Helly's Intersection Theorem

The main topic of this chapter is a famous combinatorial theorem by E. Helly (1884–1943). Let us start with a useful terminology.

Definition 0.1. Let $\mathcal{F}$ be a family of subsets of a set X, and $k \in \mathbb{N}$. We say that the family $\mathcal{F}$ is:

- *centered* if each finite (nonempty) subfamily of $\mathcal{F}$ has nonempty intersection;
- k*-centered* if each finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ with $1 \leq \operatorname{card}(\mathcal{F}_0) \leq k$ has nonempty intersection.

Let us recall the following easy and well-known characterization of compactness.

Theorem 0.2. *A topological space T is compact if and only if each centered family of closed subsets of T has nonempty intersection.*

Corollary 0.3. *Let T be a topological space, and $\mathcal{F}$ a centered family of closed sets in T. If there exists a (nonempty) finite subfamily $\mathcal{F}_0 \subset \mathcal{F}$ such that $\bigcap \mathcal{F}_0$ is compact, then $\mathcal{F}$ has nonempty intersection.*

Proof. Apply Theorem 0.2 to the compact topological space $\bigcap \mathcal{F}_0$ and the family

$$\mathcal{G} = \{F \cap \bigcap \mathcal{F}_0 : F \in \mathcal{F}\}.$$

□

As a motivation, we state the following elementary combinatorial property for intervals (that is, convex sets in $\mathbb{R}$).

Example 0.4. Each finite 2-centered family of intervals in $\mathbb{R}$ has nonempty intersection.

Proof. Let I_j $(j = 1, \ldots, n)$ be the elements of the family. Denote

$$a_j = \inf I_j\,, \quad b_j = \sup I_j\,, \quad A = \max\{a_1, \ldots, a_n\}\,, \quad B = \{b_1, \ldots, b_n\}\,.$$

For each $j, k \in \{1, \ldots, n\}$, we have $I_j \cap I_k \neq \emptyset$, and hence $a_j \leq b_k$. It follows that $A \leq B$.

If $A < B$, then clearly $(A, B) \subset \bigcap_1^n I_j$ and we are done.

Now, let $A = B$. In this case $a_j \leq A \leq b_j$ for each j. If $A \notin I_j$, we must have either $A = a_j \notin I_j$ or $A = b_j \notin I_j$. Assume, for instance, that $A = a_j \notin I_j$ (the other case is similar). Since $A = B$, there exists k such that $A = b_k$. But this implies that $I_k \cap I_j \subset (-\infty, A] \cap (A, +\infty) = \emptyset$ which is a contraddiction. This proves that $A \in \bigcap_1^n I_j$. □

Example 0.5. In $\mathbb{R}^2$, a finite 2-centered family of convex sets may have empty intersection. Indeed, consider the family of three segments that form the three sides of a triangle. (On the other hand, by the following Theorem 0.6, this cannot happen if the family is 3-centered.)

Theorem 0.6 (Helly's Theorem; Helly 1913 [publ. 1923], Radon 1921)**.** *Let $\mathcal{F}$ be a finite $(d+1)$-centered family of convex sets in $\mathbb{R}^d$. Then $\mathcal{F}$ has nonempty intersection.*

Proof. Let us proceed by induction with respect to the cardinality of $\mathcal{F}$. If $\mathrm{card}(\mathcal{F}) \leq d+1$, then obviously $\bigcap \mathcal{F} \neq \emptyset$.

Let $\mathrm{card}(\mathcal{F}) = n > d+1$, and assume that our theorem holds for families of any positive cardinality which is smaller than n. Let

$$\mathcal{F} = \{C_1, \ldots, C_n\}\,.$$

For each $i \in \{1, \ldots, n\}$, there exists a point

$$x_i \in \bigcap \{C_j :\ 1 \leq j \leq n,\ j \neq i\} = \bigcap \big(\mathcal{F} \setminus \{C_i\}\big)\,. \tag{1}$$

The linear mapping

$$T\colon \mathbb{R}^n \to \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}\,, \qquad T\lambda = \left(\sum_1^n \lambda_i x_i\,, \sum_1^n \lambda_i\right) \quad (\lambda \in \mathbb{R}^n),$$

is not injective (that is, one-to-one) since $n > d+1$. Thus there exists $\lambda \in \mathbb{R}^n \setminus \{0\}$ such that

$$\sum_1^n \lambda_i x_i = 0 \qquad \text{and} \qquad \sum_1^n \lambda_i = 0\,. \tag{2}$$

It is easy to see that the sets $P = \{i : \lambda_i > 0\}$ and $N = \{i : \lambda_i \leq 0\}$ are both nonempty. We can rewrite (2) in the form

$$\sum_{i \in P} \lambda_i x_i = \sum_{i \in N} |\lambda_i| x_i\,, \qquad \sum_{i \in P} \lambda_i = \sum_{i \in N} |\lambda_i| =: \sigma\,.$$

By (1), we have $x_i \in \bigcap_{j \in N} C_j$ whenever $i \in P$, and also $x_i \in \bigcap_{j \in P} C_j$ whenever $i \in N$. It follows that the point

$$z := \frac{1}{\sigma} \sum_{i \in P} \lambda_i x_i = \frac{1}{\sigma} \sum_{i \in N} |\lambda_i| x_i \qquad \text{(convex combinations!)}$$

belongs to both $\bigcap_{j \in N} C_j$ and $\bigcap_{j \in P} C_j$. This means that $z \in \bigcap \mathcal{F}$ and we are done. □

Theorem 0.7 (Helly' Theorem – second version)**.** *Let $\mathcal{F}$ be a (possibly infinite) $(d+1)$-centered family of closed convex sets in $\mathbb{R}^d$. Assume that there exists a (nonempty) finite subfamily $\mathcal{F}_0 \subset \mathcal{F}$ such that $\bigcap \mathcal{F}_0$ is compact. Then $\mathcal{F}$ has nonempty intersection.*

Proof. By Theorem 0.6, $\mathcal{F}$ is centered. Apply Corollary 0.3. □

Example 0.8. Given $d \in \mathbb{N}$, the sets

$$C_n = \{x \in \mathbb{R}^d : x(1) \geq n\} \qquad (n \in \mathbb{N})$$

are convex, closed and nested ($C_1 \supset C_2 \supset \ldots$). The family $\mathcal{F} = \{C_n : n \in \mathbb{N}\}$ is centered, but has empty intersection. This example shows that the word “finite” in Theorem 0.6 and the assumption about existence of $\mathcal{F}_0$ in Theorem 0.7 cannot be omitted.

Some applications.

Theorem 0.9 (Common Transversal Theorem or "Skewer theorem")**.** *Let $\mathcal{F}$ be a family of compact mutually parallel (possibly degenerate) compact segments in the plane, such that no two of them are colinear. Assume that for each triplet of elements of $\mathcal{F}$ there is a line intersecting all three of them. Then there exists a line that intersects all emenets of $\mathcal{F}$.*

Proof. We can suppose that the coordinate system is chosen so that the segments are vertical. Thus we can write $\mathcal{F} = \{J_\alpha : \alpha \in A\}$ where

$$J_\alpha = \{u_\alpha\} \times [a_\alpha, b_\alpha] \qquad (\alpha \in A).$$

Since no two of J_α's are colinear, we can consider only lines that are non-vertical. For each $\alpha \in A$, consider the set

$$\begin{aligned} C_\alpha &= \{(m,q) \in \mathbb{R}^2 : \text{ the line } y = mx + q \text{ intersects } J_\alpha\} \\ &= \{(m,q) \in \mathbb{R}^2 : \ a_\alpha - u_\alpha m \le q \le b_\alpha - u_\alpha m\}. \end{aligned}$$

Observe that each C_α is a closed strip (in the plane (m,q)) between two parallel lines of slope $-u_\alpha$. Thus each C_α is closed and convex, and if $\alpha \ne \beta$ then $C_\alpha \cap C_\beta$ is compact. By assumptions, the family $\{C_\alpha : \alpha \in A\}$ is 3-centered. By the second version of Helly's theorem, this family has nonempty intersection, which implies the statement. □

Let us use the above theorem to prove the following "sandwich-type" result.

Theorem 0.10. *Let $I \subset \mathbb{R}$ be an interval, $f, g \colon I \to \mathbb{R}$ two functions such that $f \le g$. The following assertions are equivalent.*

(i) *There exists an affine function $a \colon \mathbb{R} \to \mathbb{R}$ such that $f \le a \le g$ on I.*
(ii) *For each $x, y \in I$ and $\lambda \in [0,1]$ we have*

$$\begin{cases} f((1-\lambda)x + \lambda y) \le (1-\lambda)g(x) + \lambda g(y) \\ g((1-\lambda)x + \lambda y) \ge (1-\lambda)f(x) + \lambda f(y). \end{cases}$$

Proof. The implication $(i) \Rightarrow (ii)$ is easy: $f((1-\lambda)x + \lambda y) \le a((1-\lambda)x + \lambda y) = (1-\lambda)a(x) + \lambda a(y) \le (1-\lambda)g(x) + \lambda g(y)$, and similarly the second inequality.

Let us show the implication $(ii) \Rightarrow (i)$. Consider the family $\mathcal{F}$ whose elements are the segments

$$J_x = \{x\} \times [f(x), g(x)] \qquad (x \in I).$$

By Theorem 0.9, (i) will be proved once we show that any three of these segments are intersected by a line. Consider three points $x < z < y$ of I. It is easy to see that there is a line intersecting all three J_x, J_z, J_y if and only if the intersection I of the trapeziod of vertices

$$(x, f(x)), \quad (x, g(x)), \quad (y, f(x)), \quad (y, g(y))$$

with the vertical line at z has a common point with J_z. Write z as a convex combination

$$z = (1-\lambda)x + \lambda y$$

(with $0 < \lambda < 1$) and observe that $I = \{z\} \times [u, v]$ where $u = (1-\lambda)f(x) + \lambda f(y)$ and $v = (1-\lambda)g(x) + \lambda g(y)$. Thus $I \cap J_z \neq \emptyset$ if and only if $[f(z), g(z)] \cap [u, v] \neq \emptyset$ which is equivalent to

$$\begin{cases} f(z) \leq v \\ g(z) \geq u\,. \end{cases}$$

Since these two inequalities are exactly those in (ii), we conclude that J_x, J_z, J_y are intersected by a line. We are done by Theorem 0.9. □

Remark 0.11. Notice that (ii) in Theorem 0.10is satisfied whenever one of the functions f, g is convex and the other one concave.

Two further results. The particular case of the following Theorem 0.12 with "convex" instead of "starshaped" was proved by V. Klee in 1951 and then independently by C. Berge in 1959. The stronger version follows from a topological result due to C.D. Horvath and M. Lassonde (1997). We present here a simple geometric proof based on Klee's original argument.

Recall that a subset A of a vector space is called *starshaped* if there exists $x \in A$ such that, for any $y \in A$, the (closed) segment $[x, y]$ is contained in A. The (convex) set of all such points x is called the *kernel* of A.

Theorem 0.12. *Let $C_0, \ldots, C_n$ be convex subsets of $\mathbb{R}^d$, each n of which have a point in common, such that $\bigcup_0^n C_i$ is starshaped. Suppose that either each C_i is closed or each C_i is open. Then the intersection $\bigcap_0^n C_i$ is nonempty.*

Proof. If all C_i's are closed, we may assume that they are compact (by intersecting them with a sufficiently large closed ball). Let us proceed by induction with respect to n. For $n = 1$ the theorem follows from the fact that each starshaped set is obviously connected. Now suppose it holds for $n = k-1$ (and every d) and consider the case $n = k$. Let z_0 be a point from the kernel of the starshaped set $\bigcup_0^k C_i$. We may assume that $z_0 \in C_0$. If $\bigcap_0^k C_i = \emptyset$ then C_0 and $P = \bigcap_1^k C_i$ are nonempty disjoint compact (resp., open) convex sets, so they can be separated by a hyperplane H disjoint from both of them. Let $D_i = C_i \cap H$ $(1 \leq i \leq k)$. For an arbitrary $i_0 \in I = \{1, \ldots, k\}$, let $Q = \bigcap_{i \in I \setminus \{i_0\}} C_i$. Since each k of the C_i's have a point in common, Q intersects C_0. And since furthemore $P \subset Q$, Q must intersect H and hence $\bigcap_{i \in I \setminus \{i_0\}} D_i \neq \emptyset$. Once we show that $\bigcup_1^k D_i$ is starshaped, the theorem will be proved. Indeed, since D_i's are compact (resp., open) in H which is isomorphic to $\mathbb{R}^{d-1}$, it will follow from the inductive hypothesis that $\bigcap_1^k D_i \neq \emptyset$. But this contradicts the fact that $P \cap H = \emptyset$.

Fix an arbitrary point $p \in P$. Since $z_0 \in C_0$, the segment $[p, z_0]$ intersects H at a point z_1. Let x be an arbitrary point of $\bigcup_1^k D_i$. Then the segment $[p, x]$ is contained in $\bigcup_1^k C_i$. By the definition of z_0, $[y, z_0] \subset \bigcup_0^k C_i$ for each $y \in [p, x]$. Consequently, the triangle $\mathrm{conv}\{p, x, z_0\} = \bigcup_{y \in [p,x]} [y, z_0]$ is contained in $\bigcup_0^k C_i$. In particular, the segment $[z_1, x]$ is contained in $(\bigcup_0^k C_i) \cap H = (\bigcup_1^k C_i) \cap H = \bigcup_1^k D_i$. This proves that $\bigcup_1^k D_i$ is starshaped, as we needed. □

Theorem 0.12 immediately implies the following strengthening of Helly's theorem, due (for closed sets) to M. Breen (1990).

Theorem 0.13. *Let $\mathcal{F}$ be a family of nonempty convex sets in $\mathbb{R}^d$ such that every subfamily of $\mathcal{F}$ consisting of $d+1$ or fewer sets has a stashaped union. Suppose that at least one of the following three conditions is satisfied:*

(a) *$\mathcal{F}$ is finite and its members are closed;*
(b) *$\mathcal{F}$ is finite and its members are open;*
(c) *the members of $\mathcal{F}$ are closed and at least one of them is compact.*

Then the intersection $\bigcap \mathcal{F}$ is nonempty.

Proof. Let n be the largest integer such that $n \leq d+1$ and any n elements of $\mathcal{F}$ have a point in common. Observe that $n \geq 2$ by connectedness. Now, $n = d+1$ since otherwise Theorem 0.12 would lead to a contradiction with the maximality of n. Apply Helly's theorem. □