1. Continuity of convex functions in normed spaces

In this chapter, we consider continuity properties of real-valued convex functions defined on open convex sets in normed spaces. Recall that every infinite-dimensional normed space contains a discontinuous linear functional. Thus, in infinite-dimensional spaces, there exist discontinuous convex functions. One of the corollaries of the results of this chapter is that, in finite-dimensional spaces, this cannot happen.

Recall that $B(x,r)$ and $B^0(x,r)$ denote the open and closed ball of radius r, centered at x. A function $f\colon E \to \mathbb{R}$ is L-Lipschitz if $|f(x)-f(y)| \le L\|x-y\|$ whenever $x,y \in E$.

P:cont **Proposition 1.1.** *Let X be a normed space, $x_0 \in X$, $r>0$, $\varepsilon \in (0,r)$, $m, M \in \mathbb{R}$. Let $f\colon B^0(x_0,r) \to \mathbb{R}$ be a convex function.*

(a) *If $f(x) \le m$ on $B^0(x_0,r)$, then $|f(x)| \le |m| + 2|f(x_0)|$ on $B^0(x_0,r)$.*
(b) *If $|f(x)| \le M$ on $B^0(x_0,r)$, then f is $\left(\frac{2M}{\varepsilon}\right)$-Lipschitz on $B^0(x_0,r-\varepsilon)$.*

Proof. By translation, we can suppose that $x_0 = 0$. Denote $B = B^0(0,r)$ and $C = B^0(0,r-\varepsilon)$.
(a) Since $0 = \frac{1}{2}x + \frac{1}{2}(-x)$ $(x \in B)$, we have $f(0) \le \frac{1}{2}f(x) + \frac{1}{2}f(-x)$. Consequently, $f(x) \ge 2f(0) - f(-x) \ge 2f(0) - m$, and hence

$$|f(x)| \le \max\{m, m-2f(0)\} \le |m| + 2|f(0)| \qquad (x \in B)\,.$$

(b) Consider two distinct points $x,y \in C$. The point

$$z = y + \tfrac{\varepsilon}{\|y-x\|}(y-x)$$

belongs to B and $y \in (x,z)$. An easy calculation shows that

$$y = \tfrac{\varepsilon}{\varepsilon+\|y-x\|}x + \tfrac{\|y-x\|}{\varepsilon+\|y-x\|}z \qquad \text{(convex combination!)}.$$

Use convexity of f and multiply by the common denominator to get

$$(\varepsilon + \|y-x\|)f(y) \le \varepsilon f(x) + \|y-x\| f(z)\,.$$

Then $\varepsilon\big[f(y)-f(x)\big] \le \big[f(z)-f(y)\big]\|y-x\| \le 2M\|y-x\|$. Thus

$$f(y) - f(x) \le \tfrac{2M}{\varepsilon}\|y-x\|\,.$$

Interchanging the role of x and y, we obtain that f is $(\frac{2M}{\varepsilon})$-Lipschitz on C. □

obs **Observation 1.2.** *Let C be a convex set in a normed space X, $f\colon C \to \mathbb{R}$ a convex function, $B := B^0(x_0,r) \subset C$. Let $x,y \in C$ be such that $x = (1-\lambda)x_0 + \lambda y$ with $0<\lambda<1$. If $f \le m$ on B, then*

$$f \le \max\{m, f(y)\} \quad \textit{on} \ \operatorname{conv}\big[B \cup \{y\}\big]$$

(in particular, on $B^0(x,(1-\lambda)r)$).

Proof. Exercise. □

continuity **Theorem 1.3.** *Let C be an open convex set in a normed space X, and $f\colon C \to \mathbb{R}$ a convex function. The following assertions are equivalent:*

(i) *f is locally Lipschitz on C;*
(ii) *f is continuous on C;*
(iii) *f is continuous at some point of C;*
(iv) *f is locally bounded on C;*
(v) *f is upper bounded on a nonempty open subset of C.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (v)$ and $(ii) \Rightarrow (iv) \Rightarrow (v)$ are obvious. It remains to show that (v) implies (i).

By (v), there exists an open ball $B^0(x_0, r) \subset C$ on which f is upper bounded. Let $x \in C$. There exists $y \in C$ such that $x \in (x_0, y)$. By Observation 1.2, f is uper bounded on some ball $B^0(x, \varrho)$. By Proposition 1.1, f is Lipschitz on $B^0(x, \frac{\varrho}{2})$. □

As an easy corollary, we obtain the following result on automatic continuity of convex functions in finite-dimensional spaces.

contRd **Corollary 1.4.** *Each convex function on an open convex subset of $\mathbb{R}^d$ is locally Lipschitz (hence continuous).*

Proof. Let $C \subset \mathbb{R}^d$ be open and convex, and $f\colon C \to \mathbb{R}$ a convex function. Fix $x_0 \in C$. There exist finitely many points $c_1, \ldots, c_n \in C_0$ such that $x_0 \in U := \mathrm{int}[\mathrm{conv}\{c_1, \ldots, c_n\}]$ (take, e.g., the vertices of a small d-dimensional cube centered at x_0). By convexity, $f \le \max\{f(c_1), \ldots, f(c_n)\}$ on U. By Theorem 1.3, f is locally Lipschitz on U. □

Corollary 1.5. *Let C be a finite-dimensional convex set in a normed space X. Then every convex function $f\colon C \to \mathbb{R}$ is continuous on* $\mathrm{ri}(C)$ *(the relative interior of C).*

Proof. Ecercise. (Hint: use Corollary 1.4.) □

Continuity of semicontinuous convex functions.

Let M be a topological space, $x_0 \in M$. Recall that a function $f\colon M \to \overline{\mathbb{R}}$ is:

- *lower semicontinuous* (*l.s.c.*) at x_0 if $\forall t \in (-\infty, f(x_0))\ \exists U \in \mathcal{U}(x_0)$: $f(x) > t$ whenever $x \in U$.
- *upper semicontinuous* (*u.s.c.*) at x_0 if $\forall t \in (f(x_0), +\infty)\ \exists U \in \mathcal{U}(x_0)$: $f(x) < t$ whenever $x \in U$.

Clearly, f is u.s.c. at x_0 if and only if $-f$ is l.s.c. at x_0.

charlsc **Observation 1.6.** *Let M, f be as above. Then the following assertions are equivalent:*

(i) *f is l.s.c.;*
(ii) *for each $t \in \mathbb{R}$, the set $\{f > t\}$ is open;*
(iii) *for each $t \in \mathbb{R}$, the set $\{f \le t\}$ is closed;*
(iv) *the epigraph* $\mathrm{epi}(f) := \{(x,t) \in M \times \mathbb{R} : f(x) \le t\}$ *is closed (in $M \times \mathbb{R}$).*

Proof. Exercise. □

Notice that (iv) above immediately implies that a pointwise supremum of any family of l.s.c. functions is l.s.c.

lsc **Proposition 1.7.** *Let C be an open convex set in a normed space X, $f\colon C \to \mathbb{R}$ a convex function.*

(a) *If f is u.s.c., then f is continuous on C.*
(b) *If X is a Banach space and f is l.s.c., then f is continuous on C.*

Proof. (a) Fix $x_0 \in C$ and $t > f(x_0)$. Then the set $\{x \in C : f(x) < t\}$ is a nonempty open subset of C, on which f is bounded above. Apply Theorem 1.3.
(b) If $C = X$, put $F_n = \{x \in C : f(x) \leq n\}$; otherwise define

$$F_n = \{x \in C : \ f(x) \leq n, \ \mathrm{dist}(x, X \setminus C) \geq \tfrac{1}{n}\}\,.$$

The sets F_n ($n \in \mathbb{N}$) are closed in C; but they are also closed in X since $\overline{F_n} \subset C$. By the Baire Category Theorem, there exists $k \in \mathbb{N}$ such that F_k has a nonempty interior. This implies that f is upper bounded on a nonempty open set. Apply Theorem 1.3. □

Example 1.8. Let us show that the completeness assumption in Proposition 1.7 cannot be omitted. Consider the (incomplete) normed space $X = c_{00}$, the space of all real sequences $x = (x_n)_{n\in\mathbb{N}}$ that are eventually null, equipped with the supremum norm. The convex function $f\colon c_{00} \to \mathbb{R}$, given by $f(x) = \sum_1^\infty |x_n| = \sup_N \sum_1^N |x_n|$, is convex and l.s.c. (by the remark after Observation 1.6), but unbounded in any neighborhood of 0. Hence f is everywhere discontinuous.

Families of convex functions. Let $\mathcal{F}$ be a family of fuctions on a set E. We say that $\mathcal{F}$ is *pointwise bounded* if, for each $x \in E$, the set $\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\}$ is bounded (in $\mathbb{R}$).

BS

Theorem 1.9. *Let C be an open convex set in a Banach space X. Let $\mathcal{F}$ be a family of continuous convex functions on C. If $\mathcal{F}$ is pointwise bounded, then $\mathcal{F}$ is locally equi-Lipschitz and locally equi-bounded.*

Proof. The (real-valued!) function

$$g(x) = \sup \mathcal{F}(x) = \sup_{f\in\mathcal{F}} f(x)$$

is easily seen to be l.s.c. (e.g., by Observation 1.6). By Proposition 1.7, g is continuous on C; in particular, g is locally bounded. Hence $\mathcal{F}$ is locally equi-bounded above on C. The rest follows from Proposition 1.1. □

convergence

Theorem 1.10. *Let C be an open convex set in a Banach space X. Let $\{f_n\}$ be a sequence of continuous convex functions on C that converges pointwise on C to a (convex) function $f\colon C \to \mathbb{R}$. Then f is continuous and the convergence is uniform on compact sets.*

Proof. The sequence $\{f_n\}$ is pointwise bounded, hence, by the previous theorem, locally equi-bounded and equi-Lipschitz. Consequently, f is locally bounded and hence continuous (Theorem 1.3). Moreover, on each compact set $K \subset C$, the restrictions $f_n|_K$ ($n \in \mathbb{N}$) are equi-bounded and equi-Lipschitz. An easy application of the Ascoli-Arzelà theorem (*exercise!*) gives that they converge uniformly on K. □

Let us recall the so-called *diagonal method*, a standard argument used in many areas of Mathematical Analysis. Given a sequence $\sigma = (\sigma(1), \sigma(2), \ldots)$ of elements of a set S, and $m \in \mathbb{N}$, we denote by $\sigma|_{[m,\infty)}$ the sequence $(\sigma(m), \sigma(m+1), \ldots)$.

diagonal

Lemma 1.11 (Diagonal method). *Let S be a set, and σ_n ($n \in \mathbb{N}$) countably many sequences of elements of S such that σ_{n+1} is a subsequence of σ_n for each $n \in \mathbb{N}$. Then there exists a sequence σ_∞ of elements of S, such that, for each $n \in \mathbb{N}$,*

$$\sigma_\infty|_{[n,\infty)} \text{ is a subsequence of } \sigma_n.$$

Proof. The "diagonal" sequence $\sigma_\infty = (\sigma_1(1), \sigma_2(2), \ldots)$ has the desired property. □

subsequence

Theorem 1.12. *Let C be an open convex set in a separable Banach space X. Let $\{f_n\}$ be a pointwise bounded sequence of continuous convex functions on C. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges pointwise and uniformly on compact sets to a continuous convex function on C.*

Proof. Fix a countable dense set $D \subset C$. Since $\{f_n\}$ is pointwise bounded, an easy application of diagonal method produces a subsequence $\{f_{n_k}\}$ that converges at each point of D. Let us show that this subsequence converges pointwise on C. Given $x \in C$, there exists an open ball $U \subset C$, centered in x, on which $\{f_n\}$ is equi-Lipschitz with a certain Lipschitz constant $L > 0$. Choose $d \in U$ such that $\|d - x\| < \frac{1}{L}$. Given $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $|f_{n_k}(d) - f_{n_j}(d)| < \varepsilon$ whenever $k, j \geq k_0$. Then, for such indices, we have $|f_{n_k}(x) - f_{n_j}(x)| \leq |f_{n_k}(x) - f_{n_k}(d)| + |f_{n_k}(d) - f_{n_j}(d)| + |f_{n_j}(d) - f_{n_j}(x)| < L\|x - d\| + \varepsilon + L\|x - d\| < 3\varepsilon$. It follows that $\{f_{n_k}(x)\}$ is Cauchy and hence convergent. The rest follows from Theorem 1.10. □

Direct applications to linear mappings and functionals. Let us show that some significant results of Functional Analysis are easy consequences of the above theorems for families of convex functions. While usual proofs of the Banach-Steinhaus theorem (Corollary 1.13) go in a similar way as Theorem 1.9, the "subsequence theorem" Corollary 1.15 is usually proved using compactness of B_{X^*} (the dual unit ball) in the w^*-topology (Alaoglu's theorem) and its metrizability.

Let X be a normed space, and $x \in X$. Recall the following well-known consequence of the Hahn-Banach theorem

$$\|x\| = \sup\{x^*(x) : \ x^* \in X^*, \ \|x^*\| = 1\}.$$

UBP

Corollary 1.13 (Banach-Steinhaus Uniform Boundedness Principle). *Let X be a Banach space, Y a normed space. Let $\mathcal{T}$ be a family of coninuous linear mappings from X into Y. Suppose that $\mathcal{T}$ is pointwise bounded, that is, for each $x \in X$, the set $\{Tx : T \in \mathcal{T}\}$ is bounded in Y. Then the family $\mathcal{T}$ is bounded in the normed space $\mathcal{L}(X,Y)$ of all continuous linear mappings from X into Y.*

Proof. The family $\{y^* \circ T : \ T \in \mathcal{T}, \ y^* \in Y^*, \ \|y^*\| = 1\}$ is a pointwise bounded family of continuous linear functionals on X; indeed, $|(y^* \circ T)(x)| \leq \|y^*\| \, \|Tx\| = \|Tx\|$ ($x \in X$). By Theorem 1.9, there exist $r > 0$ and $M \in \mathbb{R}$ such that

$$y^*(Tx) = (y^* \circ T)(x) \leq M \qquad (\, x \in B(0,r), \ \|y^*\| = 1, \ T \in \mathcal{T}\,).$$

Passing to supremum w.r.t. y^*, we obtain

$$\|Tx\| \le M \qquad (\, x \in B(0,r),\ T \in \mathcal{T}\,),$$

which easily implies that $\|T\| \le \frac{M}{r}$ for each $T \in \mathcal{T}$. □

subs **Corollary 1.14.** *Let X be a Banach space, and $\{x_n^*\} \subset X^*$ a sequence that converge pointwise on X to a (linear) functional $\ell\colon X \to \mathbb{R}$. Then $\{x_n^*\}$ is bounded, $\ell \in X^*$, and the convergence is uniform on compact sets.*

Proof. By Corollary 1.13, $\{\|x_n^*\|\}$ is bounded, say, by a constant M. This easily implies that $\|\ell\| \le M$. The last part follows by Theorem 1.10. □

Let us recall that a sequence $\{x_n^*\} \subset X^*$ is *w^*-convergent* if there exists $x^* \in X^*$ such that $x_n^* \to x^*$ pointwise on X.

Corollary 1.15. *Let X be a separable Banach space. Then every pointwise bounded sequence $\{x_n^*\} \subset X^*$ is bounded and admits a w^*-convergent subsequence.*

Proof. Apply Theorem 1.9 and Theorem 1.12. □

Remarks on boundary points. Let X be a normed space, C a convex set with $\operatorname{int}(C) \neq \emptyset$, and $f\colon C \to \mathbb{R}$ a convex function which is continuous on $\operatorname{int}(C)$. The following example shows that, even for finite-dimensional X, the behaviour of f at boundary points of C can be very bad.

Example 1.16. Let $|\cdot|_e$ denote the Euclidean norm on $X = \mathbb{R}^d$, $C = \{x \in \mathbb{R}^d : |x|_e \le 1\}$. Then each function $f\colon C \to [0,+\infty)$ such that $f|_{\operatorname{int}(C)} = 0$ is convex on C.

2. Continuity of convex functions in topological vector spaces

In this additional section, we are going to show that many of the previous results can be extended to the setting of (real) topological vector spaces (t.v.s.) by adapting the proofs for normed spaces.

We shall use the following, not very standard terminology. Let X be a t.v.s., $A \subset X$, $f\colon A \to \mathbb{R}$. Given $V \in \mathcal{U}(0)$ (the family of neighborhoods of 0) and a real number $L \ge 0$, we shall say that f is *L-V-lipschitzian* if

$$|f(x) - f(y)| \le L\alpha \quad \text{whenever } \alpha \in (0,1),\ x,y \in A,\ x - y \in \alpha V.$$

We shall say that f is V-lipschitzian if it is L-V-lipschitzian for some $L \ge 0$.

Let us recall that *each t.v.s. admits a basis of neighborhoods of* 0 *whose members are balanced, that is (in real spaces), symmetric and starshaped w.r.t.* 0.

P:cont2 **Proposition 2.1.** *Let X be a t.v.s., $U \in \mathcal{U}(0)$, $f\colon U \to \mathbb{R}$ a convex function.*

(a) *If U is symmetric and $f(x) \le m$ on U, then $|f(x)| \le |m| + 2|f(0)|$ on U.*
(b) *If $|f(x)| \le M$ on U, and $V \in \mathcal{U}(0)$ is balanced and such that $V+V+V \subset U$, then f is $2M$-V-lipschitzian on V.*

Proof. The proof of (a) is the same as in Proposition 1.1(a). Let us show (b).

Let $x, y \in V$, $\alpha \in (0,1)$, $x - y \in \alpha V$. Denote $\delta = 1/\alpha$ and consider the point $z = x + (1+\delta)(y-x)$. Then

$$z \in V + (1+\delta)\alpha V \subset V + \alpha V + \delta\alpha V \subset V + V + V \subset U$$

and $y = \frac{\delta}{1+\delta}x + \frac{1}{1+\delta}z$. Thus

$$f(y) \le \tfrac{\delta}{1+\delta} f(x) + \tfrac{1}{1+\delta} f(z)\,,$$

which easily implies

$$\delta\big[f(y) - f(x)\big] \le f(z) - f(y) \le 2M\,.$$

Consequently, $f(y) - f(x) \le 2M\alpha$. The assertion now follows by interchanging the roles of x and y. □

obs2 **Observation 2.2.** *Let C be a convex set in a normed space X, $f\colon C \to \mathbb{R}$ a convex function, $V \in \mathcal{U}(0)$, $x_0 + V \subset C$. Let $x, y \in C$ be such that $x = (1-\lambda)x_0 + \lambda y$ with $0 < \lambda < 1$. If $f \le m$ on $x_0 + V$, then*

$$f \le \max\{m, f(y)\} \quad \text{on the set } \textstyle\bigcup_{b \in x_0 + V} [b, y]$$

(in particular, on $x + (1-\lambda)V$).

Proof. Exercise. □

continuity2 **Theorem 2.3.** *Let C be an open convex set in a t.v.s. X, and $f\colon C \to \mathbb{R}$ a convex function. The following assertions are equivalent:*

(i) *for each $x \in C$ there exists a balanced $V \in \mathcal{U}(0)$ such that $x + V \subset C$ and f is L-V-lipschitzian on $x + V$;*
(ii) *f is continuous on C;*
(iii) *f is continuous at some point of C;*
(iv) *f is locally bounded on C;*
(v) *f is upper bounded on a nonempty open subset of C.*

Proof. Proceed as in Theorem 1.3, using Observation 2.2 and Proposition 2.1. □

Barrelled spaces and continuity of semicontinuous convex functions. For continuity of lower semicontinuous convex functions, we need a substitute for the Baire Category Theorem. For this purpose, the following notion is useful.

A set B in a t.v.s. is a *barrel* if it is closed, convex, balanced and absorbing. It is easy to see that (in a t.v.s. over $\mathbb{R}$) this is equivalent to say that B is closed, convex, symmetric, and $0 \in \text{a-int}\,B$ (or equivalently, $\text{a-int}\,B \ne \emptyset$).

Definition 2.4. A t.v.s. X is said to be *barrelled* if every barrel $B \subset X$ is a neighborhood of 0.

Observe that X is barrelled if and only if the following implication holds for $B \subset X$:

$$B \text{ is closed, convex, symmetric, a-int}\,B \ne \emptyset \quad\Rightarrow\quad \text{int}\,B \ne \emptyset\,.$$

From the point of view of Convex Analysis, the following property is important.

barrel **Lemma 2.5.** *Let C be a closed convex set in a barrelled t.v.s. Then*

$$\text{a-int}\, C = \text{int}\, C\,.$$

Proof. Let $x \in \text{a-int}\, C$. We can (and do) suppose that $x = 0$. Since $B := C \cap (-C)$ is a barrel, we have $0 \in \text{int}\, C$. □

To be able to prove Proposition 2.9(b) below for non-Hausdorff spaces, we shall need a bit preparation. A topology on a vector space X is called a *vector topology* if it makes X a t.v.s.

R **Observation 2.6.** *The only vector topologies on $\mathbb{R}$ are the Euclidean topology τ_e and the trivial topology τ_0.*

Proof. Let τ be a vector topology on $\mathbb{R}$. Since the set $L := \overline{\{0\}}^{\tau}$, the τ-closure of the origin, is a subspace of $\mathbb{R}$, we must have either $L = \{0\}$ or $L = \mathbb{R}$. In the first case, it easily follows that the topology τ is Hausdorff, which is known to be equivalent to the fact that $\tau = \tau_e$. If $L = \mathbb{R}$, the unique nonempty τ-closed set is $\mathbb{R}$ itself; hence $\tau = \tau_0$. □

conR **Corollary 2.7.** *Let τ be a vector topology on $\mathbb{R}$, $I \subset \mathbb{R}$ a τ-open interval, and $f\colon I \to \mathbb{R}$ a τ-l.s.c. convex function. Then f is τ-continuous.*

Proof. By Observation 2.6, we have only two possibilities. If $\tau = \tau_e$, the assertion follows from Corollary 1.4. Let $\tau = \tau_0$. Then each (τ-closed in I) set $\{f \le t\}$ (with $t \in \mathbb{R}$) equals either I or $\emptyset$. This easily implies that f is constant, hence τ-continuous. □

Exercise 2.8. Consider an integer $d > 1$.

(a) Show that for $d > 1$ there are nontrivial non-Euclidean vector topologies on $\mathbb{R}^d$.
(b) Show that every vector topology on $\mathbb{R}^d$ is weaker than (or equal to) the Euclidean topology.
(c)* Try to characterize all vector topologies on $\mathbb{R}^d$.

lsc2 **Proposition 2.9.** *Let C be an open convex set in a t.v.s. X, $f\colon C \to \mathbb{R}$ a convex function.*

(a) *If f is u.s.c., then f is continuous.*
(b) *If X is barrelled and f is l.s.c., then f is continuous.*

Proof. The proof of (a) is identical to that of Proposition 1.7. Let us show (b). Assume that $0 \in C$, and consider the convex set

$$D = \{x \in \tfrac{1}{2}\overline{C} : f(x) \le f(0) + 1\}\,.$$

This set is closed in X. We claim that $0 \in \text{a-int}\, D$; this follows from the fact that, for each line L through 0, $f|_L$ is continuous on $C \cap L$ by Corollary 2.7. By Lemma 2.5, $0 \in \text{int}\, D$. Apply Theorem 2.3. □

Families of convex functions in barrelled spaces.
Now we can state the following three theorems about families of continuous convex functions in barrelled topological vector spaces, which can be proved in the same way as Theorems 1.9, 1.10 and 1.12, simply by using Proposition 2.1, Theorem 2.3 and Proposition 2.9 instead of Proposition 1.1, Theorem 1.3 and Proposition 1.7, respectively. The proofs are left to the reader as an exercise.

BS2 **Theorem 2.10.** *Let C be an open convex set in a barrelled t.v.s. X. Let $\mathcal{F}$ be a family of continuous convex functions on C. If $\mathcal{F}$ is pointwise bounded, then $\mathcal{F}$ is locally equi-bounded, and locally equi-lipschitzian in the following sense: for each $x \in C$ there exist a balanced $V_x \in \mathcal{U}(0)$ and $L_x \geq 0$ so that each element of $\mathcal{F}$ is L_x-V_x-lipschitzian on $x + V_x$.*

convergence2 **Theorem 2.11.** *Let C be an open convex set in a barrelled t.v.s. X. Let $\{f_n\}$ be a sequence of continuous convex functions on C that converges pointwise on C to a (convex) function $f\colon C \to \mathbb{R}$. Then f is continuous and the convergence is uniform on compact sets.*

subsequence2 **Theorem 2.12.** *Let C be an open convex set in a separable barrelled t.v.s. X. Let $\{f_n\}$ be a pointwise bounded sequence of continuous convex functions on C. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that converges pointwise and uniformly on compact sets to a continuous convex function on C.*