Further Results on Universal Properties in Conservative Dynamical Systems.

G. Benettin

Istituto di Fisica dell'Università - Padova
Gruppo Nazionale di Struttura della Materia del C.N.R - Padova

L. Galgani and A. Giorgilli

Istituto di Fisica e Istituto di Matematica dell'Università - Milano

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In this letter we present some results, extending the ones already reported in a previous short note (1). In ref. (1) it was established numerically that, in conservative dynamical systems (such as typically area-preserving maps of a plane) depending on a parameter μ, one could observe by varying the parameter, starting from a stable fixed point, sequences of period-doubling bifurcations analogous to those already observed by Feigenbaum (2) and others (3-5) for dissipative systems. Such sequences were found to present universal properties analogous to those already known for dissipative systems; the only difference was noticing that the corresponding universal numbers had different values: thus, the analog of Feigenbaum's $\delta = 4.669\ldots$ had the value $8.721\ldots$ and the analog of Feigenbaum's $\alpha = 2.50\ldots$ had the value $4.01\ldots$.

Precisely, let φ_μ denote a family of area-preserving diffeomorphisms of the plane or of the 2-torus, a point of which with co-ordinates x and y will be denoted by z. In particular we considered the diffeomorphisms of the 2-torus defined by

$$\varphi_\mu \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \bmod 1 \\ y + \mu\, f(x+y) \bmod 1 \end{pmatrix} \tag{1}$$

with $f(x) = \cos^2(\pi x)$, or $f(x) = (1 - 4x^2)^2$ or $f(x) = -\sin(2\pi x)$. Then, in correspondence with an initial stable periodic orbit of period p_0 for a given $\mu = \mu_{-1}$, one can determine an increasing sequence $\{\mu_k\}_{k=0}^{\infty}$ accumulating to a critical value μ_∞, such

(1) G. Benettin, C. Cercignani, L. Galgani and A. Giorgilli: *Lett. Nuovo Cimento*, **28**, 1 (1980).
(2) M. J. Feigenbaum: *J. Stat. Phys.*, **19**, 25 (1978); **21**, 6 (1979).
(3) C. Tresser and P. Coullet: *C. R. Acad. Sci. Ser. A*, **287**, 577 (1978).
(4) B. Derrida, A. Gervois and Y. Pomeau: *J. Phys. A*, **12**, 269 (1979).
(5) V. Franceschini: *J. Stat. Phys.*, **22**, 397 (1980); V. Franceschini and C. Tebaldi: *J. Stat. Phys.*, to appear (1980).

that for $\mu_{k-1} < \mu < \mu_k$ there exists a stable periodic orbit of period $p_k = 2^k p_0$; at μ_k this orbit becomes unstable and each of its p_k points produces by bifurcation a pair of homologous points, on which a new orbit runs of period $p_{k+1} = 2p_k$. In the note (1) it was remarked that $\lim_{k\to\infty} (\mu_k - \mu_{k-1})/(\mu_{k+1} - \mu_k) = \delta = 8.721 \ldots$ and, moreover, that $\lim d^+_{k-1}/d^+_k = \alpha = 4.01 \ldots$, where d^+_k is the maximal distance between pairs of homologous points, the numbers δ and α being independent of the particular family φ_μ considered.

Thus in the conservative two-dimensional case the universal number α has, in analogy with the dissipative cases, the meaning of a rescaling parameter and is actually computed as a limit of ratios of characteristic distances at successive stages of bifurcations. However, as can be seen from paper (6), in the spirit of the renormalization group approach, such a simple rescaling should be replaced, when at least two dimensions are considered, by a more complicated transformation, involving for example two different rescalings along two different directions, apart from possible rotations. In the present letter we communicate some new results obtained in this connection for two-dimensional conservative mappings. As will be seen, these results strongly indicate the necessity of a nonlinear part in the transformation quoted above.

We turn now to a description of these results, which were also obtained for the families of mappings of type (1). With the same notations as above, let us consider any fixed value ν in the interval (μ_{k-1}, μ_k). Corresponding to it there will exist p_k points on which runs a stable periodic orbit of φ_ν with period p_k, and each of them will be a stable fixed point of the mapping $\psi_k(\nu) = \varphi_\nu^{p_k}$. Around each of these fixed points one will then have invariant curves for $\varphi_k(\nu)$ with a well-defined rotation number $r(\nu)$. Conversely, having fixed a rotation number r (or more precisely its cosine), one can define a sequence of numers $\nu_k(r)$ and of mappings $\psi_k(r) = \varphi^{p_k}_{\nu_k(r)}$ with p_k stable fixed points. This prescription of taking, at all orders k, a fixed rotation number r is in a sense the analog of considering superstable cycles in the familiar one-dimensional dissipative case.

One has then in our case the problem of making, at any order k, a choice for one well-defined stable fixed point z_k among the p_k available; this will be the analog of choosing, in the familiar one-dimensional dissipative case, the stable fixed point closest to the maximum of the original mapping φ_μ. The main purpose will then be to check whether with increasing k the mappings ψ_k around z_k asymptotically differ only by a given suitable « rescaling ».

The prescription we followed (see ref. (1,5,7)) was to choose at any k the stable fixed point z_k of ψ_k which has maximal distance from the corresponding unstable point z'_k (*i.e.* from the point which, by bifurcation, produced z_k and its homologous point). From now on ψ_k will be considered to be defined around z_k and its linear part thereby will be denoted by A_k. The neighbourhood of z_k will be referred to orthogonal axes directed along the major and the minor axes of any invariant ellipse of A_k; in such a way, in the rescaling problem which we are interested in possible rotations are implicitly taken into account. Thus, if one looks only for linear transformations, as was always done in this context up to now, one only remains with the possibility of two different rescalings α_1 and α_2 along such axes.

Now, taking into consideration the linearizations A_k of ψ_k, one obviously has no possibility of estimating separately α_1 and α_2; their ratio $\beta = \alpha_2/\alpha_1$ can instead be determined, for example by requiring that the invariant ellipses of A_{k+1} superpose, through rescaling, to those of A_k. By this criterion β is then the limit of the sequence of values $\beta_k = \varrho_k/\varrho_{k+1}$, where ϱ_k is the ratio of the minor to the major axis of any invariant ellipse of A_k. As seen from column 3 of table I (which refers to $f(x) = \cos^2(\pi x)$;

(6) P. Collet, J.-P. Eckmann and H. Koch: *J. Stat. Phys.*, to appear (1980).
(7) P. Collet, J.-P. Eckmann and O. E. Lanford: *Commun. Math. Phys.*, to appear (1980).

TABLE I.

k	p_k	β_k	α'_{1k}	α'_{2k}	β'_k	β'_k/β_k
2	4	2.2939	5.0792	5.2651	1.0366	0.4519
3	8	4.5602	3.5488	19.4366	5.4769	1.2010
4	16	3.9413	4.1381	15.3762	3.7158	0.9428
5	32	4.1029	3.9852	16.3112	4.0929	0.9975
6	64	4.0648	4.0263	16.1043	3.9998	0.9840
7	128	4.0745	4.0160	16.1546	4.0225	0.9873
8	256	4.0721	4.0186	16.1426	4.0170	0.9865
9	512	4.0727	4.0179	16.1455	4.0184	0.9867

$\cos r = 0.4$ and initial fixed point as indicated below), the sequence β_k clearly converges to the value $\beta = 4.072 \ldots$.

On the other hand, taking into account a typical nonlinear feature of ψ_k, namely the position of the unstable fixed point z'_k corresponding to z_k, through its components x'_k and y'_k on the considered axes, one can define separately two rescaling parameters α'_1 and α'_2 with ratio $\beta' = \alpha'_2/\alpha'_1$. As seen from columns 4 and 5 of table I, α'_{1k} and α'_{2k} clearly converge to $\alpha'_1 = 4.018 \ldots$ and to $\alpha'_2 = 16.14 \ldots$, respectively (in particular, this fact confirms the result found in ref. (1), according to which the distances between z_k and z'_k rescale with $\alpha = \alpha'_1$). Thus the ratios β'_k, as seen from column 6, clearly converge to a value $\beta' = 4.018 \ldots$ which can be considered to be equal to α'_1; in other words we find $\alpha'_2 = (\alpha'_1)^2$, a result which is in agreement with an analogous property proven by COLLET, ECKMANN and KOCH (6) in the dissipative case.

However, the important fact should be remarked that β' is, by a small but well-definite amount, different from β. This is shown by the last column of table I, where the values β'_k/β_k are reported; they quite clearly converge to a value $0.986 \ldots$, which is distinctly different from the value 1.

As already anticipated, the results of table I refer to the family of mappings of the 2-torus defined by (1) with $f(x) = \cos^2(\pi x)$, $\cos r = 0.4$ and an initial fixed point $(x_0, 0)$ with $\mu f(x_0) = 1$, $\mu \geqslant 1$. The results appear not to depend on the particular mapping and on the rotation number. This is shown by table II, where results analogous to those

TABLE II.

k	p_k	β_k	α'_{1k}	α'_{2k}	β'_k	β'_k/β_k
2	4	3.5198	4.8613	8.2775	1.7027	0.4838
3	8	4.0859	3.6440	17.7358	4.8671	1.1912
4	16	4.0558	4.1146	15.7756	3.8341	0.9453
5	32	4.0748	3.9915	16.2141	4.0622	0.9969
6	64	4.0718	4.0248	16.1285	4.0073	0.9841
7	128	4.0727	4.0164	16.1487	4.0207	0.9872
8	256	4.0725	4.0185	16.1443	4.0175	0.9865
9	512	4.0726	4.0180	16.1454	4.0183	0.9867

of table I are reported for $f(x) = (1 - 4x^2)^2$, $\cos r = -0.2$ and an initial fixed point $(x_0, 0)$ with $\mu f(x_0) = 1$, $\mu \geqslant 1$. As one can see, the agreement is excellent. These results were confirmed for several other choices of the rotation number and also for the mappings (1) with $f(x) = -\sin(2\pi x)$. The computations were performed on a UNIVAC 1100/80 with a precision of 18 decimal digits.

In conclusion, our results indicate that, if one performs around z_{k+1} a linear transformation in order that the mapping ψ_{k+1} near it reproduce the mapping ψ_k near z_k (or more precisely in order that the linearizations A_{k+1} and A_k agree), then one cannot reproduce the features which are relevant for more distant points, such as typically the corresponding unstable points z'_{k+1} and z'_k, where the nonlinearities of ψ_{k+1} and ψ_k are sensible. As a consequence, a nonlinear transformation appears to be unavoidable; and this is at variance with all results known for the dissipative cases.

As is well known, the universal phenomena described above are in general well interpreted in terms of the renormalization group scheme [6,7], where one considers a renormalization transformation $\mathcal{R}$ in the space of the mappings of the torus, defined by

$$(\mathcal{R}\varphi)(z) = -\Lambda^{-1} \circ \varphi \circ \varphi \circ \Lambda(z) ,$$

Λ being a suitable rescaling transformation (possibly including translations) of the torus. In such a language our results then indicate that, for conservative systems, nonlinear transformations Λ are required.

The results illustrated above were obtained by us when trying to produce a kind of universal mappings which are in general well known to exist when the renormalization group scheme works. We hope to exhibit in a future paper such universal mappings, which are already available in the approximation in which the nonlinear effects described above are neglected.

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