Second category incomplete normed spaces

Let us recall that a *nowhere dense set* in a topological space is a set which is not dense in any nonempty open set. Countable unions of nowhere dense sets are called *first category sets* (or *sets of first [Baire] category*). Sets which are not of the first category are called *second category sets* (or *sets of second [Baire] category*). A topological space is a *Baire space* if it contains no nonempty open sets of the first category.

By the celebrated Baire Category Theorem, *each complete metric space, as well as each locally compact topological space, is a Baire space.*

It is easy to see that *a normed space is a Baire space if and only if it is of second category* (indeed, $X = \bigcup_n nB_X^\circ$, hence if X is of the second category then each open ball is). It is natural to ask:

do there exist incomplete normed spaces of second category?

For such spaces as domain spaces, the Uniform Boundedness Principle is still valid.

Let us show that the answer to this question is positive. Our proof is based on an e-mail by L. Zajíček to C. Zanco (we thank the latter for communicating it to us).

Let us start with a standard topological observation.

top **Observation 0.1.** *Let $A \subset B \subset C$ be three sets in a topological space. If A is nowhere dense [of first category] in B, then A is nowhere dense [of first category] in C.*

Proof. Clearly it suffices to show the statement about nowhere dense sets. If A is not nowhere dense in C, then A is dense in a nonempty set $G \subset C$ which is open in C. Since B is dense in G, the set $G \cap B$ is a nonempty relatively open set in B such that $G \cap B \subset \overline{A} \cap B = \overline{A}^B$, which implies that A is not nowhere dense in B. □

Theorem 0.2. *Every infinite-dimensional Banach space X contains a dense hyperplane which is of second category in itself.*

Proof. Fix an algebraic basis $\{a_i\}_{i \in I}$ of X. Thus every $x \in X$ can be written in a unique way as a sum

$$x = \sum_{i \in I} b_i(x) a_i$$

where at most finitely many terms are nonzero. For each $i \in I$, let H_i be the kernel of the (linear) coefficient functional b_i. We claim that only finitely many H_i's can be of first category in X. Indeed, otherwise there would exist a countably infinite set $C \subset I$ such that each H_i, $i \in C$, is of first category. But this contradicts the Baire Category Theorem since $X = \bigcup_{i \in C} H_i$. Thus, for some $j \in I$, H_j is of second category in X. By Observation 0.1, H_j is of second category in itself. Clearly, H_j cannot be closed; it follows it is dense in X. □

Another example of a similar nature is contained in the following exercise, based on a private communication by C.A. De Bernardi.

Exercise 0.3. Let $\{x_n\}$ be a linearly independent sequence in a Banach space. Then there exists a non-closed subspace $Y \subset X$ which is of second category in X (hence in itself) that contains at most finitely many x_n's.

Hint. Fix an algebraic complement V of $\operatorname{span}\{x_1, x_2, \ldots\}$ in X. Using the Baire Category Theorem, show that one of the spaces $Y = V + \operatorname{span}\{x_1, \ldots, x_n\}$ works. □

To conclude, let us remark that both above proofs are just existence proofs. We do not know of any *concrete example* of a second category incomplete normed space.

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Remark 0.4. For the validity of the Uniform Boundedness Principle in normed spaces, even the assumption that the domain space is of second category is not necessary. The Uniform Boundedness Principle is valid with a normed domain space X if and only if X is *barrelled*, that is, each closed convex set $B \subset X$, satisfying $X = \bigcup_{t>0} tB$ and $\alpha B = B$ whenever $|\alpha| \leq 1$, is a neighborhood of 0. If X is of second category then it is clearly barrelled, but not vice-versa. E.g., W. Orlicz (in *Linear Functional Analysis*, 1992) states that

$$X = \big\{x \in C[0,1] : \; x \in C^\infty\big([0,1] \setminus S_x\big) \text{ with } S_x \text{ at most countable}\big\},$$

considered as a subspace of the Banach space $C[0,1]$, is a barrelled normed space which is of first category in itself. Another example can be found in [W.L.C. Sargent, *On some theorems of Hahn, Banach and Steinhaus*, J. London Math. Soc. **28** (1953), 438–451].

This information has been taken from the lecture notes [J. Lukeš, *Zápisky z funkcionální analýzy*, Karolinum, Prague, 1998].