In this chapter, we are going to show that compact convex sets are determined by a proper subset, the set of its extreme points. Let us start with the main definition.

Definition 0.1. Let E be a set in a vector space X. We say that a point $x \in E$ is an *extreme point* of E, and write $x \in \mathrm{ext}(E)$, if x cannot belong to the relative interior of any nondegenerate line segment with endpoints in E. That is,

$$\forall\, y, z \in E,\ y \neq z : \quad x \notin (y, z)\,.$$

Observation 0.2. *Let C be a convex set in a vector space X, and $x \in C$. Then $x \in \mathrm{ext}(C)$ if and only if the following implication holds:*

$$y, z \in C,\ x = \tfrac{y+z}{2} \implies x = y = z\,.$$

Recall that $X^\sharp$ denotes the algebraic dual of a vector space X. The following simple lemma shows one of the main properties of extreme points. A hyperplane $H \subset X$ is a *support hyperplane* of a convex set $C \subset X$ at a point $x_0 \in C$ if $x_0 \in H$ and C is contained in one of the two algebraically closed halfspaces determined by H. This is equivalent to say that H is of the form $H = \ell^{-1}(\alpha)$ where $\ell \in X^\sharp \setminus \{0\}$, $\ell(x_0) = \sup \ell(C) = \alpha$. By a *support hyperplane of* C we mean a support hyperplane of C at some point of C.

Lemma 0.3. *Let C be a convex set in a vector space X, and $H \subset X$ a support hyperplane of C. Then*

$$\mathrm{ext}(C \cap H) = \mathrm{ext}(C) \cap H\,.$$

Proof. The inclusion "$\supset$" follows directly from the definition of extreme point. Let us show the reverse inclusion. Assume that $x_0 \in \mathrm{ext}(C \cap H)$, $x_0 = \frac{y+z}{2}$ with $y, z \in C$. Write $H = \ell^{-1}(\alpha)$ where $\ell \in X^\sharp \setminus \{0\}$, $\ell(x_0) = \sup \ell(C) = \alpha$. Since $\ell(y) \leq \alpha$, $\ell(z) \leq \alpha$ and $\frac{\ell(y)+\ell(z)}{2} = \alpha$, we must have $\ell(y) = \ell(z) = \alpha$, and hence $y, z \in C \cap H$. Consequently, $x_0 = y = z$. □

Extreme points of finite-dimensional compact convex sets.

Theorem 0.4 (Minkowski)**.** *Let K be a finite-dimensional compact convex set in some t.v.s. Then*

$$K = \mathrm{conv}[\mathrm{ext}(K)]\,.$$

Proof. Let us proceed by induction with respect to the dimension of K. The case of $\dim(K) = 0$ is trivial. Now, assume that our theorem holds for all compact convex sets of dimension less or equal to m. Let K be a compact convex set of dimension $m+1$. By translation, we can suppose that $0 \in K$. In this case, $L := \mathrm{span}(K) = \mathrm{aff}(K)$ has dimension $m+1$ and $\mathrm{int}_L(K) \neq \emptyset$ by the Relative Interior Theorem. For a point $x_0 \in K$, we have two possibilities (the boundary and the interior are considered in L).

(a) $x_0 \in \partial K$. By the H-B Separation Theorem, there exists a support hyperplane $H \subset L$ of C at x_0. Then $K_1 := K \cap H$ is a compact convex set of dimension at most m. By our assumption, $x_0 \in \mathrm{conv}[\mathrm{ext}(K_1)]$, but the last set is contained in $\mathrm{conv}[\mathrm{ext}(K)]$ by Lemma 0.3.

(b) $x_0 \in \mathrm{int}(K)$. In this case, there exist $y, z \in \partial K$ such that $x_0 \in (y, z)$. Since $y, z \in \mathrm{conv}[\mathrm{ext}(K)]$ by (a), we have also $x_0 \in \mathrm{conv}[\mathrm{ext}(K)]$.

□

Corollary 0.5 (Elementary Maximum Principle). *Let K be a (nonempty) finite-dimensional compact convex set in some t.v.s., and $f\colon K \to (-\infty, +\infty]$ a convex function. If f attains its maximum over K, then the maximum is attained at some extreme point of K.*

Proof. Let $x \in K$ be a point of maximum for f. By Theorem 0.4, we can write $x = \sum_{i=1}^n \lambda_i e_i$ where $n \in \mathbb{N}$, $e_i \in \mathrm{ext}(K)$, $\lambda_i > 0$ and $\sum_1^n \lambda_j = 1$. Since

$$f(e_i) \le f(x) \text{ for each } i\,, \quad \text{and} \quad f(x) \le \textstyle\sum_1^n \lambda_i f(e_i)\,,$$

we must have $f(e_i) = f(x)$ for some i (even for each i if $f(x) < +\infty$). □

The following theorem contains the Minkowski's theorem together with a kind of its converse. It says that $\mathrm{ext}(K)$ is the smallest subset of K whose convex hull equals K.

Theorem 0.6. *Let K be a finite-dimensional compact convex set in some t.v.s., and $A \subset K$ a set. Then the following assertions are equivalent:*

(i) $\mathrm{ext}(K) \subset A$;

(ii) $K = \mathrm{conv}(A)$.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Minkowski's Theorem 0.4. Let us prove the reverse one. Let (ii) hold and $x \in \mathrm{ext}(K)$. Let $n \in \mathbb{N}$ be the smallest integer such that x is a convex combination of n elements of A. If $n = 1$, we have $x \in A$ and we are done. Let $n > 1$. Write x as a convex combination of n elements of A: $x = \sum_{i=1}^n \lambda_i a_i$. Notice that $\lambda_i \in (0, 1)$ for each i (by minimality of n), and write

$$x = \lambda_1 a_1 + (1 - \lambda_1) \left[\sum_{i=2}^n \frac{\lambda_i}{1 - \lambda_1} a_i \right] .$$

Since the sum in square brackets is an alement of K, and x is an extreme point of K, we must have $x = a_1 = \sum_{i=2}^n \frac{\lambda_i}{1-\lambda_1} a_i$. But this contradiction with minimality of n means that the case $n > 1$ is impossible. □

Extreme points of arbitrary compact convex sets. We are going to show that the infinite-dimensional compact convex sets are determined by their sets of extreme points, too. However, the convex hull is not enough in general (see Example 0.7); we have to replace it with the *closed convex hull.* Let us stress that the assumption of local convexity of the space is crutial in these results (see Example 0.8).

Example 0.7. Consider the Banach space

$$\ell_1 = (c_0)^*$$

and the corresponding w^*-topology $\sigma(\ell_1, c_0)$. The space (ℓ_1, w^*) is a locally convex t.v.s. in which the closed unit ball $B := B_{\ell_1}$ is compact (by Alaoglu's theorem). However,

$$B \neq \operatorname{conv}[\operatorname{ext}(B)]\,.$$

Proof. Let $x \in \partial B$, that is, $\sum_{i=1}^{+\infty} |x(i)| = 1$. Suppose that x has at least two nonzero coordinates, say, $x(j)$ and $x(k)$. Fix $0 < \varepsilon < \min\{|x(j)|, |x(k)|\}$ and denote $\sigma_j = \operatorname{sign}(x(j))$, $\sigma_k = \operatorname{sign}(x(k))$. Notice that $|x(j) \pm \sigma_j \varepsilon| = |x(j)| \pm \varepsilon$ and similarly for k in place of j. Define $y, z \in \ell_1$ by

$$\begin{aligned} &y(i) = z(i) = x(i) \text{ for } i \notin \{j, k\}, \\ &y(j) = x(j) + \sigma_j \varepsilon,\ z(j) = x(j) - \sigma_j \varepsilon, \\ &y(k) = x(k) - \sigma_k \varepsilon,\ z(k) = x(k) + \sigma_k \varepsilon. \end{aligned}$$

Then $y, z \in \partial B$, $y \neq z$ and $x = \frac{y+z}{2}$. Thus x is not an extreme point of B.

It follows that an extreme point of B must have exactly one nonzero coordinate, that is, $x = \pm e_i$ for some i, where e_i is the i-th canonical unit vector. It is easy to see that such points are actually extreme points of B. Thus we have

$$\operatorname{ext}(B) = \{\pm e_i : i \in \mathbb{N}\}\,.$$

Observe that each element of $\operatorname{conv}[\operatorname{ext}(B)]$ has a finite support, and hence this set is strictly contained in B. $\square$

The following example shows that the assumption of local convexity in the subsequent theorems cannot be omitted.

Example 0.8 (Roberts, 1977)**.** There exist a Hausdorff t.v.s. X which is metrizable by a complete metric, and a nonempty compact convex set $K \subset X$ such that $\operatorname{ext}(K) = \emptyset$.

Let us start with the following auxiliary theorem.

Theorem 0.9. *Let K be a compact convex set in some Hausdorff locally convex t.v.s. If $U \subset K$ is a nonempty convex relatively open set in K such that*

$$\operatorname{ext}(K) \subset U\,,$$

then $U = K$.

Proof. Assume that $U \neq K$. Consider the family

$$\mathcal{V} = \{V \subset K :\ V \text{ is convex and open in } K,\, V \neq K,\, U \subset V\}$$

which is partially ordered by inclusion. Using compactness of K, it is easy to see that each chain in $\mathcal{V}$ has an upper bound in $\mathcal{V}$. By Zorn's lemma, $\mathcal{V}$ contains a maximal element V_0.

For $x \in V_0$ and $t \in (0,1)$, we define

$$W_{x,t} = \{y \in K : (1-t)x + ty \in V_0\}\,.$$

Notice that $W_{x,t}$ is an open (in K) convex set containing V_0. We claim that $W_{x,t}$ contains $\overline{V_0}$. Indeed, if $y \in \overline{V_0}$, then $(1-t)x + ty \in [x, y) \subset \mathrm{int}(\overline{V_0}) = \mathrm{int}(V_0) = V_0$.

Since $\emptyset \neq V_0 \neq K$, connectedness of K implies that $V_0 \neq \overline{V_0}$. By maximality of V_0, we must have $W_{x,t} = K$.

In particular, we have proved that

$$x \in V_0,\, y \in K \quad \Longrightarrow \quad [x, y) \subset V_0\,.$$

Consequently, if $A \subset K$ is a convex open (in K) set, then $V_0 \cup A$ is convex. This fact, together with maximality of V_0, easily implies that $K \setminus V_0 = \{z_0\}$ for some $z_0 \in K$. Obviously, $z_0 \in \mathrm{ext}(K)$. But then $z_0 \in K \setminus V_0 \subset K \setminus U$ which is in contradiction with our assumption on U. □

Theorem 0.10 (Krein–Milman, 1940)**.** *Let K be a compact convex set in some Hausdorff locally convex t.v.s. Then*

$$K = \overline{\mathrm{conv}}[\mathrm{ext}(K)]\,.$$

Proof. Let K contain more than one point (otherwise everything is trivial). Fix an arbitrary $y_0 \in K$ and define

$$K_0 = \begin{cases} \overline{\mathrm{conv}}[\mathrm{ext}(K)] & \text{if } \mathrm{ext}(K) \neq \emptyset \\ \{y_0\} & \text{otherwise} \end{cases}.$$

Assume there exists $x \in K \setminus K_0$. By the H-B Strong Separation Theorem, there exists $\ell \in X^*$ such that $\ell(x) > \sup \ell(K_0)$. Then the set

$$U = \{y \in K : \ell(y) < \ell(x_0)\}$$

is a nonempty convex open (in K) proper subset of K, containing $\mathrm{ext}(K)$. But this contradicts Theorem 0.9. Thus we must have $K_0 = K$ which implies the statement. □

Corollary 0.11. *Let X be a normed space.*

(a) $B_{X^*} = \overline{\mathrm{conv}}^{w^*}[\mathrm{ext}(B_{X^*})]$.
(b) *If X is a reflexive Banach space and $C \subset X$ is a bounded closed convex set, then $C = \overline{\mathrm{conv}}^{\|\cdot\|}[\mathrm{ext}(C)]$.*

Proof. (a) follows from the Krein-Milman Theorem applied to the t.v.s. (X^*, w^*).
(b) Since C is compact in (X, w), the Krein-Milman Theorem implies that $C = \overline{\mathrm{conv}}^{w}[\mathrm{ext}(C)]$. Since a convex set is w-closed if and only if it is closed, we can substitute the w-closure with the closure in the norm topology. □

The possibility to use the *norm*-closure remains valid also in some nonreflexive Banach spaces. Let us state without proof the following theorem of such kind.

Theorem 0.12. *Let X be a Banach space for which $\mathrm{ext}(B_{X^*})$ is separable. Then*

$$B_{X^*} = \overline{\mathrm{conv}}^{\|\cdot\|}[\mathrm{ext}(B_{X^*})]\,.$$

In particular, X^* is separable. (And this implies that X is separable, too.)

Theorem 0.13 (Bauer's Maximum Principle)**.** *Let K be a nonempty compact convex set in some Hausdorff locally convex t.v.s., and $f\colon K \to \mathbb{R}$ a convex u.s.c. function. Then f attains its maximum over K at some extreme point of K.*

Proof. Denote $m = \max f(K)$ (it exists since K is compact and f is u.s.c.). If the statement is false, then the set $U = \{x \in K : f(x) < m\}$ is an open (in K) convex set containing all extreme points of K. Since $U \neq K$, this contradicts Theorem 0.9. □

Theorem 0.14 (Milman)**.** *Let K be a compact convex set in a Hausdorff locally convex t.v.s., and $A \subset K$. Then the following assertions are equivalent:*

(i) $\mathrm{ext}(K) \subset \overline{A}$;
(ii) $K = \overline{\mathrm{conv}}(A)$.

Proof. The implication $(i) \Rightarrow (ii)$ follows immediately from the Krein-Milman Theorem (Theorem 0.10). Let us show the reverse one.

Let (ii) hold. We want to show that $\mathrm{ext}(K) \subset A + V$ for each $V \in \mathcal{U}(0)$. Since our space is locally convex, we can limit ourselves to the neighborhoods V of 0 which are closed, convex and symmetric.

Since $\overline{A}$ is compact, A is totally bounded. Thus there exist $a_1, \ldots, a_n \in A$ such that $A \subset \bigcup_1^n (a_i + V)$. Consider the compact convex sets

$$K_i = \overline{\mathrm{conv}}[A \cap (a_i + V)]$$

contained in K. Then $\mathrm{conv}\left(\bigcup_1^n K_i\right)$ is compact, and hence

$$K = \overline{\mathrm{conv}}(A) = \overline{\mathrm{conv}}\left(\bigcup_1^n K_i\right) = \mathrm{conv}\left(\bigcup_1^n K_i\right).$$

Now, each $x \in \mathrm{ext}(K)$ can be written as a convex combination $x = \sum_1^n \lambda_i y_i$ where $y_i \in K_i$ for each i. Since x is an extreme point, we must have $x = y_j$ for some j, that is, $x \in K_j \subset a_j + V \subset A + V$. The proof is complete. □

Examples in concrete spaces.

Example 0.15. Let C be a closed convex set in a normed space X. If C is strictly convex (in the sense that ∂C does not contain any nontrivial line segment), then $\mathrm{ext}(C) = \partial C$. The closed unit ball of any $L_p(\mu)$-space with $1 < p < +\infty$ is strictly convex. (Such spaces include all Hilbert spaces, ℓ_p and $L_p[0,1]$ for $1 < p < +\infty$.)

Example 0.16. For the closed unit balls of the spaces c_0 and $C[0,1]$, we have:

(a) $\mathrm{ext}(B_{c_0}) = \emptyset$;
(b) $\mathrm{ext}(B_{C[0,1]}) = \{\pm e\}$ where $e(t) \equiv 1$.

Consequently, these spaces are not dual spaces (Corollary 0.11(a)).

Proof. (a) Let $x \in c_0$ be such that $\|x\| = 1$. Fix $k \in \mathbb{N}$ such that $|x(k)| < 1$, and a $\delta > 0$ so small that $|x(k) \pm \delta| < 1$. Then $x = \frac{(y_+) + (y_-)}{2}$ where $y_\pm(k) = x(k) \pm \delta$ and $y_\pm(i) = x(i)$ for $i \neq k$. Since $\|y_\pm\| = 1$, x is not an extreme point of B_{c_0}.
(b) It is easy to see that the points $\pm e$ are extreme points of $B_{C[0,1]}$. If $x \in C[0,1]$ is such that $\|x\| = 1$ and $x \neq \pm e$, there exists $t_0 \in (0,1)$ such that $|x(t_0)| < 1$. By continuity, there exists $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subset [0,1]$, and $|x(t)| < 1 - \delta$

whenever $t \in (t_0 - \delta, t_0 + \delta)$. Consider the function $\Delta \in C[0,1]$ such that $\Delta(t_0) = \delta$, $\Delta(t) = 0$ for $t \in [0,1] \setminus (t_0 - \delta, t_0 + \delta)$, and Δ is affine on $[t_0 - \delta, t_0]$ and on $[t_0, t_0 + \delta]$. Then $x = \frac{1}{2}(x + \Delta) + \frac{1}{2}(x - \Delta)$ and $\|x \pm \Delta\| = 1$. Consequently, $x \notin \text{ext}(B_{C[0,1]})$. $\square$

Exercise 0.17. Show that $\text{ext}(B_{\ell_\infty}) = \{x \in \ell_\infty : \ |x(i)| = 1 \text{ for each } i\}$.

Exercise 0.18. Show that $\text{ext}(B_{L_1[0,1]}) = \emptyset$.
(Hint: use the idea of the first part of the proof of Example 0.7.)

Exercise 0.19. Determine the set of extreme points of B_c (where c is the Banach space of all convergent real sequences, equipped with the supremum norm). Then show that

$$B_c = \overline{\text{conv}}[\text{ext}(B_c)]\,.$$

(Remark: it is known that c is not a dual space, but, because of the above equality, this fact cannot be proved using the Krein-Milman Theorem.)

It is known that the dual of $C(K)$, where K is a Hausdorff compact topological space, can be represented as the space

$$\mathcal{M}^r(K) = \{\mu\colon \mathit{Borel}(K) \to \mathbb{R} \ : \ \ \mu \text{ regular (signed) measure of finite variation}\}\,,$$

equipped with the norm $\|\mu\| = \text{Var}(\mu, K)$ (the total variation of μ), and the isometric correspondence between $\ell \in C(K)^*$ and $\mu \in \mathcal{M}^r(K)$ is given by the formula

$$\ell(x) = \int_K x(t)\, d\mu(t) \qquad (x \in C[0,1]).$$

This is the famous *Riesz Representation Theorem.* Let $\mathcal{M}_1^r(K)$ denote the set of all probability measures in $\mathcal{M}^r(K)$. Let us state without proof the following proposition.

Proposition 0.20. *Let K be a compact Hausdorff topological space. Then*

$$\text{ext}(B_{\mathcal{M}^r(K)}) = \{\pm\delta_t : t \in K\}\,, \qquad \text{ext}(\mathcal{M}_1^r(K)) = \{\delta_t : t \in K\}\,.$$

Since $\mathcal{M}_1^r(K)$ is easily seen to be a w^*-closed convex subset of $B_{\mathcal{M}^r(K)}$, it is w^*-compact. By the Krein-Milman Theorem and the above proposition, the set of all convex combinations of Dirac measures is w^*-dense in $\mathcal{M}_1^r(K)$.

Corollary 0.21. *Let K be a compact Hausdorff topological space. Let $\mu \in \mathcal{M}^r(K) \setminus \{0\}$, $\mu \geq 0$, $F \subset C(K)$ a finite set and $\varepsilon > 0$ be given. Then there exist finitely many points $t_1, \ldots, t_n \in K$ and real numbers $\lambda_1, \ldots, \lambda_n > 0$ such that $\sum_1^n \lambda_i = \mu(K)$ and*

$$\left| \int_K x\, d\mu - \sum_{i=1}^n \lambda_i x(t_i) \right| < \varepsilon \quad \textit{for each } x \in F.$$

Proof. The measure $\hat{\mu} = \frac{\mu}{\mu(K)}$ belongs to $\mathcal{M}_1^r(K)$. The set

$$W = \{\nu \in \mathcal{M}^r(K) : \ |\textstyle\int_K x\, d\nu - \int_K x\, d\hat{\mu}| < \frac{\varepsilon}{\mu(K)} \ \forall x \in F\}$$

is a w^*-neighborhood of $\hat{\mu}$, and hence it contains a convex combination of Dirac measures of the form $\nu = \sum_1^n \hat{\lambda}_i \delta_{t_i}$ (of course, we can suppose that $\hat{\lambda}_i > 0$ for each i). In other words, we have $\sum_1^n \hat{\lambda}_i = 1$ and

$$\left| \int_K x\, d\hat{\mu} - \sum_1^n \hat{\lambda}_i x(t_i) \right| < \frac{\varepsilon}{\mu(K)} \quad \text{for each } x \in F.$$

Now, multiplying by $\mu(K)$, we obtain the desired formula for $\lambda_i = \hat{\lambda}_i \mu(K)$. $\square$

Let us remark that in the same way one can get a similar result for an arbitrary signed mesure $\mu \in \mathcal{M}^r(K) \setminus \{0\}$, with the change that the numbers λ_i can have any sign and $\sum_1^n |\lambda_i| = \mathrm{Var}(\mu, K)$.

Topological properties of the set of extreme points.

Lemma 0.22. *Let K be a (nonempty) finite-dimensional compact convex set in some t.v.s.*

(a) *If* $\dim(K) \le 2$, *then* $\mathrm{ext}(K)$ *is closed.*
(b) *For* $\dim(K) > 2$, $\mathrm{ext}(K)$ *is not necessarily closed.*

Proof. (a) The case of $\dim(K) \le 1$ is obvious. Let $\dim(K) = 2$. We can suppose that K is contained in $\mathbb{R}^2$; then $\mathrm{int}(K) \neq \emptyset$. The set of non-extreme boundary points of K, being a union of open segments, is open in ∂K. Thus, $\mathrm{ext}(K)$ is closed in ∂K, and hence in $\mathbb{R}^2$.
(b) Consider $\mathbb{R}^d$ with $d \ge 3$. Let $B \subset \mathbb{R}^d$ be the Euclidean unit ball, $x_0 \in \partial B$ a point, and $[a, b] \subset \mathbb{R}^d$ a segment tangent to B at $x_0 = \frac{a+b}{2}$. Let $H \subset \mathbb{R}^d$ be the hyperplane through x_0, orthogonal to $[a, b]$ (that is, $H = (b-1)^\perp$). The set $K = \mathrm{conv}(B \cup [a, b])$ is compact and convex. It is easy to see that $x_0 \notin \mathrm{ext}(K)$ while $(H \cap \partial B) \setminus \{x_0\} \subset \mathrm{ext}(K)$. Thus $\mathrm{ext}(K)$ is not closed. □

The following example shows that the set of extreme points can be topologically quite complicated.

Example 0.23 (Bishop and de Leeuw, 1959)**.** There exists a compact convex set K in a locally convex t.v.s. such that $\mathrm{ext}(K)$ is not a Borel set.

However, the situation is much better in the case of *metrizable* compact convex sets.

Theorem 0.24 (Choquet)**.** *Let K be a metrizable compact convex set in a t.v.s. Then $\mathrm{ext}(K)$ is a G_δ subset of K.*

Proof. Let K be metrizable by a metric d. Observe that

$$K \setminus \mathrm{ext}(K) = \{x \in K :\ x = \tfrac{y+z}{2} \text{ with } y, z \in K,\ d(x, y) > 0\} = \bigcup_{n=1}^{+\infty} F_n\,,$$

where

$$F_n = \{x \in K :\ x = \tfrac{y+z}{2} \text{ with } y, z \in K,\ d(x, y) \ge \tfrac{1}{n}\}\,.$$

Compactness of K easily implies that the sets F_n are closed. Consequently, $K \setminus \mathrm{ext}(K)$ is an F_σ set, which implies the statement. □