2

THE POINCARÉ–SIEGEL CENTER PROBLEM

- *The Poincaré–Siegel problem:* Dynamics in the neighbourhood of an equilibrium of complex differential equations.
- *The non linear case:* resonances and small divisors.

2.1 The Poincaré center problem

Consider the holomorphic system

$$\dot{z} = V(z) \ , \quad V(z) = \Lambda z + V_1(z) + V_2(z) + \dots \ , \quad z \in \mathbb{C}^n \ , \tag{2.1}$$

with $V_s(z)$ a homogeneous polynomial of degree $s+1$.

- The series is assumed to be convergent in some neighbourhood of the origin.

Problem (Poincaré, 1879)
Find a near the identity transformation of the form (2.3) which conjugates the system (2.1) to its linear part, i.e., such that the transformed system reads

$$\dot{w} = \Lambda w \ . \tag{2.2}$$

- Near the identity transformation: a power series

$$w_j = z_j + \varphi_{1,j}(z) + \varphi_{2,j}(z) + \dots \ , \quad j = 1, \dots, n \ , \tag{2.3}$$

  where $\varphi_{s,j}(z)$ is a homogeneous polynomial of degree $s+1$.

- Formal expansions (in rough terms): calculate the coefficients of a power series disregarding convergence.

- The problem of convergence of the series comes later.

2.1.1 *Formal scheme of solution*

Look for n power series

$$w_j = z_j + \varphi_{1,j}(z) + \varphi_{2,j}(z) + \dots \ , \quad j = 1, \dots, n \ ,$$

where $\varphi_{s,j}(z)$ is a homogeneous polynomial of degree $s+1$, which are solutions of the equation

$$L_V w_j = \lambda_j w_j \ . \tag{2.4}$$

- Suppose a solution exists. Recall that $\dot{w}_j = L_V w_j$. Then:
  - The functions $w_1(z), \dots, w_n(z)$ are formally independent;
  - the transformation (2.3) conjugates the system to its linear part, i.e.,

$$\dot{w} = \Lambda w \ .$$

- Try to solve eq. (2.4) by comparison of coefficients. Recall (linearity):

$$L_V = L_{\Lambda z} + L_{V_1} + L_{V_2} + \dots$$

  Set

$$\partial_\lambda = L_{\Lambda z} = \sum_{j=1}^{n} \lambda_j z_j \frac{\partial}{\partial z_j} \ .$$

- Split the system $L_V w_j = \lambda_j w_j$ as

$$\big(\partial_\lambda - \lambda_j\big) z_j = 0 \ ,$$

$$\big(\partial_\lambda - \lambda_j\big) \varphi_{s,j} = \psi_{s,j} \ , \quad \psi_{s,j} = -\sum_{l=1}^{s} L_{V_l} \varphi_{s-l,j} \ , \quad s \geq 1 \ .$$

  This is a recurrent system.

Problem: *Solve for φ the equation*

$$\big(\partial_\lambda - \lambda_j\big)\varphi = \psi$$

with ψ a known homogeneous polynomial of degree $s \geq 2$, with a given complex vector λ.

Lemma 2.1: *The linear operator ∂_λ is diagonal over the basis of monomials, since*

$$\partial_\lambda z^k = \langle k, \lambda \rangle z^k \ .$$

Corollary 2.2: *For every $j \in \{1, \ldots, n\}$ the linear operator $\big(\partial_\lambda - \lambda_j\big)$ is diagonal on the basis of monomials, since*

$$\big(\partial_\lambda - \lambda_j\big) z^k = \big(\langle k, \lambda \rangle - \lambda_j\big) z^k \ .$$

- Solution of the equation $\big(\partial_\lambda - \lambda_j\big)\varphi = \psi$.
  - Write

    $$\psi = \sum_{k \in \mathbb{N}_0^n} \psi_k z^k$$

    with known (complex) coefficients ψ_k .
  - The *formal* solution is $\varphi = \sum_k \varphi_k z^k$ with

    $$\varphi_k = \frac{\psi_k}{\langle k, \lambda \rangle - \lambda_j} \ ,$$

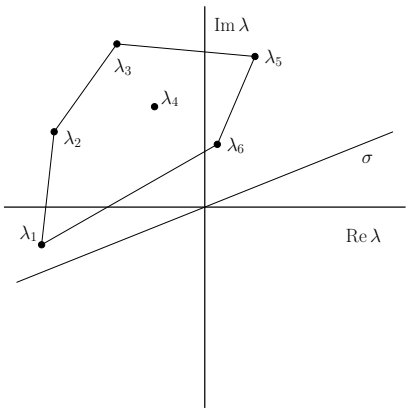
. . . BUT . . .

- *Cave canem:* some divisors might be zero!

2.1.2 *Resonances: the case of Poincaré and the case of Siegel*

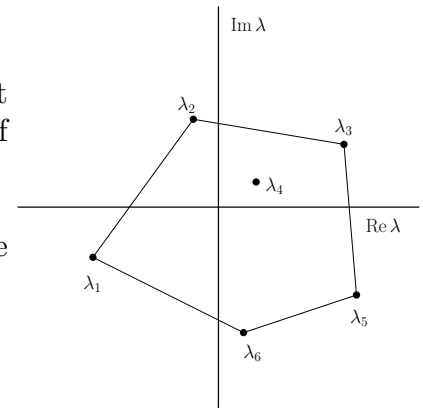
A vector $\lambda \in \mathbb{C}^n$ is said to be *resonant* in case one has $\langle k, \lambda \rangle - \lambda_j = 0$ for some $k \in \mathbb{N}_0^n$ with $|k| > 1$ and some $j \in \{1, \ldots, n\}$. The integer $|k| = k_1 + \ldots + k_n$ is called the *order* of the resonance.

The plane $\langle k, \lambda \rangle - \lambda_j = 0$ is called a *resonant plane*.



Two different cases:

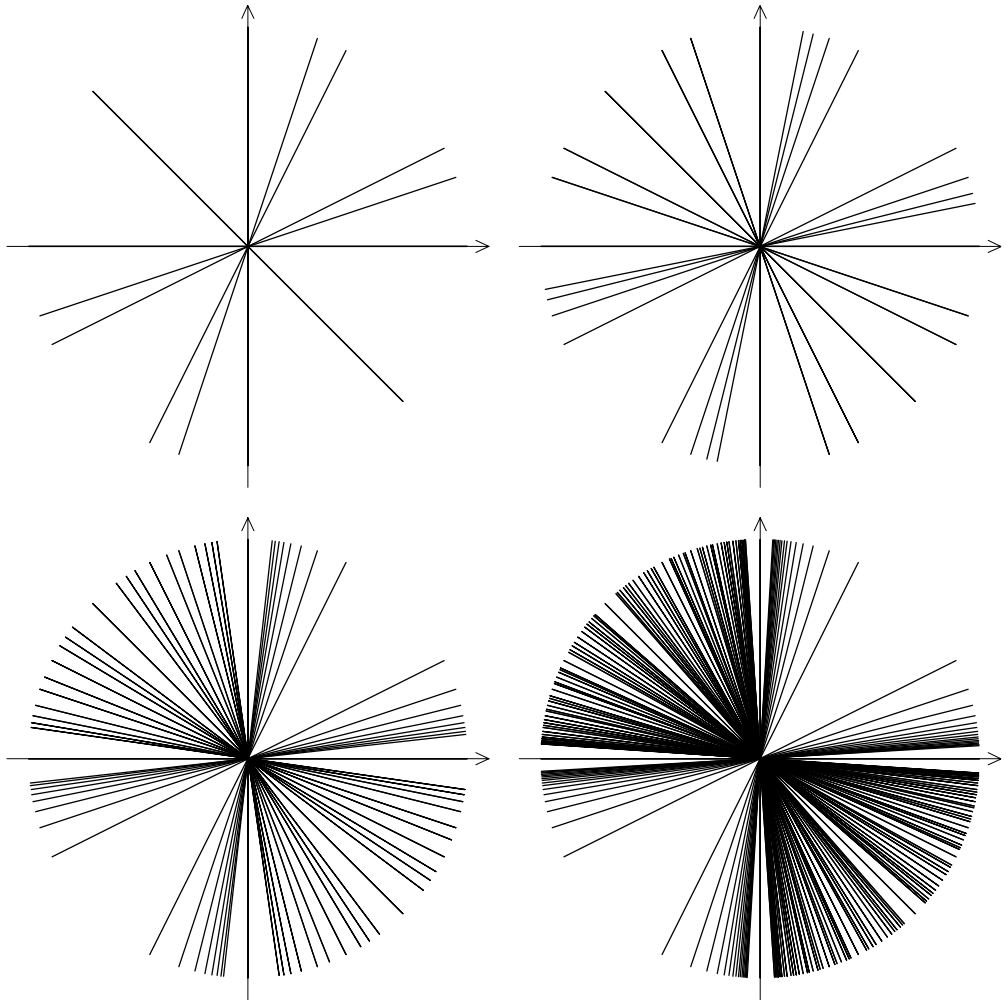
- *Poincaré domain:* the convex hull of the set $\{\lambda_1, \ldots, \lambda_n\}$ does not contain the origin of $\mathbb{C}$ (left).
- *Siegel domain:* The complement in $\mathbb{C}$ of the Poincaré domain (right).



- ∘ Every k with $|k| = 1$ is resonant (obvius), but it is harmless because we must consider only $|k| > 1$.

Example 2.3: *resonant and non resonant vectors.* Let $n = 2$.

- ∘ $\lambda_1 = 2\lambda_2$ is a resonance of order 2. Set $j = 1$ and $k = (0, 2)$.
- ∘ $\lambda_1 = m\lambda_2$ with $m > 1$ is a resonance of order m. Set $j = 1$ and $k = (0, m)$.
- ∘ $\lambda_1 = -\lambda_2$ satisfies infinitely many resonance relations or order $2s + 1$ for $s \geq 1$. Set either $j = 1$ and $k = (s + 1, s)$, or $j = 1$ and $k = (s, s + 1)$.
- ∘ $\lambda_1 = \lambda_2$ is *not* a resonance, in view of $|k| > 1$.
- ∘ $r\lambda_1 = s\lambda_2$ with $r, s > 1$ is not a resonance relation.



Example 2.4: *Resonant planes in the real plane.* All resonant planes for:

- $2 \leq |k| \leq 3$ (upper left);
- $2 \leq |k| \leq 5$ (upper right);
- $2 \leq |k| \leq 9$ (lower left);
- $2 \leq |k| \leq 17$ (lower right).

- Poincaré domain: the first and third quadrant.
- Siegel domain: the second and fourth quadrant, including the axes.

2.1.3 *The Poincaré domain*

- The original condition of Poincaré:
  There is a straight line σ through the origin that leaves all eigenvalues $\lambda_1, \ldots, \lambda_n$ on the same side.

  (Equivalent to the condition on the convex hull).

Proposition 2.5: *Let $\lambda \in \mathbb{C}^n$ be in the Poincaré domain. Then the following statements hold true:*

(i) There are positive constants K and C such that

$$\frac{\big|\langle k, \lambda\rangle - \lambda_j\big|}{|k|} > C \ , \quad \text{for } |k| \geq K \ .$$

(ii) The eigenvalues λ satisfy at most a finite number of resonance relations.

(iii) If λ is nonresonant then there exists a positive constant γ such that

$$\frac{\big|\langle k, \lambda\rangle - \lambda_j\big|}{|k|} > \gamma \ , \quad \text{for all } k \in \mathbb{N}_0^n \ , \ |k| > 1 \ .$$

(iv) If λ is resonant then the claim (iii) holds true for k non resonant (i.e., for all k such that $\langle k, \lambda\rangle \neq \lambda_j$).

(v) If λ is nonresonant then there is a neighbourhood of λ which is nonresonant (i.e., the resonant planes form a discrete set in the Poincaré domain).

Proof. Main remark: $\langle k, \lambda\rangle / |k|$ is the barycenter of n masses $k_1, \ldots, k_n$ placed at the points $\lambda_1, \ldots, \lambda_n$ of the plane. Hence it lies inside the convex hull of $\lambda_1, \ldots, \lambda_n$.

Claim (i).

- Define
$$d = \min_l \big\{\mathrm{dist}(\lambda_l, \sigma)\big\} \ , \quad D = \max_l \big\{\mathrm{dist}(\lambda_l, \sigma)\big\}$$
and set $K = 2\lceil D/d \rceil$.
- Hence
$$D \geq \frac{\mathrm{dist}\big(\langle k, \lambda\rangle, \sigma\big)}{|k|} \geq d \quad \text{for } |k| \geq 1$$
(true for every point of the convex hull of $\lambda_1, \ldots, \lambda_n$).
- For $|k| \geq K$ and $j = 1, \ldots, n$ we have
$$\frac{\mathrm{dist}(\lambda_j, \sigma)}{|k|} \leq \frac{\mathrm{dist}(\lambda_j, \sigma)}{K} \leq \frac{D}{K} \leq \frac{d}{2}$$
by definition of K.
- Conclude
$$\frac{\big|\langle k, \lambda\rangle - \lambda_j\big|}{|k|} \geq \frac{\mathrm{dist}\big(\langle k, \lambda\rangle - \lambda_j, \sigma\big)}{|k|} \geq \frac{\mathrm{dist}\big(\langle k, \lambda\rangle, \sigma\big)}{|k|} - \frac{\mathrm{dist}(\lambda_j, \sigma)}{|k|} \geq \frac{d}{2} \ .$$
Hence (i) holds true with $C = d/2$.

Claim (ii).

- By (i), $\langle k, \lambda\rangle - \lambda_j = 0$ may occur only if $|k| < K$, which represents a finite set.

Claim (iii).

- Let

$$C' = \min_{|k|<K} \min_{1\le j\le n} \frac{\big|\langle k,\lambda\rangle - \lambda_j\big|}{|k|} \ . \tag{2.5}$$

- By non–resonance $C' > 0$ (the minimum over a finite set).
- Hence may set $\gamma = \min(C, C')$, with C as in the proof of (i).

Claim (iv).

- Set C' as in(2.5), but taking the minimum over the non resonant k's.
- Repeat the proof of (iii).

Claim (v).

- For $\mu \in \mathbb{C}^n$ with $|\mu| < \gamma/2$ and for $|k| > 1$

$$\frac{\big|\langle k,\mu\rangle\big| - \mu_j}{|k|} \le |\mu| + \frac{|\mu_j|}{|k|} < \frac{3\gamma}{4} \ .$$

- Then

$$\frac{\big|\langle k,\lambda+\mu\rangle - \lambda_j - \mu_j\big|}{|k|} \ge \frac{\big|\langle k,\lambda\rangle - \lambda_j\big|}{|k|} - \frac{\big|\langle k,\mu\rangle - \mu_j\big|}{|k|} > \frac{\gamma}{4} \ ,$$

  and the claim follows since $\gamma > 0$.

Q.E.D.

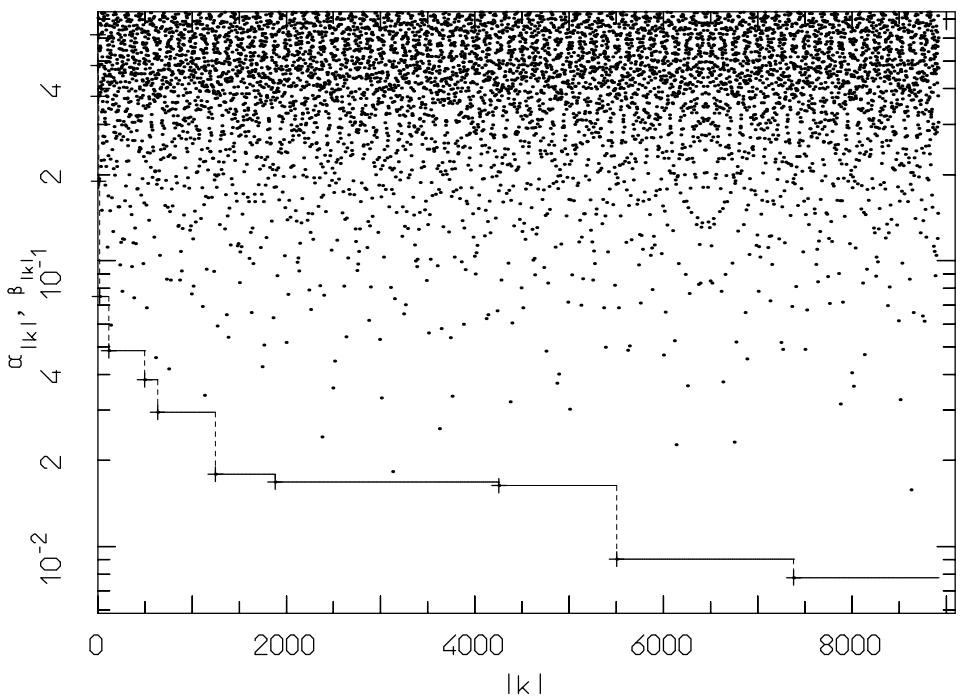
2.1.4 The Siegel domain

Proposition 2.6: *Let λ be in the Siegel domain. Then the following statements hold true:*

(i) the quantity $\left|\langle k,\lambda\rangle - \lambda_j\right|$ may assume an arbitrarily small value, i.e.,

$$\inf_{|k|>1} \min_{1\le j\le n} \left|\langle k,\lambda\rangle - \lambda_j\right| = 0 \ ;$$

(ii) there is a resonant plane arbitrarily close to every λ, i.e., the resonant planes are dense in the Siegel domain.



Example 2.7: *small divisors.* Let

$$\beta_0 = 1$$
$$\beta_s = \min_{|k|=s+1} \min_{1\le j\le n} \left|\langle k,\lambda\rangle - \lambda_j\right| \ .$$

Let also

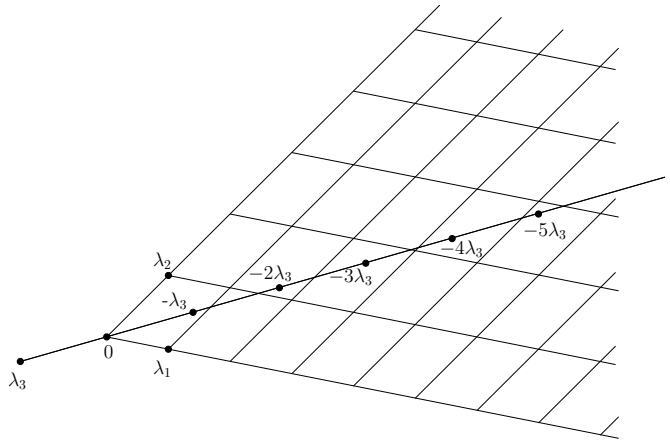
$$\alpha_0 = \beta_0 \ , \quad \alpha_s = \min(\alpha_{s-1}, \beta_s) \ .$$

- The sequences $\{\beta_s\}_{s\ge1}$ (dots) and $\{\alpha_s\}_{s\ge1}$ (crosses) for the complex vector

$$\lambda = (1+i, -i\phi, -\sqrt{2}) \ ,$$
$$\phi = \frac{\sqrt{5}-1}{2} \ .$$

Proof of proposition 2.6.

Claim (i)



- Enough to consider the case $n = 3$.
- Let $\mathscr{A}$ be the sector determined by the half–lines $O\lambda_1$ and $O\lambda_2$.
- Say that $\mu, \mu' \in \mathscr{A}$ are equivalent in case $\mu - \mu' = s\lambda_1 + r\lambda_2$ for some integers s and r.
- Consider the infinite sequence $-\lambda_3$, $-2\lambda_3$, $-3\lambda_3, \ldots$, as in figure. Let $\mu_1, \mu_2, \mu_3, \ldots$ be the image of the sequence inside the parallelogram of sides λ_1, λ_2 .
- If two of the μ's coincide, then λ is resonant, and belongs to a resonant plane: nothing else.
- If they are all distinct, then there is an accumulation point, i.e., there are $|\mu_{s+r} - \mu_s| < \varepsilon$ with ε arbitray.
- Thus, $-r\lambda_3 = j_1\lambda_1 + j_2\lambda_2 + \mu_{s+r} - \mu_s$ for some j_1, j_2, and so (claim (i))

$$|j_1\lambda_1 + j_2\lambda_2 + (r+1)\lambda_3 - \lambda_3| = |\mu_{s+r} - \mu_s| < \varepsilon \ .$$

Claim (ii).

- Let $\lambda_3' = \lambda_3 - \frac{\mu_{r+s} - \mu_r}{r}$. The vector $\lambda' = (\lambda_1, \lambda_2, \lambda_3')$ is resonant and satisfies (claim (ii))

$$|\lambda' - \lambda| = \frac{|\mu_{r+s} - \mu_r|}{r} < \frac{\varepsilon}{r} \ .$$

Q.E.D.

2.1.5 *Diophantine inequalities*

Definition 2.8: *A vector $\lambda \in \mathbb{C}^n$ is said to be diophantine of class (γ, ν) with $\gamma > 0$ and $\nu \in \mathbb{R}$ in case*

$$\big|\langle k, \lambda\rangle - \lambda_j\big| \geq \frac{\gamma}{|k|^\nu}$$

for all $k \in \mathbb{N}_0^n$, $|k| \geq 2$ and for $1 \leq j \leq n$.

Denote by $\Gamma_{\gamma,\nu}$ the set of diophantine vectors of class (γ, ν).

- Poincaré domain: if λ is non resonant then it is diophantine of class $(\gamma, -1)$.
- Siegel domain: the diophantine inequality introduces a condition of *strong non resonance*.

Proposition 2.9: *Let $\nu > (n-2)/2$. Then for every ball $B_R \subset \mathbb{C}^n$ we have*

$$\mathrm{Vol}\big(B_R \setminus \Gamma_{\gamma,\nu}\big) \leq c\gamma^2 \ ,$$

where c is a positive constant depending on the radius R of the ball, on the dimension n and on ν.

- Hint for the proof:
  - Recall the proof that rational numbers on the interval $[0, 1]$ have measure zero.
  - Adapt it to the n–dimensional case (see next page).

Proof.

- Pick $k \in \mathbb{N}_0^n$ ($\mathbb{N}_0$ the set of non negative integers). Let $\alpha = \gamma/|k|^\nu$.
- The set
$$G_{k,j} = \big\{\lambda \in \mathbb{C}^n \ : \ |\langle k, \lambda\rangle - \lambda_j| < \alpha\big\}$$
is a strip of width less that $\alpha/\|k\| < \sqrt{n}\alpha/|k|$ around the (k,j)–resonant plane.
- Counting the number of different k's:
$$\#\{k \in \mathbb{N}_0^n \ : \ |k| = s\} = \binom{s+n-1}{n-1} < s^{n-1}$$
(see next page).
- Add up the contributions of all $k \in \mathbb{N}_0^n$, $|k| \geq 2$ and of $1 \leq j \leq n$, and get
$$\begin{aligned}
\mathrm{Vol}\Big(\bigcup_{\substack{|k|\geq 2\\ 1\leq j\leq n}} G_{k,j}\Big) &\leq C_1 n\gamma^2 \sum_{|k|\geq 2} \frac{1}{|k|^{2(\nu+1)}} \\
&= C_1 n\gamma^2 \sum_{s\geq 2}\sum_{|k|=s} \frac{1}{s^{2(\nu+1)}} \\
&= C_1 n\gamma^2 \sum_{s\geq 2} \binom{s+n-1}{n-1} \frac{1}{s^{2(\nu+1)}} \\
&< C_1 n\gamma^2 \sum_{s\geq 2} \frac{1}{s^{2(\nu+1)-(n-1)}} \ .
\end{aligned}$$
- The sum is finite if $2(\nu+1) - (n-1) > 1$, i.e., if $\nu > (n-2)/2$. Hence a constant c is readily found.

Q.E.D.

Counting integer vectors: the number of $k \in \mathbb{N}_0^n$ with $|k| = s$.

I.e., counting the number of independent monomials of degree s in n variables.

Lemma 2.10: *We have*

$$\#\big\{k \in \mathbb{N}_0^n \,:\, |k| = s\big\} = \binom{s+n-1}{n-1} \ .$$

Proof.

- Take s white balls and $n-1$ black balls. Align them as, e.g.,

$$\underbrace{\circ\,\circ\cdots\circ}_{k_1}\bullet\underbrace{\circ\cdots\circ\,\circ}_{k_2}\bullet\ \cdots\cdots\ \bullet\underbrace{\circ\cdots\circ}_{k_n}$$

  - Every vector k generates a single alignment.
  - Every alignment corresponds to a single vector.
- There is a one–to–one correspondence between the set of k's and the alignments of balls.
- From combinatorial analysis: the number of distinct alignments of s white balls and $n-1$ black balls is

$$\frac{(s+n-1)!}{s!\,(n-1)!} = \binom{s+n-1}{n-1} \ .$$

Q.E.D.

The case $n = 4, s = 2$.

$$\begin{array}{lcl}
k = (2,0,0,0) & \longleftrightarrow & \circ\circ\bullet\bullet\bullet \\
k = (1,1,0,0) & \longleftrightarrow & \circ\bullet\circ\bullet\bullet \\
k = (1,0,1,0) & \longleftrightarrow & \circ\bullet\bullet\circ\bullet \\
k = (1,0,0,1) & \longleftrightarrow & \circ\bullet\bullet\bullet\circ \\
k = (0,2,0,0) & \longleftrightarrow & \bullet\circ\circ\bullet\bullet \\
k = (0,1,1,0) & \longleftrightarrow & \bullet\circ\bullet\circ\bullet \\
k = (0,1,0,1) & \longleftrightarrow & \bullet\circ\bullet\bullet\circ \\
k = (0,0,2,0) & \longleftrightarrow & \bullet\bullet\circ\circ\bullet \\
k = (0,0,1,1) & \longleftrightarrow & \bullet\bullet\circ\bullet\circ \\
k = (0,0,0,2) & \longleftrightarrow & \bullet\bullet\bullet\circ\circ
\end{array}$$

Exercise 2.11: Prove that

$$\#\big\{k \in \mathbb{N}_0^n \,:\, |k| \le s\big\} = \binom{s+n}{n} \ .$$

2.2 Formal statements

Back to the initial problem:

Transform the non linear system

$$\dot{z} = \Lambda z + V_1(z) + V_2(z) + \dots \tag{2.6}$$

into a simpler (possibly linear) form.

2.2.1 The homological equation

- Want to solve $\big(\partial_\lambda - \lambda_j\big)\varphi = \psi$ with $\psi(z)$ a known homogeneous polynomial of degree $s \geq 2$.
- Introducing suitable linear spaces:
  - $\mathscr{P}^s$: the space of homogeneous polynomials of degree s in the n variables $z_1, \dots, z_n$.
  - $\mathscr{N}_j^s = \big(\partial_\lambda - \lambda_j\big)^{-1}\big(\{0\}\big)$: the kernel of the linear operator $\big(\partial_\lambda - \lambda_j\big)$.
  - $\mathscr{R}_j^s = \big(\partial_\lambda - \lambda_j\big)\big(\mathscr{P}^s\big)$: the range of the linear operator $\big(\partial_\lambda - \lambda_j\big)$.
  - Remark: both $\mathscr{N}_j^s$ and $\mathscr{R}_j^s$ are subspaces of $\mathscr{P}^s$.

Lemma 2.12: *Let $j \in \{1, \dots, n\}$ be arbitray but fixed. For any $s > 2$ the following statements are true:*

(i) The subspaces $\mathscr{N}_j^s$ and $\mathscr{R}_j^s$ satisfy

$$\mathscr{N}_j^s \cap \mathscr{R}_j^s = \{0\} \ , \quad \mathscr{N}_j^s \cup \mathscr{R}_j^s = \mathscr{P}^s \ .$$

(ii) If $\lambda_1 \dots, \lambda_n$ are non resonant then $\mathscr{N}^s = \{0\}$ for any $s > 1$.

(iii) The operator $\big(\partial_\lambda - \lambda_j\big)$ restricted to its range $\mathscr{R}^s$ is uniquely inverted. I.e., for any $\psi \in \mathscr{R}_j^s$ there is a unique $\varphi \in \mathscr{R}^s$ such that $\big(\partial_\lambda - \lambda_j\big)\varphi = \psi$.

An obvious statement, since $\big(\partial_\lambda - \lambda_j\big)$ is a diagonal operator.

- The projection operators

$$\Pi_{\mathscr{R}} = \big(\partial_\lambda - \lambda_j\big)^{-1} \circ \big(\partial_\lambda - \lambda_j\big) \ , \quad \Pi_{\mathscr{N}} = 1 - \Pi_{\mathscr{R}} \ , \tag{2.7}$$

satisfy $\Pi_{\mathscr{R}} \circ \Pi_{\mathscr{N}} = \Pi_{\mathscr{N}} \circ \Pi_{\mathscr{R}} = 0$ and $\Pi_{\mathscr{R}} + \Pi_{\mathscr{N}} = 1$.

2.2.2 *The non resonant case of Poincaré and Siegel*

Denote $V = \Lambda z + V_1 + V_2 + \dots$ the vector field in eq. (2.6).

Lemma 2.13: *Let $\lambda_1, \dots, \lambda_n$ be non resonant. Then for every $j = 1, \dots, n$ eq. $L_V \varphi = \lambda_j \varphi$ possesses a unique formal solution $\varphi = \varphi_0 + \varphi_1 + \dots$ with $\varphi_0 = z_j$. The n solutions so found are independent.*

Proof. Recall that $L_V \varphi = \lambda_j \varphi$ splits into the recurrent system

$$\big(\partial_\lambda - \lambda_j\big)\varphi_0 = 0 \ ,$$

$$\big(\partial_\lambda - \lambda_j\big)\varphi_s = \psi_s \ , \quad \psi_s = -\sum_{l=1}^{s} L_{V_l} \varphi_{s-l} \ , \quad s \geq 1 \ .$$

- The first equation has the obvious solution $\varphi_0 = z_j$, which is unique.
- Let $s > 0$ and proceed by induction.
  - If $\varphi_0, \dots \varphi_{s-1}$ are known then so is ψ_s.
  - $\psi_s \in \mathscr{R}^s$, i.e., $\Pi_{\mathscr{N}} \psi_s = 0$, in view of non resonance (lemma 2.1).
  - Hence $\varphi_s = \big(\partial_\lambda - \lambda_j\big)\psi_s$ is uniquely determined, thus completing the induction.
- The independence follows from the independence of the linear parts $z_1, \dots, z_n$.

Q.E.D.

Proposition 2.14: *Let $\lambda_1, \dots, \lambda_n$ be non resonant. Then there exists a formal near the identity transformation $w_j = z_j + w_{1,j}(z) + \dots$ that conjugates the system to its linear part, i.e., $\dot{w}_j = \lambda_j w_j$. The transformation is unique.*

Proof. The n functions $w_j(z)$ are the solutions found in lemma 2.13. *Q.E.D.*

2.2.3 *The resonant case of Dulac*

- Problem: the property $\Pi_{\mathscr{N}}\psi_s = 0$ fails to be true because there are resonant monomials.

- Way out (to be explored): change the homological equation to

$$\big(\partial_\lambda - \lambda_j\big)\varphi_s + \eta_s = \psi_s \ , \tag{2.8}$$

  with unknown φ_s and η_s.
  - Split $\psi_s = \Pi_{\mathscr{N}}\psi_s + \Pi_{\mathscr{R}}\psi_s$.
  - Solve the homological equation by setting

$$\eta_s = \Pi_{\mathscr{N}}\psi_s \ , \quad \varphi_s = \big(\partial_\lambda - \lambda_j\big)^{-1}\psi_s \ .$$

  - The solution is not unique; it is made unique by asking $\varphi_s \in \mathscr{R}^s$;
  - an arbitrary term $\overline{\varphi}_s \in \mathscr{N}^s$ may be added, if there are good reasons.

- Reformulate the initial problem as:
  Find a near the identity transformation which changes the system (2.1) into the non linear normal form

$$\dot{w}_j = \lambda_j w_j + \eta^{(j)}(w) \ , \quad j = 1, \dots, n \tag{2.9}$$

  with $\big(\partial_\lambda - \lambda_j\big)\eta^{(j)}(w) = 0$.

- Problem: show that the equation $L_V\varphi = \lambda_j\varphi$ can be consistently solved.
  - Reformulate in proper manner the recursive equations;
  - find a formal solution.

Assume that $\lambda_1, \ldots, \lambda_n$ are resonant, but belong to the domain of Poincaré.

- Recall: there is a straight line σ trough the origin that leaves $\lambda_1, \ldots, \lambda_n$ on the same side.
- Reorder the eigenvalues so that

  $$\operatorname{dist}(\lambda_1, \sigma) \leq \operatorname{dist}(\lambda_j, \sigma) \leq \ldots \leq \operatorname{dist}(\lambda_n, \sigma) \ . \tag{2.10}$$

  (the resonance relations are not affected).
- Remark: the resonance relation $\langle k, \lambda \rangle = \lambda_j$ may occur only if $k_j = \ldots = k_n = 0$.

Proposition 2.15: *(Dulac, 1912) Let the eigenvalues $\lambda_1, \ldots, \lambda_n$ belong to the Poincaré domain, and be ordered as in (2.10). Then there exists a formal near the identity transformation that gives the system $\dot{z} = \Lambda z + V_1(z) + \ldots$ the normal form*

$$\begin{aligned} \dot{w}_1 &= \lambda_1 w_1 \ , \\ \dot{w}_j &= \lambda_j w_j + \eta^{(j)}(w_1, \ldots, w_{j-1}) \ , \quad j = 2, \ldots, n \ , \end{aligned} \tag{2.11}$$

where $\eta^{(j)}(w_1, \ldots, w_{j-1})$ is a (non homogeneous) polynomial of finite degree belonging to the kernel of the operator $\big(\partial_\lambda - \lambda_j\big)$. The normal form for $j = 2, \ldots, n$ needs not be unique.

Proof. Proceed step by step.

- Let $j = 1$.
  - ∘ None of the divisors $\langle k, \lambda\rangle - \lambda_1$ can be zero.
  - ∘ Solve the recurrent system as in the non resonant case. Get

$$w_1 = z_1 + \varphi_{1,1}(z) + \varphi_{2,1}(z) + \dots \ .$$

  - ∘ Invert (standard procedure for analytic functions)

$$z_1 = w_1 + \tilde{\varphi}_{1,1}(w_1, z_2, \dots, z_n) + \tilde{\varphi}_{2,1}(w_1, z_2, \dots, z_n) + \dots$$

  - ∘ Replace in the initial system of equations, and get the new system

$$(2.12) \quad \dot{w}_1 = \lambda_1 w_1 \ ; \quad \dot{z}_j = \lambda_j z_j + V'_{1,j}(w_1, z_2, \dots, z_n) + V'_{2,j}(w_1, z_2, \dots, z_n) + \dots \ , \quad j = 2, \dots, n \ ,$$

  with $n - 1$ equations to be solved.
- Let $j = 2$, assume that $k_1\lambda_1 = \lambda_2$ for some $k_1 \geq 2$. (If not, there are no resonances: nothing new.)
  - ∘ There is only one resonance relation (obvious).
  - ∘ Split the system (2.12) as

$$\big(\partial_\lambda - \lambda_2\big)\varphi_{s,2} = \psi_s \text{ for } s \neq k_1 \ ; \quad \big(\partial_\lambda - \lambda_2\big)\varphi_{k_1,2} + \eta^{(2)} = \psi_{k_1} \ ,$$

  with $\eta^{(2)}$ an unknown polynomial of degree $k_1 + 1$, and

$$\psi_1 = -V'_{1,2} \ , \quad \psi_s = -V'_{s,2} - \sum_{l=1}^{s-1} L_{V'_{l,2}} \varphi_{s-l,2} \ .$$

  - ∘ For $s \neq k_1$ there are no resonances; nothing new.
  - ∘ For $s = k_1$ the monomial $w_1^{k_1}$, and only that, is resonant: move it into $\eta^{(2)}$ (depends only on w_1).
  - ∘ Get the solution

$$w_2 = z_2 + \varphi_{1,2}(w_1, z_2, \dots, z_n) + \varphi_{2,2}(w_1, z_2, \dots, z_n) + \dots \quad \text{so that} \quad \dot{w}_2 = \lambda_1 w_2 + \eta^{(2)}(w_1) \ .$$

- By induction, let $1 < j \leq n$, and assume we know

$$w_l = z_l + \varphi_{1,2}(w_1, \dots, w_{l-1}, z_l, \dots, z_n) + \varphi_{2,2}(w_1, \dots, w_{l-1}, z_l, \dots, z_n) + \dots$$

that change the first $j-1$ equations to

$$\dot{w}_1 = \lambda_1 \ , \ \dots \ , \dot{w}_{j-1} = \lambda_{j-1} w_{j-1} + \eta^{(j-1)}(w_1, \dots, w_{j-1}) \ , \tag{2.13}$$

  - Invert (standard procedure)

$$z_l = w_l + \tilde{\varphi}_{1,2}(w_1, \dots, w_{l-1}, z_l, \dots, z_n) + \tilde{\varphi}_{2,2}(w_1, \dots, w_{l-1}, z_l, \dots, z_n) + \dots \ , \quad l = 1, \dots, j-1 \ .$$

  - Replace in the initial system and get the new system (2.13) plus the $n-j+1$ equations

$$\dot{z}_l = \lambda_l z_l + V''_{1,l}(w_1, \dots, w_{j-1}, z_j, \dots, z_n) + V''_{2,l}(w_1, \dots, w_{j-1}, z_j, \dots, z_n) + \dots \ , \quad l = j \dots, n$$

  - Split the equation for z_j as

$$\big(\partial_\lambda - \lambda_j\big)\varphi_{s,j} + \eta_s^{(j)} = \psi_s \ , \quad s \geq 2 \ ,$$

  - There is a finite number of resonant terms, hence only a finite number of $\eta_s^{(j)}$ may be non zero.
  - Resonant monomials: $w_1^{k_1} \cdots w_{j-1}^{k_{j-1}}$ with $k = (k_1, \dots, k_{j-1}, 0, \dots, 0)$ resonant. I.e., they depend only on $w_1, \dots, w_{j-1}$.
  - Hence the equation for w_j may be solved as

$$w_j = z_j + \varphi_{1,2}(w_1, \dots, w_{j-1}, z_j, \dots, z_n) + \varphi_{2,2}(w_1, \dots, w_{j-1}, z_j, \dots, z_n) + \dots$$

- Iterate the procedure for $j = 2, \dots, n$.

Q.E.D.

2.3 Quantitative methods

The basic tools for investigating the convergence of the formal expansions.

2.3.1 Polynomial norm

- For a homogeneous polynomial $f(z) = \sum_{|k|=r} f_k z^k$ of any degree r define the *polynomial norm* as

$$\|f\| = \sum_{|k|=r} |f_k| \ .$$

- For a homogeneous polynomial vector field $X(z) = \big(X_1(z), \ldots, X_n(z)\big)$ define the norm as

$$\|X\| = \max_j \|X_j\| \ ,$$

- Consider a (closed) polydisk Δ_ϱ of radius ϱ around the origin of $\mathbb{C}^n$, i.e.,

$$\Delta_\varrho = \{z \in \mathbb{C}^n \ : \ |z_j| \leq \varrho \, , \ j = 1, \ldots, n\}$$

  - For a homogeneous polynomial $f_r(z)$ of degree r we have

$$|f_r(z)| \leq \|f_r\| \varrho^r \quad \text{for all} \quad z \in \Delta_\varrho \ .$$

  - Let $f = \sum_{r \geq 0} f_r(z)$ be a power series. If $\|f_r\| \leq C^{r-1} F$ for some $C \, , F > 0$ then the series is absolutely and uniformly convergent in any polydisk of radius $\varrho < 1/C$.

2.3.2 *Technical estimates*

Estimate of the Lie derivative.

Lemma 2.16: *Let X_r by a homogeneous polynomial vector field of degree $r+1$ and f_m be a homogeneous polynomial of degree $m+1$. Then $L_{X_r} f_m$ is homogeneous of degree $s+m+1$, and*

$$\left\| L_{X_r} f \right\| \leq (m+1) \, \|X\| \, \|f\| \ .$$

Proof. Write $X_{r,l} = \sum_{|k'|=r+1} c_{l,k'} z^{k'}$ and $f_m = \sum_{|k|=m+1} b_k z^k$.

- Get

$$L_{X_r} f_m = \sum_{l=1}^{n} \sum_{k,k'} \frac{c_{j,k'} \, k_l b_k}{z_l} z^{k+k'} \ ,$$

- By definition of norm

$$\left\| L_{X_r} f_m \right\| = \sum_{l=1}^{n} \sum_{k} |k_l| \, |b_k| \sum_{k'} |c_{l,k'}| \leq \sum_{k} \sum_{l=1}^{n} |k_l| \, |b_k| \, \|X_{r,l}\| \ .$$

- Use

$$\sum_{k'} |c_{l,k'}| = \|X_{r,l}\| \leq \|X_r\| \ , \quad \sum_{l} |k_l| = |k| = m+1 \ . \quad \text{and} \quad \sum_{k} |b_k| = \|f_m\| \ ,$$

  which gives the claim.

Q.E.D.

Solution of the homological equation.

- Define the sequence $\{\beta_s\}_{s\geq 0}$ as

$$\beta_0 = 1 \ , \quad \beta_s = \min_{|k|=s+1} \min_{1\leq j\leq n} \big|\langle k,\lambda\rangle - \lambda_j\big| \ , \ s \geq 1 \ . \tag{2.14}$$

  - Poincaré domain, λ non resonant:

$$\beta_s \geq \gamma(s+1) \ , \quad \gamma > 0 \ .$$

  - Siegel domain, λ non resonant and diophantine:

$$\beta_s \geq \frac{\gamma}{(s+1)^\nu} \ , \quad \gamma > 0 \ , \ \nu > \frac{n-2}{2} \ .$$

Lemma 2.17: *Let λ be non resonant, and let ψ_s be a homogeneous polynomial of degree $s+1$. Then we have*

$$\big\|\big(\partial_\lambda - \lambda_j\big)^{-1}\psi_s\big\| \leq \frac{\|\psi_s\|}{\beta_s}$$

with the sequence $\{\beta_s\}_{s>1}$ defined by (2.14).

Proof. Straightforward in view of the diagonal form of the operator $\big(\partial_\lambda - \lambda_j\big)$ stated by corollary 2.2.

- Recall

$$\big(\partial_\lambda - \lambda_j\big)z^k = \big(\langle k,\lambda\rangle - \lambda_j\big)z^k \ .$$

- Just divide the coefficient of every monomial z^k by the corresponding eigenvalue $\langle k,\lambda\rangle - \lambda_j$, which has lower bound β_s.

Q.E.D.

2.3.3 The theorem of Poincaré

Theorem 2.18: *(Poincaré, 1879) Let the system $\dot{z} = \Lambda z + V_1 + \dots$ be analytic in a neighbourhood of the origin of $\mathbb{C}^n$, and let the eigenvalues of the matrix $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ belong to the Poincaré domain and be non resonant. Then in a neighbourhood of the origin there exists an analytic diffeomorphism that conjugates the system to its linear normal form (2.2), i.e., $\dot{w} = \Lambda w$.*

Proof. By Proposition 2.14 there are n independent formal series $w_j = z_j + \varphi_{1,j}(z) + \dots$ which give the wanted normal form. Prove that the formal series are convergent in a polydisk.

- Recall the recurrent system

$$(\partial_\lambda - \lambda_j)\varphi_0 = 0 \ ,$$

$$(\partial_\lambda - \lambda_j)\varphi_s = \psi_s \ , \quad \psi_s = -\sum_{l=1}^{s} L_{V_l}\varphi_{s-l} \ , \quad s \geq 1 \ .$$

  to be solved for $\varphi_0 = z_j$, hence $\|\varphi_0\| = 1$.
- By analyticity there are $B \geq 0$ and $E > 0$ such that

$$\|V_s\| \leq B^{s-1}E \ , \quad s > 0$$

- By lemmas 2.16 and 2.17 and using $\beta_s \geq \gamma s$ get

$$\|\varphi_0\| = 1 \ ; \quad \|\varphi_s\| \leq \frac{1}{\gamma}\sum_{l=1}^{s}\frac{(s-l+1)}{(s+1)}\|V_l\|\|\varphi_{s-l}\| \leq \frac{E}{\gamma}\sum_{l=1}^{s}B^{l-1}\|\varphi_{s-l}\| \ .$$

- Hence $\|\varphi_s\| \leq \mu_s$ with the arithmetic sequence

$$\mu_0 = 1 \ , \quad \mu_s = \frac{E}{\gamma}\sum_{l=1}^{s}B^{l-1}\mu_{s-l} \ ; \quad \text{hence} \quad \mu_s \leq \frac{E}{\gamma}\left(B + \frac{E}{\gamma}\right)^{s-1} \ .$$

- Conclude: the series for φ is absolutely convergent in any polydisk Δ_ϱ with $\varrho < \left(B + \frac{E}{\gamma}\right)^{-1}$.

Q.E.D.

2.3.4 *On the case of Siegel*

Consider the simplified model

$$\dot{z} = \Lambda z + V_1(z) \ .$$

- The recurrent system simplifies to

$$\big(\partial_\lambda - \lambda_j\big) z_j = 0 \ , \quad \big(\partial_\lambda - \lambda_j\big)\varphi_{s,j} = -L_{V_1}\varphi_{s-1,j} \ , \quad s \geq 1 \ .$$

- The recurrent estimate gives

$$\|\varphi_s\| \leq \frac{s!}{\beta_1 \cdots \beta_s} \ .$$

- Two sources of divergence:
  - the factorial at the numerator, due to estimates of derivatives;
  - the accumulation of divisors at the denominator estimated via the diophantine inequality.
- Accumulation of divisors controlled by a number theoretical lemma due to Siegel:

$$\frac{1}{\beta_1 \cdots \beta_s} \leq K^s$$

  for some $K > 1$.
- Need to change the algorithm in order to remove the factorial.

The obstacle will be removed using the normal form theory.