

Let us recall that a linear mapping $T: X \rightarrow Y$ between two normed spaces is continuous if and only if it is bounded, that is, there exists a constant $M \geq 0$ such that $\|Tx\| \leq M\|x\|$ for all $x \in X$.

Definition 0.1.

- (a) We say that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space X are *equivalent* if there exist two constants $b \geq a > 0$ such that $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$ for all $x \in X$.
- (b) We say that two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are *isomorphic* if there exists an algebraic isomorphism (i.e., a linear bijection) $T: X \rightarrow Y$ such that both T and T^{-1} are continuous.

Note that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent if and only if the normed spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are isomorphic. If two normed spaces are isomorphic and one of them is a Banach space, then the other one is a Banach space, too.

Let us recall the famous Open Mapping Theorem: *If X, Y are Banach spaces and $T: X \rightarrow Y$ is a continuous linear operator which is onto, then T is an open mapping, that is the image of any open set in X is an open set in Y .* (In particular, if T is also one-to-one, then T is an isomorphism between X and Y .) It follows that two Banach norms on X are either equivalent or incomparable.

Proposition 0.2. *Any two norms on \mathbb{R}^d are equivalent.*

Proof. It suffices to show that each norm $\|\cdot\|$ on \mathbb{R}^d is equivalent to the Euclidean norm $|\cdot|_e$. Let e_i ($1 \leq i \leq d$) denote the i -th vector of the standard basis of \mathbb{R}^d . Given $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we have

$$\|x\| = \left\| \sum_{i=1}^d x_i e_i \right\| \leq \sum_{i=1}^d |x_i| \|e_i\| \leq c \sum_{i=1}^d |x_i|,$$

where $c = \max_{1 \leq i \leq d} \|e_i\|$. Using the Cauchy-Schwarz inequality, we get

$$\|x\| \leq c\sqrt{d} |x|_e.$$

In particular, this implies that $\|\cdot\|$ is continuous in $(\mathbb{R}^d, |\cdot|_e)$, since $|\|x\| - \|y\|| \leq \|x - y\| \leq (c\sqrt{d})|x - y|_e$. Since the sphere

$$S = \{y \in \mathbb{R}^d : |y|_e = 1\}$$

is $|\cdot|_e$ -compact and the function $\|\cdot\|$ is strictly positive on S , there exists a constant $a > 0$ such that $\|y\| \geq a$ for all $y \in S$. Now, for every $x \in \mathbb{R}^d \setminus \{0\}$, we can write

$$\|x\| = \left\| |x|_e \frac{x}{|x|_e} \right\| = |x|_e \left\| \frac{x}{|x|_e} \right\| \geq a|x|_e.$$

This completes the proof. □

As a corollary we have the following important theorem.

Theorem 0.3. *Every two normed spaces of the same finite dimension are isomorphic.*

Proof. It suffices to show that each d -dimensional normed space X is isomorphic to \mathbb{R}^d (with the Euclidean norm). Let $T: \mathbb{R}^d \rightarrow X$ be an algebraic isomorphism. Then the formula

$$\|\xi\| = \|T\xi\| \quad (\xi \in \mathbb{R}^d)$$

defines a norm on \mathbb{R}^d . By Proposition 0.2, this norm is equivalent to the Euclidean norm on \mathbb{R}^d . It follows that there exist constants $a, b > 0$ such that $a|\xi|_e \leq \|T\xi\| \leq b|\xi|_e$ ($\xi \in \mathbb{R}^d$). Thus T is an isomorphism between \mathbb{R}^d and X . \square

Corollary 0.4. *Let X be a finite-dimensional normed space. Then:*

- (a) X is a Banach space.
- (b) Every linear mapping from X into any normed space Y is continuous.

Proof. (a) follows immediately from the fact that X is isomorphic to the Euclidean \mathbb{R}^d which is known to be complete.

(b) By previous theorem, we can suppose that $X = (\mathbb{R}^d, \|\cdot\|_1)$. For each $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we have

$$\|Tx\| = \left\| \sum_1^d x_i T e_i \right\| \leq \sum_1^d |x_i| \|T e_i\| \leq a \|x\|_1$$

where $a = \max_{1 \leq i \leq d} \|T e_i\|$. We are done. \square

For the definition and properties of affine mappings see the chapter “Affine mappings and convex functions...”.

Theorem 0.5. *Each finite-dimensional affine set A in a normed space is closed. Moreover, each affine mapping on A (with values in some normed space) is continuous.*

Proof. By translation, we can suppose that our affine set is linear. But then it is closed since it is complete by Corollary 0.4(a). The second part follows from Corollary 0.4(b). \square

Some characterizations of finite dimensional normed spaces. It is a well-known fact of Functional Analysis that for every infinite-dimensional normed space X :

- a) X contains a bounded sequence which has no convergent subsequences;
- b) there exists a linear functional $f: X \rightarrow \mathbb{R}$ that is not continuous.

Thus we obtain the following characterizations of finite-dimensionality.

Theorem 0.6. *For a normed space X , the following assertions are equivalent.*

- (i) X is finite-dimensional.
- (ii) Every linear functional on X is continuous.
- (iii) Every closed bounded set $D \subset X$ is compact.
- (iv) X contains a compact set with a nonempty interior.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ follows from Corollary 0.4(b) and b) above. Let us prove the equivalence of (i) , (iii) and (iv) .

$(i) \Rightarrow (iv)$ follows from Theorem 0.3 and from the fact that the closed unit ball of \mathbb{R}^n is a compact set with interior points.

$(iv) \Rightarrow (iii)$. If (iv) holds, there exists a compact set $K \subset X$ containing 0 as an interior point. Chose $r > 0$ such that the closed ball $B(0, r)$ is contained in K . This ball is compact; hence, by homogeneity and translation, every closed ball in X is compact. This implies (iii) .

$(iii) \Rightarrow (i)$. If (i) is false, X contains a bounded sequence with no convergent subsequences (see a) above). Take a closed ball D containing this sequence. It is not compact; hence (iii) is false. \square

Relative interior.

Definition 0.7. Let C be a convex set in a normed space X . The *relative interior* of C is the set

$$\text{ri}(C) = \text{int}_{\text{aff}(C)}(C),$$

that is, the interior of C in its affine hull.

A closed convex set can have empty relative interior. Consider, e.g., a compact convex set whose affine hull is infinite-dimensional. The following theorem shows that this is impossible for finite-dimensional convex sets.

Definition 0.8. The *dimension* of a convex set is defined as the dimension of its affine hull.

Theorem 0.9. *Every finite-dimensional convex set has a nonempty relative interior.*

Proof. Let C be a d -dimensional convex set in a normed space X . By translation, we can suppose that $0 \in C$, and hence $\text{aff}(C) = \text{span}(C)$. Let v_1, \dots, v_d be linearly independent elements of C . The mapping $T: \text{span}(C) \rightarrow \mathbb{R}^d$, given by $T(\sum_1^d \alpha_i v_i) = (\alpha_1, \dots, \alpha_d)$, is an isomorphism (Corollary 0.2). The convex set $T(C)$ contains 0 and the vectors e_1, \dots, e_d of the canonical basis of \mathbb{R}^d . Consequently, it contains the convex hull of these $d+1$ points, which is the set $\{\xi \in \mathbb{R}^d : \xi_i \geq 0, \sum_1^d \xi_j \leq 1\}$. This set has a nonempty interior in \mathbb{R}^d . Since T is an isomorphism, C has a nonempty interior in $\text{span}(C)$. \square