Spherelike Twist Functors

Andreas Hochenegger

University of Cologne, Germany

ICRA 2012
This talk is about a joint work with Martin Kalck and David Ploog.
**Protagonists**

**Notation**

Let $\mathcal{D}$ be a $k$-linear algebraic triangulated category with Serre functor. All triangles are meant to be distinguished, and all functors exact.

**Definition**

An object $F$ in $\mathcal{D}$ is called

- **$d$-spherelike** $\iff$ $\text{Hom}^\bullet(F, F) = k \oplus k[-d]$  
- **$d$-spherical** $\iff$ $F$ is $d$-spherelike and $\text{Hom}^\bullet(F, \cdot) = \text{Hom}^\bullet(\cdot, F[d])^*$

**$d$-Sphere**  

**$d$-Calabi-Yau**
Definition

For any object $F$ in $\mathcal{D}$ there is the evaluation map
\[
\text{Hom}^\bullet(F, \cdot) \otimes F \to \text{id}.
\]

We define the twist functor $T_F$ as its cone. So the functor fits into the triangle
\[
\text{Hom}^\bullet(F, \cdot) \otimes F \to \text{id} \to T_F.
\]

If $F$ spherelike, we call $T_F$ a spherelike twist functor.

Analogously, $T_F$ is called spherical twist functor, if $F$ is spherical.

Theorem (Paul Seidel and Richard Thomas)

Let $F$ be non-zero.

$F$ is spherical $\iff T_F$ is an auto-equivalence.
Paul Seidel and Richard Thomas were motivated by mirror symmetry. In algebraic geometry, a typical example is a \((-2)\)-curve \(C\) on a smooth projective surface \(X\). Then \(F = \mathcal{O}_C\) is 2-spherical and \(T_F\) is an auto-equivalence of \(\mathcal{D}^b(X)\). On the symplectic side, this twist corresponds to a Dehn twist.
There are some immediate similarities:

\[ T_F(F) = F[1-d] \quad \leftrightarrow \quad \text{rotating red circle} \]

\[ T_F = \text{id when restricted to } F^\perp \quad \leftrightarrow \quad \text{outside red circle} \quad \text{essentially nothing happens} \]
Spherical Example: Quiver Algebra $A = \mathbb{k}Q/I$

$A = \mathbb{k}Q/I$

$I = \langle a_2a_1 \rangle$

The simple $A$-module $S(1)$ has the projective resolution

$$0 \rightarrow P(1) \rightarrow P(2) \rightarrow P(1) \rightarrow S(1) \rightarrow 0$$

Easy calculation:

$$\text{Hom}^\bullet(S(1), S(1)) = \mathbb{k} \oplus \mathbb{k}[-2]$$

$S(1)$ is 2-Calabi-Yau

$S(1)$ is 2-spherical
Spherelike Example: Quiver Algebra $A' = kQ'/I'$

$Q'$:

Choosing idempotent $e = e_1 + e_{r+1}$:

$$\mathcal{D}^b(A) \cong \mathcal{D}^b(eA'e)$$

Fully faithful embedding:

$$j : \mathcal{D}^b(A) \hookrightarrow \mathcal{D}^b(A')$$

(induced by functor $A'e \otimes_{eA'e} \cdot$)

Result

$\Rightarrow j(S(1))$ is still 2-spherelike, but not 2-spherical.

$\Rightarrow T_j(S(1))$ is not an equivalence.
The Spherical Subcategory

Let $F$ be $d$-spherelike but not $d$-spherical. Denote the Serre functor of $\mathcal{D}$ by $S$. Then

$$\text{Hom}^\bullet(F, F) = \text{Hom}^\bullet(F, S(F))^* \not\cong \text{Hom}^\bullet(F, F[d])^*$$

We can compare $\omega(F) := S(F)[-d]$ and $F$. By Serre duality

$$\text{Hom}^\bullet(F, \omega(F)) = \text{Hom}^\bullet(F, F)^*[-d] = k \oplus k[-d]$$

$\Rightarrow$ canonical map

$$w : F \rightarrow \omega(F)$$

(even for the case $d = 0$, which needs more care)
Using the canonical map, we define the

\[ F \xrightarrow{w} \omega(F) \rightarrow Q_F \quad \text{aspherical triangle} \]

Properties of \( Q_F \)

- \( F \) spherical \( \iff \) \( Q_F \) is zero
- \( \text{Hom}^\bullet(F, Q_F) \) vanishes

Main Definition

Let \( F \) be a spherelike object in \( \mathcal{D} \).

- \( \mathcal{D}_F := \bot Q_F \) spherical subcategory
- \( Q_F := \mathcal{D}_F^\perp \) aspherical subcategory
Main Theorem

**Theorem**
Let $F$ be a $d$-spherelike object in $\mathcal{D}$. Then $F$ is $d$-spherical in $\mathcal{D}_F$. Moreover, $T_F$ induces auto-equivalences of $\mathcal{D}_F$ and $\mathcal{Q}_F$.

**Sketch of the Proof**
Easy: $T_F|_{\mathcal{Q}_F} = \text{id}_{\mathcal{Q}_F}$ by $F \in \mathcal{D}_F = \perp \mathcal{Q}_F$.

$T_F|_{\mathcal{D}_F}$: Apply $\text{Hom}^\bullet(A, \cdot)$ to the aspherical triangle

\[
\text{Hom}^\bullet(A, F) \to \text{Hom}^\bullet(A, \omega(F)) \to \text{Hom}^\bullet(A, \mathcal{Q}_F) = \text{Hom}^\bullet(F, A[d])^* = 0
\]

so $F$ is $d$-spherical in $\mathcal{D}_F$.

Warning: $\mathcal{D}_F$ has no Serre functor in general. Theorem of Seidel and Thomas does not apply here.
Spherelike Example, Revisited

\[ A = kQ/I \]

\[ Q: \]

\[ \begin{array}{c}
1 \\
\vdots \\
2 \\
\end{array} \]

\[ \xrightarrow{a_2} \]

\[ l = \langle a_2a_1 \rangle \]

\[ \]

\[ A' = kQ'/I' \]

\[ Q': \]

\[ \begin{array}{c}
1 \\
\vdots \\
2 \\
\end{array} \]

\[ \xrightarrow{a_1} \]

\[ \xrightarrow{a_{r-1}} \]

\[ l' = \langle a_r \cdots a_n \rangle \]

Special situation: \( j : \mathcal{D}^b(A) \hookrightarrow \mathcal{D}^b(A') \)

has right adjoint \( i \) such that \( i \circ j = \text{id} \).

**Proposition**

For \( F = j(S(1)) \) holds

\[ \mathcal{D}^b(A'_F) = \langle \mathcal{D}^b(A)^\perp \cap \perp F, \mathcal{D}^b(A) \rangle \]

Using this proposition, we calculate

\[ \mathcal{D}^b(A'_F) \cong \langle S(k), k = 2, \ldots, r - 1 \rangle \times \]

\[ \times \langle S(k), k = r + 2, \ldots, n - 1 \rangle \times \mathcal{D}^b(A) \]

\[ \cong \mathcal{D}^b(\tilde{A}_{r-2}) \times \mathcal{D}^b(\tilde{A}_{n-r-2}) \times \mathcal{D}^b(A) \]

with \( \tilde{A}_m \) the path algebra of \( 1 \rightarrow \cdots \rightarrow m \).