

In this section,  $X, Y$  denote real vector spaces, unless otherwise specified.

**Affine mappings.**

**Definition 0.1.** Let  $X, Y$  be vector spaces,  $A \subset X$  an affine set. A mapping  $F: A \rightarrow Y$  is *affine* if  $F((1-t)x + ty) = (1-t)F(x) + tF(y)$  whenever  $t \in \mathbb{R}$  and  $x, y \in A$ .

**Proposition 0.2.** Let  $F: X \rightarrow Y$ .

- (a)  $F$  is linear if and only if  $F$  is affine and  $F(0) = 0$ .
- (b)  $F$  is affine if and only if there exist a linear mapping  $T: X \rightarrow Y$  and a vector  $y_0 \in Y$  such that  $F(x) = Tx + y_0$  ( $x \in X$ ). Moreover,  $T$  is unique in this case.

*Proof.* Exercise. □

**Corollary 0.3.** Let  $A \subset X$  be an affine set. If  $F: A \rightarrow Y$  is affine, then

$$F(\sum_1^n \lambda_i x_i) = \sum_1^n \lambda_i F(x_i) \quad \text{whenever } x_i \in A, \lambda_i \in \mathbb{R}, \sum_1^n \lambda_j = 1.$$

*Proof.* Fix  $a_0 \in A$  and consider the linear set  $L = A - a_0$ . Since  $A = L + a_0$ , we can define an affine mapping  $G: L \rightarrow Y$  by  $G(y) = F(y + a_0)$ . By Proposition 0.2, we can write  $G(y) = Ty + y_0$ , where  $T: L \rightarrow Y$  is linear and  $y_0 \in Y$  is fixed. Then our  $F$  is of the form

$$F(x) = G(x - a_0) = T(x - a_0) + y_0.$$

Now, an easy calculation completes the proof. □

Now, we can consider the following notion of affine mappings defined on *convex* (not necessarily affine) sets.

**Definition 0.4.** Let  $X, Y$  be vector spaces,  $C \subset X$  a convex set. We say that a mapping  $F: C \rightarrow Y$  is *c-affine* if  $F((1-t)x + ty) = (1-t)F(x) + tF(y)$  whenever  $t \in [0, 1]$  and  $x, y \in C$ .

**Lemma 0.5.** Let  $A \subset X$  be an affine set and  $F: A \rightarrow Y$  a mapping. Then  $F$  is affine if and only if  $F$  is c-affine.

*Proof.* Each affine mapping is clearly c-affine. To prove the vice-versa, suppose that  $F$  is c-affine and consider  $x, y \in A$ ,  $t \in \mathbb{R}$ ,  $z = (1-t)x + ty$ . We want to show that  $F(z) = (1-t)F(x) + tF(y)$ . For  $t \in [0, 1]$  this is true. Suppose  $t > 1$ . Then  $y$  is a convex combination of  $x$  and  $z$ :

$$y = \frac{1}{t}z + \frac{t-1}{t}x.$$

Consequently,  $F(y) = \frac{1}{t}F(z) + \frac{t-1}{t}F(x)$  which easily implies  $F(z) = (1-t)F(x) + tF(y)$ . The case  $t < 0$  can be proved in a similar way by expressing  $x$  by means of  $y, z$ . □

**Lemma 0.6.** Let  $C$  be a convex set in a vector space  $X$ . Then

$$\text{aff}(C) = \{\alpha u - \beta v : \alpha, \beta \geq 0, \alpha - \beta = 1, u, v, \in C\}.$$

*Proof.* Let us prove the inclusion “ $\subset$ ” (the other one is trivial). Take  $x \in \text{aff}(C)$  and write it as an affine combination  $x = \sum_{i=1}^n \lambda_i u_i$  of elements  $u_i \in C$  ( $1 \leq i \leq n$ ). We can suppose that  $\lambda_i \neq 0$  for each  $i$ . Denote

$$I = \{i : \lambda_i > 0\}, \quad J = \{i : \lambda_i < 0\}.$$

Clearly,  $I \neq \emptyset$ . If  $J = \emptyset$ , we have  $x \in C$ , and hence  $x = \alpha x - \beta x$  with  $\alpha = 1, \beta = 0$ . Now, let  $J \neq \emptyset$ . Then  $x = \alpha u - \beta v$  where  $\alpha = \sum_{j \in I} \lambda_j$ ,  $\beta = \sum_{j \in J} (-\lambda_j)$ ,  $u = \sum_{i \in I} \frac{\lambda_i}{\alpha} u_i \in C$  and  $v = \sum_{i \in J} \frac{-\lambda_i}{\beta} u_i \in C$ .  $\square$

**Theorem 0.7.** *Let  $X, Y$  be vector spaces,  $C \subset X$  a convex set,  $A = \text{aff}(C)$ . If  $F: C \rightarrow Y$  is  $c$ -affine, then there exists an affine mapping  $G: A \rightarrow Y$  such that  $G|_C = F$ . Moreover, such  $G$  is unique.*

*Proof.* By Lemma 0.6, each  $x \in A$  can be written in the form

$$(1) \quad x = \alpha u - \beta v \quad \text{where } \alpha, \beta \geq 0, \alpha - \beta = 1, u, v \in C.$$

Then we put  $G(x) = \alpha F(u) - \beta F(v)$ .

*Claim.*  $G(x)$  does not depend on the representation (1).

To see this, consider two such representations  $\alpha u - \beta v = \alpha' u' - \beta' v'$ . Then  $\alpha u + \beta' v' = \alpha' u' + \beta v$  and  $\alpha' + \beta = \alpha + \beta' =: \Delta$ . We must have  $\Delta > 0$ , since otherwise we would get  $\alpha = \alpha' = \beta = \beta' = 0$  which is impossible. Since  $\frac{\alpha}{\Delta} u + \frac{\beta'}{\Delta} v' = \frac{\alpha'}{\Delta} u' + \frac{\beta}{\Delta} v$  are convex combinations, we get  $\frac{\alpha}{\Delta} F(u) + \frac{\beta'}{\Delta} F(v') = \frac{\alpha'}{\Delta} F(u') + \frac{\beta}{\Delta} F(v)$  which easily implies  $\alpha F(u) - \beta F(v) = \alpha' F(u') - \beta' F(v')$ . Our Claim is proved.

Thus we have defined a mapping  $G: A \rightarrow Y$ . Moreover,  $G$  is an extension of  $F$ . Since any affine extension of  $F$  has to satisfy the same definition,  $F$  has at most one affine extension to  $A$ .

In view of Lemma 0.5, it suffices to show that  $G$  is  $c$ -affine on  $A$ . Let  $x, y \in A$ ,  $t \in (0, 1)$ ,  $p = (1-t)x + ty$ . By Lemma 0.6, we can write

$$x = \alpha u - \beta v, \quad y = \gamma w - \delta z,$$

where  $\alpha, \beta, \gamma, \delta \geq 0$ ,  $\alpha - \beta = \gamma - \delta = 1$ ,  $u, v, w, z \in C$ . Then

$$p = (1-t)\alpha u + t\gamma w - [(1-t)\beta v + t\delta z].$$

Put  $a = (1-t)\alpha + t\gamma$  and  $b = (1-t)\beta + t\delta$  and observe that  $a - b = 1$ . If  $a \neq 0$  and  $b \neq 0$ , we can write

$$p = a \left[ \frac{(1-t)\alpha}{a} u + \frac{t\gamma}{a} w \right] - b \left[ \frac{(1-t)\beta}{b} v + \frac{t\delta}{b} z \right].$$

The expressions in square brackets are points of  $C$ . Thus

$$\begin{aligned} G(p) &= aF \left( \frac{(1-t)\alpha}{a} u + \frac{t\gamma}{a} w \right) - bF \left( \frac{(1-t)\beta}{b} v + \frac{t\delta}{b} z \right) \\ &= a \left[ \frac{(1-t)\alpha}{a} F(u) + \frac{t\gamma}{a} F(w) \right] - b \left[ \frac{(1-t)\beta}{b} F(v) + \frac{t\delta}{b} F(z) \right] \\ &= (1-t)[\alpha F(u) - \beta F(v)] + \lambda[\gamma F(w) - \delta F(z)] \\ &= (1-t)G(x) + tG(y). \end{aligned}$$

$\square$

By the above theorem, every  $c$ -affine mapping on a convex set is a restriction of an affine mapping. From this reason, *c-affine mappings on a convex set will be called affine*.

**Exercise 0.8.** Let  $X, Y$  be vector spaces,  $C \subset X$  a convex set, and  $F: C \rightarrow Y$  an affine mapping. Let  $A \subset C$  and  $B \subset Y$  be sets.

- (a) If  $A$  is convex, then  $F(A)$  is convex. If  $B$  is convex, then  $F^{-1}(B)$  is convex.
- (b) Is it true that  $\text{conv}(F(A)) = F(\text{conv}(A))$ ?
- (b) Is it true that  $\text{conv}(F^{-1}(B)) = F^{-1}(\text{conv}(B))$ ?

### Convex functions.

By  $\overline{\mathbb{R}}$  we mean the extended real line  $[-\infty, +\infty]$ .

**Definition 0.9.** Let  $C \subset X$  be a convex set,  $f: C \rightarrow \overline{\mathbb{R}}$  a function.

- (a)  $f$  is called *proper* (on  $C$ ) if its *effective domain*

$$\text{dom}(f) = \{x \in C : f(x) \in \mathbb{R}\}$$

is nonempty.

- (b)  $f$  is called *convex* (on  $C$ ) if its *epigraph*

$$\text{epi}(f) = \{(x, t) \in C \times \mathbb{R} : t \geq f(x)\}$$

is a convex set in  $X \times \mathbb{R}$ .

- (c)  $f$  is called *concave* (on  $C$ ) if  $-f$  is convex.

It is clear that  $f$  is concave on  $C$  if and only if its *hypograph*  $\text{hypo}(f) = \{(x, t) \in C \times \mathbb{R} : t \leq f(x)\}$  is a convex set. Moreover,  $f: C \rightarrow \overline{\mathbb{R}}$  is both convex and concave if and only if  $f$  is affine.

Notice that if  $f: C \rightarrow [-\infty, +\infty]$  is a convex function, then the function

$$\hat{f}(x) = \begin{cases} f(x) & (x \in C) \\ +\infty & (x \in X \setminus C) \end{cases}$$

is convex on  $X$ .

**Observation 0.10.** If  $f_i: C \rightarrow \overline{\mathbb{R}}$  ( $i \in I$ ) are convex functions, then also their *pointwise supremum*

$$f(x) = \sup_{i \in I} f_i(x)$$

is a convex function. Indeed,  $\text{epi}(f) = \bigcap_{i \in I} \text{epi}(f_i)$ .

**Theorem 0.11.** Let  $C$  be a convex set in a vector space  $X$ ,  $f: C \rightarrow \overline{\mathbb{R}}$  a function. Then  $f$  is convex if and only if

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$

whenever  $x, y \in C$ ,  $\lambda \in (0, 1)$  and the right-hand side is defined.

*Proof.* “ $\Rightarrow$ ” Fix  $x, y \in C$  and  $\lambda \in (0, 1)$  such that  $(1 - \lambda)f(x) + \lambda f(y)$  is defined (i.e.,  $f(x), f(y)$  are not infinite with opposite signs). If  $f(x) = +\infty$  or  $f(y) = +\infty$ , the inequality is obviously satisfied. Let now  $f(x) < +\infty$  and  $f(y) < +\infty$ . For any  $t, s \in \mathbb{R}$  such that  $t \geq f(x)$  and  $s \geq f(y)$ , the points  $(x, t)$  and  $(y, s)$  belong to the convex set  $\text{epi}(f)$ . Thus also  $((1 - \lambda)x + \lambda y, (1 - \lambda)t + \lambda s) \in \text{epi}(f)$ , which implies

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)t + \lambda s.$$

The formula follows since  $t \geq f(x)$  and  $s \geq f(y)$  were arbitrary.

“ $\Leftarrow$ ” Let  $(x, t)$  and  $(y, s)$  be two points of  $\text{epi}(f)$ , and  $\lambda \in (0, 1)$ . Then  $f(x) \leq t < +\infty$  and  $f(y) \leq s < +\infty$ . By our formula,  $f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) \leq (1 - \lambda)t + \lambda s$ . Hence  $(1 - \lambda)(x, t) + \lambda(y, s) = ((1 - \lambda)x + \lambda y, (1 - \lambda)t + \lambda s) \in \text{epi}(f)$ .  $\square$

**Observation 0.12.** *If  $f: C \rightarrow \overline{\mathbb{R}}$  is a convex function on a convex set  $C$ , then the sets  $\text{dom}(f)$ ,  $\{x \in C : f(x) \leq \alpha\}$  and  $\{x \in C : f(x) < \alpha\}$  ( $\alpha \in \overline{\mathbb{R}}$ ) are convex.*

Convex functions that assume the value  $-\infty$  are not particularly interesting. For example, if  $f: \mathbb{R} \rightarrow \overline{\mathbb{R}}$  is convex with  $J := f^{-1}(-\infty) \neq \emptyset$ , then  $J$  is an interval,  $\text{dom}(f) \subset \partial I$ , and  $f$  is identically  $+\infty$  on  $\mathbb{R} \setminus \overline{J}$ . From this reason, we shall consider convex functions with values in  $(-\infty, +\infty]$  only. In this case, the right-hand side of the formula in Theorem 0.11 is always defined.

**Observation 0.13.**

- (a) *If  $f, g: C \rightarrow (-\infty, +\infty]$  are convex functions and  $\alpha, \beta \geq 0$ , then also the function  $\alpha f + \beta g$  is convex.*
- (b) *If  $f_n: C \rightarrow (-\infty, +\infty]$  ( $n \in \mathbb{N}$ ) are convex functions, then also the sum  $f = \sum_{n=1}^{+\infty} f_n$  is a convex function.*

**Theorem 0.14** (disuguaglianza di Jensen). *Let  $X$  be a vector space,  $C \subset X$  a convex set, and  $f: C \rightarrow (-\infty, +\infty]$  a convex function. Then*

$$(2) \quad f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i) \quad \text{whenever } n \in \mathbb{N}, x_i \in C, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1.$$

*Proof.* The proof, based on an induction argument (as in the proof of the fact that convex sets are closed under making arbitrary convex combinations), is left to the reader as an exercise.  $\square$

**Midconvex functions.** Let  $C \subset X$  be a convex set. A function  $f: C \rightarrow (-\infty, +\infty]$  is called *midconvex* (or *Jensen convex*, or *J-convex*) if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \quad \text{whenever } x, y \in C.$$

It is clear that  $f$  is midconvex on  $C$  if and only if it is midconvex on each segment  $C \cap L$  where  $L \subset X$  is a line intersecting  $C$ .

Each convex function is also midconvex, but not vice-versa. Examples of nonconvex midconvex functions on  $\mathbb{R}$  can be constructed using the so-called Hamel basis: the basis of  $\mathbb{R}$  as a vector space over the rationals  $\mathbb{Q}$ .

However, it can be proved that midconvex functions satisfy the Jensen inequality with rational coefficients. Moreover, a midconvex function  $f$  on an open interval  $I \subset \mathbb{R}$  is convex provided it satisfies at least one of the following conditions:

- (a)  $f$  is continuous;
- (b)  $f$  is locally bounded on  $I$ ;
- (c)  $f$  is Lebesgue measurable;
- (d)  $f$  is bounded above on a set of positive measure;
- (e)  $f$  is bounded above on a set of the second Baire category.

Let us conclude this section with some important examples of convex functions.

**Indicator function.** Given a set  $A$  in a vector space  $X$ , the *indicator function*  $\delta_A$  of  $A$  is defined by

$$\delta_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{if } x \in X \setminus A \end{cases}.$$

It is easy to see that  $\delta_A$  is convex if and only if  $A$  is a convex set.

**Sublinear functions.** A function  $p: X \rightarrow (-\infty, +\infty]$  is called *sublinear* if  $p(0) = 0$ ,  $p(tx) = tp(x)$  and  $p(x + y) \leq p(x) + p(y)$  whenever  $t > 0$ ,  $x, y \in X$ .

Observe that every sublinear function is convex and its effective domain is a convex cone (that is a convex set which is invariant under multiplication by nonnegative scalars). Any norm is a sublinear function.

**Distance function.** Given a nonempty set  $A$  in a normed linear space, its *distance function* is the function

$$d_A: X \rightarrow [0, +\infty), \quad d_A(x) = d(x, A) = \inf\{\|x - a\| : a \in A\}.$$

Observe that  $d_A = d_{\bar{A}}$ . Moreover,  $d_A$  is 1-Lipschitz (i.e., Lipschitz with constant 1); this follows easily from the fact that  $d_A$  is the pointwise infimum of 1-Lipschitz functions  $\|\cdot - a\|$  ( $a \in A$ ).

**Exercise 0.15.** Let  $X$  be a normed space,  $x_i \in X$ ,  $r_i > 0$ ,  $\lambda_i > 0$  ( $i = 1, \dots, n$ ). Then  $\sum_{i=1}^n \lambda_i B(x_i, r_i) = B(\sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \lambda_i r_i)$ .

**Proposition 0.16.** Let  $X$  be a normed space,  $A \subset X$  a nonempty closed set.

- (a)  $d_A$  is convex if and only if  $A$  is a convex set.
- (b)  $d_A$  is concave on  $X \setminus A$  if and only if  $X \setminus A$  is a convex set.

*Proof.* (a) If  $d_A$  is convex, then the set  $A = \{x : d_A(x) \leq 0\}$  is convex. Now, let  $A$  be convex. Given  $x, y \in X$  and  $\varepsilon > 0$ , choose  $a, b \in A$  so that  $\|x - a\| < d_A(x) + \varepsilon$  and  $\|y - b\| < d_A(y) + \varepsilon$ . Then for each  $t \in [0, 1]$  we have  $d_A((1-t)x + ty) \leq \|(1-t)x + ty - [(1-t)a + tb]\| \leq (1-t)\|x - a\| + t\|y - b\| \leq (1-t)d_A(x) + td_A(y) + \varepsilon$ . By arbitrariness of  $\varepsilon > 0$ ,  $d_A$  is a convex function.

(b) If  $d_A$  is concave on  $X \setminus A$ , then the set  $X \setminus A = \{x : d_A(x) > 0\}$  is convex. Now, let  $X \setminus A$  be convex. Let  $x, y \in X \setminus A$  and  $t \in [0, 1]$ . The open balls  $B^0(x, d_A(x))$  and  $B^0(y, d_A(y))$  are contained in the convex set  $X \setminus A$ . Thus  $X \setminus A \supset \text{conv}[B^0(x, d_A(x)) \cup B^0(y, d_A(y))] \supset (1-t)B^0(x, d_A(x)) + tB^0(y, d_A(y)) = B^0((1-t)x + ty, (1-t)d_A(x) + td_A(y))$ . It follows that  $d_A((1-t)x + ty) \geq (1-t)d_A(x) + td_A(y)$ . Thus  $d_A$  is concave on  $X \setminus A$ .  $\square$

**Support functions.** Let  $X$  be a normed space with the continuous dual  $X^*$  (the normed space of all continuous linear functionals on  $X$ ). Given nonempty sets  $A \subset X^*$  and  $B \subset X$ , we can define the following so-called *support functions*:

$$s_A: X \rightarrow (-\infty, +\infty], \quad s_A(x) = \sup_{a^* \in A} a^*(x) = \sup x(A);$$

$$\sigma_B: X^* \rightarrow (-\infty, +\infty], \quad \sigma_B(x^*) = \sup_{b \in B} x^*(b) = \sup x^*(B).$$

These two functions are easily seen to be sublinear and hence convex.

**Minkowski functional.** Let  $X$  be a vector space. By the *cone hull* of a set  $A \subset X$  we mean the set

$$\text{cone}(A) = \{0\} \cup \bigcup_{t>0} tA.$$

The cone hull of a nonempty set is just the set of all nonnegative multiples of the elements of the set. Observe that a cone hull of a convex set  $C$  containing 0 is the set  $\bigcup_{n \in \mathbb{N}} nC$ .

The *Minkowski functional* (or the *Minkowski gauge*) of a set  $A$  is the function

$$p_A: X \rightarrow (-\infty, +\infty], \quad p_A(x) = \inf\{t > 0 : x \in tA\} = \inf\{t > 0 : \frac{x}{t} \in A\},$$

with the usual convention that  $\inf \emptyset = +\infty$ .

**Example 0.17.** Let  $B^0(0, r)$  and  $B(0, r)$  denote the open and closed ball of radius  $r$ , centered in 0. Then  $p_{B^0(0, r)}(x) = \inf\{t > 0 : \|x\| < tr\} = \inf\{t > 0 : \frac{\|x\|}{r} < t\} = \frac{1}{r} \|x\|$ , and analogously  $p_{B(0, r)}(x) = \frac{1}{r} \|x\|$ .

**Proposition 0.18.** Let  $C \subset X$  be a convex set containing 0.

- (a)  $\text{dom}(p_C) = \text{cone}(C)$ .
- (b)  $p_C$  is sublinear.
- (c)  $\{x : p_C(x) < 1\} \subset C \subset \{x : p_C(x) \leq 1\}$ .
- (d) If, for each line  $L \subset \text{span}(C)$  through 0,  $C \cap L$  is closed in  $L$ , then  $C = \{x : p_C(x) \leq 1\}$ .
- (e) If, for each line  $L \subset \text{span}(C)$  through 0,  $C \cap L$  is open in  $L$ , then  $C = \{x : p_C(x) < 1\}$ .

*Proof.* (a),(c),(d),(e) are easy exercises. Let us prove (b). First,  $p_C$  is positively homogeneous since, for each  $t > 0$ , we have

$$p_C(tx) = \inf\{\lambda > 0 : \frac{tx}{\lambda}\} \stackrel{\lambda=t\mu}{=} \inf\{t\mu : \mu > 0, \frac{x}{\mu} \in C\} = tp_C(x).$$

It remains to show that  $p_C$  is subadditive. Let  $x, y \in \text{dom}(C)$ ,  $\varepsilon > 0$ . Choose  $\alpha, \beta > 0$  such that  $\frac{x}{\alpha}, \frac{y}{\beta} \in C$ ,  $\alpha < p_C(x) + \varepsilon$ ,  $\beta < p_C(y) + \varepsilon$ . Then

$$\frac{x+y}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta} \left(\frac{x}{\alpha}\right) + \frac{\beta}{\alpha+\beta} \left(\frac{y}{\beta}\right) \in C.$$

Consequently,  $p_C(x+y) \leq \alpha + \beta < p_C(x) + p_C(y) + 2\varepsilon$ . Since  $\varepsilon > 0$  was arbitrary, we conclude that  $p_C(x+y) \leq p_C(x) + p_C(y)$ .  $\square$

**Observation 0.19.** Let  $C, D \subset X$  be two convex sets containing 0. If  $K \subset X$  is a cone such that  $C \cap K \subset D$ , then  $p_C(x) \geq p_D(x)$  for each  $x \in K$ .

**Theorem 0.20.** *Let  $X$  be a normed space, and  $C \subset X$  a convex set such that  $0 \in C$ . Then the following assertions are equivalent:*

- (i) *there exists  $c > 0$  such that  $p_C(x) \leq c\|x\|$  for each  $x \in \text{cone}(C)$ ;*
- (ii) *the restriction  $(p_C)|_{\text{cone}(C)}$  is continuous in 0;*
- (iii) *0 belongs to the relative interior of  $C$  in  $\text{cone}(C)$ , that is, there exists  $r > 0$  such that  $B^0(0, r) \cap \text{cone}(C) \subset C$ .*

*If moreover  $\text{cone}(C)$  is a linear set, the above equivalent conditions imply that  $p_C$  is Lipschitz on  $X$ .*

*Proof.* (i) obviously implies (ii). If (ii) holds, there exists  $r > 0$  such that  $p_C(x) < 1$  whenever  $x \in B^0(0, r) \cap \text{cone}(C)$ . Then, by Proposition 0.18(c),  $B^0(0, r) \cap \text{cone}(C) \subset C$ . Finally, if (iii) holds, then for each  $x \in \text{cone}(C)$  we have (by Observations 0.19 and 0.17) that  $p_C(x) \leq p_{B^0(0, r)}(x) = \frac{1}{r}\|x\|$ . This proves that (i)-(iii) are equivalent.

Now, let  $\text{cone}(C)$  be a linear set,  $x, y \in \text{cone}(C)$ . Then  $p_C(x) = p_C(y + (x - y)) \leq p_C(y) + p_C(x - y)$ . Consequently,  $p_C(x) - p_C(y) \leq p_C(x - y) \leq c\|x - y\|$  where  $c$  is as in (i). By interchanging the role of  $x$  and  $y$ , we obtain that  $p_C$  is  $c$ -Lipschitz on  $\text{cone}(C)$ .  $\square$

As an application of Minkowski functionals, we prove the following theorem.

**Theorem 0.21.** *Let  $X$  be a normed linear space,  $C \subset X$  and  $D \subset X$  bounded convex sets with nonempty interiors. Let the sets  $C, D$  be either both closed or both open. Then there exists a homeomorphism  $\Phi$  of  $X$  onto itself such that  $\Phi(C) = D$ . Moreover, given two points  $x_0 \in \text{int}(C)$  and  $y_0 \in \text{int}(D)$ , the homeomorphism  $\Phi$  can be taken so that it satisfies the property*

$$\Phi(x_0 + tv) = y_0 + t\Phi(v) \quad \text{whenever } v \in X, t \geq 0.$$

*(In particular,  $\Phi(x_0) = y_0$ .)*

*Proof.* The sets  $C_0 = C - x_0$  and  $D_0 = D - y_0$  contain 0 as an interior point. Define a mapping  $F: X \rightarrow X$  by

$$F(0) = 0, \quad F(x) = \frac{p_{C_0}(x)}{p_{D_0}(x)} x \quad (x \neq 0).$$

Then  $F$  is positively homogeneous, and continuous on  $X \setminus \{0\}$ , since  $p_{C_0}$  and  $p_{D_0}$  are continuous by Theorem 0.20. To show that  $F$  is continuous at 0, observe that  $D_0 \subset B(0, R)$  for some  $R > 0$ , and hence  $p_{D_0} \geq p_{B(0, R)} = \frac{1}{R}\|\cdot\|$  (Observation 0.19). Thus  $\frac{\|x\|}{p_{D_0}(x)} \leq R$  ( $x \neq 0$ ), and this easily implies continuity of  $F$  at 0. A direct calculation shows that  $G(y) = \frac{p_{D_0}(y)}{p_{C_0}(y)}y$  is the inverse function of  $F$ . Since  $G$  is continuous,  $F$  is a homeomorphism of  $X$ . Moreover,  $p_{D_0}(F(x)) = p_{C_0}(x)$  for each  $x \in X$ . By Proposition 0.18(d,e),  $F(C_0) = D_0$ . Now, it is easy to verify that the mapping  $\Phi(x) := y_0 + F(x - x_0)$  has the desired properties.  $\square$

**Exercises.** Let  $C \subset X$  be a convex set,  $f: C \rightarrow (-\infty, +\infty]$  a function.

1. Let  $p \geq 1$ . If  $f$  is convex and nonnegative, then  $f^p$  is convex, too. Is it possible to omit "nonnegative" for some integer values of  $p$ ?

2. We say that  $f$  is *quasiconvex* if all sets  $f^{-1}((-\infty, \alpha))$  and  $f^{-1}((-\infty, \alpha])$  ( $\alpha \in \mathbb{R}$ ) are convex. Show that a quasiconvex function is not necessarily convex. Characterize quasiconvex functions by an inequality analogous to the one in Theorem 0.11.