Exercises on Helly's Intersection Theorem

Helly's Intersection Theorem, whose one version states that *each* $(d+1)$*-centered family of compact convex sets in* $\mathbb{R}^d$ *has nonempty intersection*, has numerous applications. We collect here some of them as exercises for the reader. First four of them are relatively easy, while the last two (marked with an asterisk) are more challenging.

Exercise 0.1. *Let* $1 \leq k \leq d+1$ *be integers, and* $\mathcal{F}$ *a* k*-centered family of convex sets in* $\mathbb{R}^d$ *such that either* $\mathcal{F}$ *is finite, or the elements of* $\mathcal{F}$ *are all closed and at least one of them compact. Let* $Y \subset \mathbb{R}^d$ *be a subspace of dimension at least* $d+1-k$*. Then there exists a translate of* Y *that intersects each member of* $\mathcal{F}$*.*

Exercise 0.2. *Let* $C \subset \mathbb{R}^d$ *be a convex set, and* $\mathcal{H}$ *a family of halfspaces in* $\mathbb{R}^d$ *such that* $C \subset \bigcup \mathcal{H}$*. Assume that either* $\mathcal{H}$ *is finite, or* C *is compact and all members of* $\mathcal{H}$ *are open. Then there exists a subfamily* $\mathcal{H}_0 \subset \mathcal{H}$ *of cardinality at most* $d+1$ *such that* $C \subset \bigcup \mathcal{H}_0$*.*

Exercise 0.3. *Let* $C \subset \mathbb{R}^d$ *be a compact convex set. There exists* $x \in \mathbb{R}^d$ *such that* $x - \frac{1}{d}C \subset C$*.*

Exercise 0.4.

(A) *Let* E *be a finite set,* $\varepsilon \geq 0$*,* $m \in \mathbb{N}$*, and* $f_1, \dots, f_m, g$ *real-valued functions on* E*. Assume that for each* $E_0 \subset E$ *of cardinality at most* $m+1$ *there exists* $\varphi \in \operatorname{span}\{f_1, \dots, f_m\}$ *such that*

$$|\varphi(x) - g(x)| \leq \varepsilon \quad \text{for each } x \in E_0.$$

Then there exists $\bar{\varphi} \in \operatorname{span}\{f_1, \dots, f_m\}$ *such that*

$$|\bar{\varphi}(x) - g(x)| \leq \varepsilon \quad \text{for each } x \in E.$$

(B)* *By adding appropriate assumptions on* E *and on the functions* $f_1, \dots, f_m, g$*, formulate and prove a variant of* (A) *for infinite* E*.*

Exercise* 0.5. *Let* $\mathcal{F}$ *be a finite* d*-centered family of compact convex sets in* $\mathbb{R}^d$*. Given* $x_0 \in \mathbb{R}^d$*, there exists a line through* x_0 *that intersects each member of* $\mathcal{F}$*.*
(For a proof see the book [Lay, *Convex Sets and their Applications*].)

Exercise* 0.6 (Krasnoselsky)**.** *Let* $K \subset \mathbb{R}^d$ *be an infinite compact set. Assume that each* $d+1$ *points of* K *are visible (in* K*) from some point of* K*, that is:*

$$\forall\, x_0, \dots, x_d \in K \;\exists\, y \in K : \; [x_i, y] \subset K, \; i = 0, \dots, k.$$

Then K *is star-shaped.*
(For a proof see the book [Lay, *Convex Sets and their Applications*].)