A short account of topological vector spaces

Normed spaces, and especially Banach spaces, are basic ambient spaces in Infinite-Dimensional Analysis. However, there are situations in which it is necessary to use more general structures: the so-called topological vector spaces.

For example, the space $\mathcal{C}(\mathbb{R})$ of all continuous real-valued functions on $\mathbb{R}$ is not normable in an easy way, but there is a natural topology —the topology of uniform convergence on compact sets— on $\mathcal{C}(\mathbb{R})$, that is metrizable by a complete metric.

Another examples are the so-called weak topologies on a normed space. Infinite-dimensional normed spaces do not satisfy the compactness criterion valid in $\mathbb{R}^d$, since their closed balls are never compact. To have compacness of all closed balls, we need different, non-normable, topologies.

In this chapter, we give a minimum introduction to topological vector spaces. The interested reader is referred to standard text-books of Functional Analysis, like W. Rudin's *Functional Analysis*, for a more systematic study of such spaces.

Let us recall that we consider here only the *real* case, that is, the field of scalars is $\mathbb{R}$. Although complex spaces are very important, in Convex Analysis we use only real spaces (we need an ordered field of scalars to define convex functions!).

Basic definitions. A *topological vector space* (*t.v.s.*, for short) is a couple (X, τ), where X is a vector space and τ is a topology on X which agrees with the vector structure in the following way: the mappings

$$S\colon X \times X \to X \qquad \text{and} \qquad M\colon \mathbb{R} \times X \to X\,,$$

defined by

$$S(x,y) = x + y \qquad \text{and} \qquad M(t,x) = tx\,,$$

are continuous (when X is equipped with the topology τ).

It is easy to see that every normed space is a t.v.s.

Observation 0.1. *Let X be a t.v.s.*

(a) *For each $n \in \mathbb{N}$, the mapping $(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n) \mapsto \sum_{i=1}^n \alpha_i x_i$ is continuous on $X^n \times \mathbb{R}^n$.*
(b) *For each $v \in X$, the mapping $x \mapsto x + v$ is a homeomorphism of X.*
(c) *For each $\alpha \in \mathbb{R} \setminus \{0\}$, the mapping $x \mapsto \alpha x$ is a homeomorphism of X.*

By a *neighborhood* of a point $x \in X$ we mean any set $V \subset X$ such that $x \in \mathrm{int}(V)$. The family of all neighborhoods of x will be denoted by $\mathcal{U}(x)$. By Observation 0.1, we have

$$\mathcal{U}(x) = x + \mathcal{U}(0)\,.$$

Thus it suffices to know $\mathcal{U}(0)$ to know the neighborhoods of any point of X.

A *basis of neighborhoods* of 0 is a family $\mathcal{B} \subset \mathcal{U}(0)$ such that:

$$\forall U \in \mathcal{U}(0)\ \exists V \in \mathcal{B}:\ V \subset U\,.$$

Consequently, $\mathcal{U}(0) = \{U \subset X : U \supset V \text{ for some } V \in \mathcal{B}\}$.

For instance, the family of all open neighborhoods of 0 is a basis of neighborhoods of 0 in every t.v.s. Another example: if X is a normed space, then the family $\mathcal{B} = \{B^0(0, \frac{1}{n}) : n \in \mathbb{N}\}$ is a basis of neighborhoods of 0.

Definition 0.2. A set $A \subset X$ is said to be *balanced* if $tA \subset A$ whenever $|t| \leq 1$. (Equivalently, A is a balanced set if A is symmetric and starshaped w.r.t. the origin.)

Proposition 0.3. *Let X be a t.v.s.*

(a) *For each $U \in \mathcal{U}(0)$ there exists $V \in \mathcal{U}(0)$ such that $V + V \subset U$.* (This property substitutes the triangular inequality of the norm.)
(b) *X has a basis of neighborhoods of 0 whose all elements are open and balanced.*
(c) *If $\mathcal{B}$ is a basis of neighborhoods of 0, then $\overline{\mathcal{B}} = \{\overline{B} : B \in \mathcal{B}\}$ is a basis of neighborhoods of 0, too.*

Proof. (a) Since the mapping $(x, y) \mapsto x + y$ is continuous at $(0, 0) \in X \times X$, there exists $V \in \mathcal{U}(0)$ such that $x + y \in U$ whenever $(x, y) \in V \times V$.
(b) Fix $U \in \mathcal{U}(0)$. Since the mapping $(t, x) \mapsto tx$ is continuous at $(0, 0) \in \mathbb{R} \times X$, there exist $\delta > 0$ and $V \in \mathcal{U}(0)$ such that $tx \in U$ whenever $|t| \leq \delta$ and $x \in V$. We can suppose that V is open (otherwise substitute it with its interior). Then the set

$$B = \bigcup_{|t| \leq \delta} tV$$

is an open set (since $B = \bigcup_{0<|t|\leq\delta} tV$ is a union of open sets) which is balanced (*exercise!*), contains 0, and is contained in U.
(c) Given $U \in \mathcal{U}(0)$, let $V \in \mathcal{U}(0)$ be as in (a). Let $B \in \mathcal{B}$ be such that $B \subset V$. For $x \in \overline{V}$ there exists $v \in V$ such that $v \in x - V$ (since $x - V$ is a neighborhood of x!); hence $x \in v + V \subset V + V$. Consequently,

$$\overline{B} \subset \overline{V} \subset V + V \subset U .$$

□

Definition 0.4. A topological vector space X is said to be *locally convex* it it has a basis of neighborhoods of 0 whose all elements are convex.

Observation 0.5. *Every locally convex t.v.s. has a basis of neighborhoods of 0 made of open convex balanced sets.*

Proof. Given $U \in \mathcal{U}(0)$, let $V \subset U$ be a convex neighborhood of 0. We can suppose that V is open. Then $V \cap (-V)$ is a symmetric open convex neighborhood of 0, and hence it is also balanced. □

Continuity of linear functionals. Let X, Y be topological vector spaces, $E \subset X$. A mapping $F \colon E \to Y$ is *uniformly continuous* on E if

$$\forall V \in \mathcal{U}_Y(0)\ \exists U \in \mathcal{U}_X(0) : \quad F(x) - F(y) \in V \text{ whenever } x - y \in U.$$

Theorem 0.6. *Let X be a t.v.s., and $\ell \colon X \to \mathbb{R}$ a linear functional which is not identically zero. Then the following assertions are equivalent.*

(i) *ℓ is uniformly continuous on X.*
(ii) *ℓ is continuous.*
(iii) *ℓ is continuous at 0.*

(iv) ℓ *is continuous at some point.*
(v) ℓ *is bounded on a neighborhood of* 0.
(vi) ℓ *is upper bounded on a nonempty open set.*
(vii) $\mathrm{Ker}(\ell)$ *is closed.*
(viii) $\mathrm{Ker}(\ell)$ *is not dense in* X.

Proof. The following implications are obvious:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (vi)\,, \quad (iii) \Rightarrow (v) \Rightarrow (vi)\,, \quad (i) \Rightarrow (vii) \Rightarrow (viii)\,.$$

To complete the proof, it suffices to show the implications $(vi) \Rightarrow (v)$, $(v) \Rightarrow (i)$ and $(viii) \Rightarrow (vi)$.

$(vi) \Rightarrow (v)$. If (vi) holds, there exist $V \in \mathcal{U}(0)$, $x_0 \in X$ and a constant m such that $\ell \le m$ on $x_0 + V$. By Proposition 0.3, we can suppose that V is symmetric. Since $\ell \le m - \ell(x_0)$ on the symmetric set V, we have also $|\ell| \le m - \ell(x_0)$ on V.

$(v) \Rightarrow (i)$. Let $|\ell| < M$ on some $U \in \mathcal{U}(0)$. Given $\varepsilon > 0$, we have $V := \frac{\varepsilon}{M} U \in \mathcal{U}(0)$ and, if $x - y \in V$ then $\frac{M}{\varepsilon}(x-y) \in U$ and

$$\bigl|\ell(x) - \ell(y)\bigr| = \tfrac{\varepsilon}{M}\bigl|\ell\bigl(\tfrac{M}{\varepsilon}(x-y)\bigr)\bigr| < \tfrac{\varepsilon}{M} M = \varepsilon\,.$$

$(viii) \Rightarrow (vi)$. If (viii) holds, there exist $x_0 \in X$ and a neighborhood W of x_0 such that $W \cap \mathrm{Ker}(\ell) = \emptyset$. By Proposition 0.3, we can suppose that W is starshaped w.r.t. x_0. For each $y \in W$, the segment $[x_0, y]$ is contained in W and its image $\ell([x_0, y])$ is a compact segment (in $\mathbb{R}$) that does not contain 0. It follows that $\ell(y)$ and $\ell(x_0)$ have the same sign. Thus either $\ell < 0$ or $\ell > 0$ on W. Consequently, ℓ is upper bounded either on W or on $-W$. □

Finite-dimensional topological vector spaces.

Two topological vector spaces X and Y are *isomorphic* if there exists a linear homeomorphism of X onto Y. The following results, together with their proofs, are analogous to the case of finite-dimensional *normed* spaces. In many of them, the assumption that the space is Hausdorff cannot be omitted.

Theorem 0.7. *Any two Hausdorff topological vector spaces of the same finite dimension are isomorphic.*

Proof. Let us show that every d-dimensional t.v.s. X is isomorphic to $\mathbb{R}^d$ (equipped with the Euclidean norm $|\cdot|_e$). Fix a basis $\{v_1, \ldots, v_d\}$ of X and consider the algebraic isomorphism

$$T\colon \mathbb{R}^d \to X\,, \quad T(\alpha_1, \ldots, \alpha_d) = \sum_{i=1}^{d} \alpha_i v_i\,.$$

Observe that T is continuous by Observation 0.1(a). To show that also its inverse is continuous, consider the sets

$$S = \{\alpha \in \mathbb{R}^d : |\alpha|_e = 1\} \quad \text{and} \quad \Sigma = T(S).$$

Then Σ is compact in X since S is compact in $\mathbb{R}^d$. Moreover, Σ is closed (since X is Hausdorff) and $0 \notin \Sigma$. By Proposition 0.3, there exists an open balanced $V \in \mathcal{U}(0)$

such that $V \cap \Sigma = \emptyset$. The fact that V is connected and contains 0 easily implies that $|T^{-1}x|_e < 1$ whenever $x \in V$. Now, given $\varepsilon > 0$, we have $|T^{-1}x|_e < \varepsilon$ whenever $x \in \varepsilon V$. Consequently, T^{-1} is continuous at 0, and hence continuous on X. □

Corollary 0.8. *Let X be a finite-dimensional Hausdorff t.v.s., and Y an arbitrary t.v.s. Then each linear mapping $T\colon X \to Y$ is continuous.*

Proof. By Theorem 0.7, we can suppose that $X = \mathbb{R}^d$. Let e_i be the i-th vector of the canonical orthonormal basis of $\mathbb{R}^d$. Then, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, we have

$$T(\alpha_1, \ldots, \alpha_d) = \sum_{i=1}^{d} \alpha_i \, Te_i \,,$$

and the continuity of T follows from Observation 0.1(a) (applied to Y). □

It is easy to see that $\bigcap \mathcal{B} = \{0\}$ whenever $\mathcal{B}$ is a basis of neighborhoods of 0 in a Hausdorff t.v.s.

Corollary 0.9. *Every finite-dimensional subspace of a Hausdorff t.v.s. X is closed in X.*

Proof. Let $Y \subset X$ be a d-dimensional subspace, and $x_0 \in \overline{Y}$. Let $T\colon Y \to \mathbb{R}^d$ be an isomorphism (Theorem 0.7). Let B be the closed unit ball of $\mathbb{R}^d$ (e.g., in the Euclidean norm). The set $T^{-1}(B)$ is compact, hence closed in Y, and it is a neighborhood of 0 in Y. Let V be an open neighborhood of 0 such that $V \cap Y \subset T^{-1}(B)$. There exists $\lambda > 0$ such that $x_0 \in \lambda V$ (indeed, by continuity of $t \mapsto tx_0$ at 0, we have that $tx_0 \in V$ whenever t is sufficiently small). Then $\lambda V \cap Y = \lambda(V \cap Y) \subset \lambda T^{-1}(B)$ (where the last set is closed in Y) implies that

$$x_0 \in \overline{\lambda V \cap Y} \subset \lambda T^{-1}(B) \subset Y \,.$$

□

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For general, possibly non-Hausdorff, topological vector spaces, we have the following theorem which we state without proof; see Section 7, Problem A in [J.L. Kelley and I. Namioka, Linear Topological Spaces]. Recall that a topological space T is called *topologically trivial* if the only nonempty open set in T is T itself.

Theorem 0.10. *Let X be a t.v.s. of a finite dimension d. Then X is isomorphic to the Cartesian product $\mathbb{R}^m \times V$, where $m \leq d$ and V is a topologically trivial t.v.s. (of dimension $d - m$).*

Let us conclude this chapter with the following important theorem which will not be needed in the sequel.

Theorem 0.11. *A Hausdorff topological vector space is finite-dimensional if and only if it contains a compact neighborhood of* 0.

Proof. The implication "⇒" follows from Theorem 0.7 and from the fact that the closed unit ball of $\mathbb{R}^d$ is compact. For a proof of the reverse implication we refer the reader to Theorem 1.22 in [W. Rudin, Functional Analysis]. □

Remark 0.12. Using Theorem 0.10, it is easy to show that *each finite-dimensional t.v.s. contains a compact neighborhood of* 0. The converse is not true for non-Hausdorff spaces; as an example, consider any infinite-dimensional t.v.s. which is topologically trivial (hence compact).