*, in print.
(10) G. CONTOPOULOS and M. ZIKIDES: *Astron. Astrophys.*, in print.