

In what follows, C is a convex set in a vector space X .

Topological interior. Let us start with the following easy observation.

Observation 0.1. *Let C be a convex set in a normed space X . If C contains an open ball $B^0(z_0, r)$, then*

$$B^0((1-t)z_0 + ty, (1-t)r) \subset C \quad \text{whenever } y \in C, t \in [0, 1].$$

Proof. Fix $y \in C$, $t \in [0, 1]$. Since C is convex, it contains the set $(1-t)B^0(z_0, r) + ty = B^0((1-t)z_0 + ty, (1-t)r)$. \square

Corollary 0.2. *Let C be a convex set in a normed space X . If $x \in \text{int}(C)$ and $y \in C$, then $[x, y) \subset \text{int}(C)$. In particular, $\text{int}(C)$ is a convex set.*

Theorem 0.3. *Let C be a convex set in a normed space X .*

- (a) *If either $\text{int}(C) \neq \emptyset$ or C is finite-dimensional, then $\text{int}(C) = \text{int}(\overline{C})$.*
- (b) *If $\text{int}(C) \neq \emptyset$, then $\overline{C} = \overline{\text{int}(C)}$.*

Proof. First part of (a). Suppose that $\text{int}(C) \neq \emptyset$. Thus C contains an open ball $B^0(z_0, r)$. One inclusion is obvious. Let $x \in \text{int}(\overline{C})$. There exists $\rho > 0$ such that $B^0(x, \rho) \subset \overline{C}$. If $x = z_0$, we are done. If $x \neq z_0$, fix $y \in B^0(x, \rho)$ such that $x \in (z_0, y)$, that is, there exists $t \in (0, 1)$ with $x = (1-t)z_0 + ty$. Since $y \in \overline{C}$, there exists $y' \in C$ be such that $\|y - y'\| < r$. For $x' = (1-t)z_0 + ty'$, Observation 0.1 says that $B^0(x', (1-t)r) \subset C$. Then $x \in \text{int}(C)$, since $\|x - x'\| = (1-t)\|y - y'\| < (1-t)r$. Second part of (a). Let C be finite-dimensional. If $\text{aff}(C) = X$, then $\text{int}(C) \neq \emptyset$ by the ‘‘relative interior theorem’’; hence we can apply the first part of (a). If $\text{aff}(C) \neq X$, the affine hull of C is closed (since it is a translate of a finite-dimensional linear set) and has no interior points. Since $\text{int}(C) \subset \text{int}(\overline{C}) \subset \text{int}(\overline{\text{aff}(C)}) = \text{int}(\text{aff}(C)) = \emptyset$, we have $\text{int}(C) = \text{int}(\overline{C}) = \emptyset$.

(b) Obviously, $\overline{C} \supset \overline{\text{int}(C)}$. If $x \in \overline{C}$, fix $z_0 \in \text{int}(C)$. Arbitrarily near to x we can find some $y \in C$; but since $[z_0, y) \subset \text{int}(C)$ (Corollary 0.2), arbitrarily near to x we can find points of $\text{int}(C)$. Thus $x \in \overline{\text{int}(C)}$. \square

Algebraic interior. Let C be a convex set in a vector space X , $x \in C$. We say that x is an *algebraically interior point* of C if for every $v \in X$ there exists $t > 0$ such that $x + tv \in C$. In this case, we shall write $x \in \text{a-int}(C)$.

(For a nonconvex set $E \subset X$, an algebraically interior point is a point $x \in E$ such that for each $v \in X$ there exists $t > 0$ such that $[x, x + tv] \subset E$.)

In other words, $x \in \text{a-int}(C)$ if and only if $x \in \text{int}_L(C \cap L)$ for each line $L \subset X$ passing through x .

The following proposition is an analogue of Corollary 0.2.

Proposition 0.4. *Let C be a convex set in a vector space X . If $x \in \text{a-int}(C)$ and $y \in C$, then $[x, y) \in \text{a-int}(C)$. In particular, the set $\text{a-int}(C)$ is convex.*

Proof. Let $0 < \lambda < 1$, $z = (1 - \lambda)x + \lambda y$, $v \in X$. Take $t > 0$ such that $x + tv \in C$. Then $z + (1 - \lambda)tv \in C$ since $z + (1 - \lambda)tv = (1 - \lambda)(x + tv) + \lambda y$. Now, the assertion easily follows. \square

If X is a normed space, then obviously $\text{int}(C) \subset \text{a-int}(C)$. The following theorem states some sufficient conditions for equality of the two sets. Before stating it we recall the following fundamental theorem of R.L. Baire, which we state in the following form.

Theorem 0.5 (Baire Category Theorem). *Let T be either a complete metric space or a Hausdorff locally compact topological space. If F_n ($n \in \mathbb{N}$) are closed sets without interior points, then also the union $\bigcup_{n \in \mathbb{N}} F_n$ has no interior points.*

Exercise 0.6. Use the Baire Category Theorem to prove the following two facts.

1. Every compact countable set in a Hausdorff topological space has at least one isolated point.
2. The algebraic dimension of a Banach space is either finite or uncountable.

Theorem 0.7. *Let C be a convex set in a normed linear space X . Then*

$$\text{int}(C) = \text{a-int}(C)$$

provided at least one of the following conditions is satisfied.

- (a) $\text{int}(C) \neq \emptyset$;
- (b) C is finite-dimensional;
- (c) X is Banach and C is an F_σ -set.

Proof. (a) Let $x \in \text{a-int}(C)$. Fix $z_0 \in \text{int}(C)$. If $x \neq z_0$, there exists $y \in C$ such that $x \in (z_0, y)$. Then $x \in \text{int}(C)$ by Corollary 0.2.

(b) Let $x \in \text{a-int}(C)$. This implies that $\text{aff}(C) = X$ and hence X is finite-dimensional. By the “relative interior theorem”, $\text{int}(C) = \text{ri}(C)$ is nonempty. Apply the part (a).

(c) Let $x \in \text{a-int}(C)$. We can suppose that $x = 0$. Write $C = \bigcup_k F_k$ where each F_k is a closed set. Then

$$X = \bigcup_{t>0} tC = \bigcup_n nC = \bigcup_{n,k} nF_k.$$

By the Baire Category Theorem, there exist n, k such that nF_k has a nonempty interior. Thus $\text{int}(F_k) \neq \emptyset$ and hence $\text{int}(C) \neq \emptyset$. Apply (a). \square

Directions from a point. Let C be a convex set in a vector space X , $x_0 \in C$. Let us consider two sets of directions: the set $\text{c}(x_0, C)$ of directions of the half-lines from x_0 to the points of C , and the set $\text{i}(x_0, C)$ of directions of the lines L such that $L \cap C$ is a nondegenerate segment containing x_0 in its relative interior. More precisely,

$$\begin{aligned} \text{c}(x_0, C) &= \{v \in V : x_0 + tv \in C \text{ for some } t > 0\}, \\ \text{i}(x_0, C) &= \{v \in V : x_0 \pm tv \in C \text{ for some } t > 0\}. \end{aligned}$$

Let us state main properties of these two sets.

Proposition 0.8. *Let X, C, x_0 be as above.*

- (a) $x_0 \in x_0 + i(x_0, C) \subset x_0 + c(x_0, C) \subset \text{aff}(C)$.
- (b) $c(x_0, C) = \text{cone}(C - x_0)$.
- (c) $c(x_0, C)$ is a convex cone, and $i(x_0, C)$ is a linear set.
- (d) $i(x_0, C) = c(x_0, C) \cap [-c(x_0, C)]$.
- (e) $x_0 + i(x_0, C)$ is the largest affine set A such that $x_0 \in \text{int}_A(C \cap A)$.
- (f) $\text{aff}(C) = x_0 + [c(x_0, C) - c(x_0, C)]$.

Proof. (b) follows directly from the definitions; hence $c(x_0, C)$ is a convex cone. Obviously, we have $0 \in i(x_0, C) \subset c(x_0, C)$. Since $x_0 + c(x_0, C)$ is the set of all closed half-lines from x_0 to the points of C (or $\{x_0\}$ if there are no such half-lines), it is contained in $\text{aff}(C)$; hence (a) holds. The part (d) follows easily from definitions. Since $K \cap (-K)$ is a linear set whenever K is a convex cone, we have proved (c), too.

Let us show (e). Clearly, $x_0 + i(x_0, C)$ is an affine set. Let A be an affine set such that $x_0 \in \text{int}(C \cap A)$. For every point $x \in A \setminus x_0$ the set $C \cap \overrightarrow{x_0 x}$ is a nondegenerate segment containing x_0 in its relative interior. Since $x_0 + i(x_0, C)$ is the union of all possible lines L with this property, we have $A \subset x_0 + i(x_0, C)$.

It remains to prove (f). By translation, we can suppose that $x_0 = 0$. Thus, by (b), we have to show that, if $0 \in C$, then $\text{span}(C) = \text{cone}(C) - \text{cone}(C)$. The inclusion “ \supset ” is obvious. To show the other one, consider a (finite) linear combination $x = \sum_i \alpha_i c_i$ with $\alpha_i \neq 0$, $c_i \in C$. Then

$$x = \sum_{\alpha_i > 0} \alpha_i c_i - \sum_{\alpha_i < 0} (-\alpha_i) c_i \in \text{cone}(C) - \text{cone}(C).$$

□

Algebraic relative interior. Let us recall that a *relative interior* of a convex set C in a normed space is the set $\text{ri}(C) = \text{int}_{\text{aff}(C)}(C)$. Recall also the fact, proved in the chapter on finite-dimensional spaces, that every finite-dimensional convex set has a nonempty relative interior. The *algebraic relative interior* of C is defined analogously in the following definition.

Definition 0.9. Let C be a convex set in a vector space X . The *algebraic relative interior* of C is the set $\text{a-ri}(C) = \text{a-int}_{\text{aff}(C)}(C)$.

Observation 0.10. Let C be a convex set in a vector space X , $x \in C$. Then

$$x \in \text{a-ri}(C) \iff x + i(x, C) = \text{aff}(C) \iff x + c(x, C) = \text{aff}(C).$$

Proof. Easy exercise. □

Observe that Theorem 0.7 immediately gives the following

Corollary 0.11. Let C be a convex set in a normed space. Then $\text{a-ri}(C) = \text{ri}(C)$ whenever at least one of the following conditions is satisfied: (a) $\text{ri}(C) \neq \emptyset$; (b) $\dim(C) < \infty$; (c) X is a Banach space, $\text{aff}(C)$ is closed, and C is an F_σ -set.

Inside points. Let C be a convex set in a normed space X . For $x \in C$, let \mathcal{L}_x be the family of all lines $L \subset X$, for which $C \cap L$ is a nondegenerate segment containing x in its relative interior.

It is easy to see that

$$x \in \text{a-ri}(C) \iff \bigcup_{L \in \mathcal{L}_x} (C \cap L) = C.$$

Thus the following notion of an “inside point” weakens the notion of an algebraically interior point.

Definition 0.12. Let X, C, x, \mathcal{L}_x be as above. We say that x is an *inside point* of C (or x is *almost surrounded* by C) if $\bigcup_{L \in \mathcal{L}_x} (C \cap L)$ is dense in C .

Since $\bigcup_{L \in \mathcal{L}_x} L = x + \text{i}(x, C)$ whenever $\mathcal{L}_x \neq \emptyset$, we have the following reformulation of the above definition.

Observation 0.13. A point x is an inside point of C if and only if $[x + \text{i}(x, C)] \cap C$ is dense in C .

Proposition 0.14. Let C be a convex set in a normed linear space X .

- (a) If x is an inside point of C and $y \in C$, then each point of $[x, y]$ is an inside point of C .
- (b) The set of all inside points of C is convex.
- (c) If x is an inside point of C , then $\bigcup_{L \in \mathcal{L}_x} L$ is dense in $\text{aff}(C)$.
- (d) If x is an inside point of C , then every continuous affine function $a: C \rightarrow \mathbb{R}$ that attains its maximum over C at x is constant on C .
- (e) If C is finite-dimensional, then each inside point of C belongs to $\text{ri}(C)$.

Proof. (a) Fix $z \in (0, 1)$, $u \in C$ and $\varepsilon > 0$. Since x is an inside point of C , there exist two distinct points $c, d \in C$ such that $x \in (c, d)$ and $\text{dist}(u, [c, d]) < \varepsilon$. Consider the line $L = \overleftrightarrow{cd}$. If $y \in L$, then obviously $z \in \text{ri}(C \cap L)$. If $y \notin L$, then $\Delta := \text{conv}\{c, d, y\}$ is a nondegenerate triangle containing z in its relative interior. Thus there exist $c', d' \in \Delta$ such that $z \in (c', d')$ and $\text{dist}(u, [c', d']) < \varepsilon$ (indeed, it suffices take $c' \in [c, d]$ so that $\|c' - u\| < \varepsilon$). We have proved that z is an inside point of C .

(b) is an immediate corollary of (a).

(c) The set $D = \bigcup_{L \in \mathcal{L}_x} (L \cap C)$ is dense in C . Thus $\text{aff}(D)$ is dense in $\text{aff}(C)$. Observe that the set $A = \bigcup_{L \in \mathcal{L}_x} L = x + \text{i}(x, C)$ is affine (Proposition 0.8(c)), it contains D , and every affine set containing D must contain A . It follows that $A = \text{aff}(D)$. So A is dense in $\text{aff}(C)$.

(d) The function a must be constant on each segment $C \cap L$ ($L \in \mathcal{L}_x$); so it is constant on $\bigcup_{L \in \mathcal{L}_x} (C \cap L)$. By density, a is constant on C .

(e) If C is finite-dimensional, the set affine set $\bigcup_{L \in \mathcal{L}_x} L$ is closed (since it is contained in the finite-dimensional affine set $\text{aff}(C)$). Hence $\bigcup_{L \in \mathcal{L}_x} (L \cap C)$ is closed in C . Thus $\bigcup_{L \in \mathcal{L}_x} (L \cap C) = C$ which means that $x \in \text{a-ri}(C) = \text{ri}(C)$. (The equality follows from Corollary 0.11). \square

Now, we are going to prove a theorem about existence of inside points. Before proving it, we shall need an important lemma about infinite convex combinations. We say that a point $x \in X$ is an *infinite convex combination of elements of a set* $A \subset X$ if

$$x = \sum_{n=1}^{+\infty} \lambda_n y_n$$

with $y_n \in A$, $\lambda_n \geq 0$, $\sum_1^{+\infty} \lambda_j = 1$.

Lemma 0.15. *Let C be a convex set in a normed space X . Suppose that C is either closed or open. Then every infinite convex combination of elements of C belongs to C .*

Proof. We can (and do) suppose that $\lambda_n > 0$ for each n . Let C be closed. Denoting $\sigma_N = \sum_1^N \lambda_j$ ($N \in \mathbb{N}$), we have

$$x = \lim_N \sum_{n=1}^N \lambda_n c_n = \lim_N \sigma_N \sum_{n=1}^N \frac{\lambda_n}{\sigma_N} c_n = \lim_N \sum_{n=1}^N \frac{\lambda_n}{\sigma_N} c_n \in \overline{C} = C.$$

Now, let C be open. By translation, we can suppose that $0 \in C$. Suppose that $x \neq 0$ (otherwise we are done). There exists $\delta > 0$ such that $\tilde{c}_1 := c_1 + \delta x \in C$. Then, by the first part of the proof,

$$\tilde{x} := (1 + \lambda_1 \delta)x = \lambda_1 \tilde{c}_1 + \sum_{n=2}^{+\infty} \lambda_n c_n \in \overline{C}.$$

Since $0 \in \text{int}(\overline{C})$, $\tilde{x} \in \overline{C}$ and $x \in (0, \tilde{x})$. Applying Corollary 0.2 and Theorem 0.3(a), we get $x \in \text{int}(\overline{C}) = \text{int}(C) = C$. \square

The following proposition is both an immediate corollary and a generalization of Lemma 0.15.

Proposition 0.16. *Let C be a convex set in a normed space X . If C is the intersection of a family of open convex sets, then every infinite convex combination of elements of C belongs to C .*

Remark 0.17. The sets that are intersections of families of open convex sets are called *evenly convex* sets. An easy application of the Hahn-Banach theorem gives that a convex set is evenly convex if and only if it is the intersection of a family of open halfspaces. There exist convex sets C that are not evenly convex but still they are closed under making infinite convex combinations of their elements: consider, in \mathbb{R}^2 , the union of an open halfplane and one of its boundary points.

The following theorem is essentially due to V. Klee.

Theorem 0.18 (Klee, 1955). *Let X be a Banach space. If $C \subset X$ is a separable non-singleton set which is the intersection of a family of open convex sets (e.g., a separable closed convex set), then C has an inside point.*

Proof. There exist countably many pairwise distinct points $c_n \in C$ ($n \in \mathbb{N}$) such that the set $\{c_n\}_{n \in \mathbb{N}}$ is dense in C . Fix a sequence $\{\lambda_n\} \subset (0, 1)$ such that $\sum_1^{+\infty} \lambda_n = 1$ and $\sum_1^{+\infty} \lambda_n \|c_n\| < +\infty$. Then the series $\sum_1^{+\infty} \lambda_n c_n$ converges in X (since it

converges absolutely and X is complete). Let us prove that the point $x = \sum_1^{+\infty} \lambda_n c_n$ is an inside point of C .

For each $n \in \mathbb{N}$, $x = \lambda_n c_n + (1 - \lambda_n)x_n$ where $x_n = \sum_{k \neq n} \frac{\lambda_k}{1 - \lambda_n} c_k \in C$ (Lemma 0.15). If $c_n = x_n$ for some n , then $c_n = x$, and hence such n is unique; put $n_0 = n + 1$ in this case. Otherwise define $n_0 = 1$. The segments $[c_n, x_n]$ ($n \geq n_0$) are nondegenerate and contained in C , and their union is dense in C . Since $x \in (c_n, x_n)$ for each $n \geq n_0$, the proof is complete. \square

Corollary 0.19. *Let X be a normed space, and $C \subset X$ a separable complete convex set. Then C has an inside point.*

Proof. Consider C as a closed convex set in the completion \overline{X} of X , and apply the previous theorem. \square

Comparison of several properties.

We can define another two properties of a point x of a convex set C in a normed linear space X . Let \mathcal{L}_x be as above. We shall say that:

- x is a *quasi-inside point* (“quasi inside point”) of C if $\bigcup_{L \in \mathcal{L}_x} L$ is dense in $\text{aff}(C)$;
- x is a *non-support point* of C if every $x^* \in X^*$ that attains its supremum over C at x is constant on C .

Theorem 0.20. *Let X, C, x be as above. Let us consider the following properties:*

- (a) $x \in \text{ri}(C)$;
- (b) $x \in \text{a-ri}(C)$;
- (c) x is an inside point of C ;
- (d) x is a q -inside point of C ;
- (e) x is a non-support point of C .

Then the following implications hold

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e),$$

and none of these implications can be reversed. However, for finite-dimensional C , the properties (a)-(e) are all equivalent.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are easy, while $(c) \Rightarrow (d)$ is contained in Proposition 0.14(c).

$(d) \Rightarrow (e)$. Let $x^* \in X^*$ be such that $x^*(x) = \sup x^*(C)$. Then necessarily x^* is constant on $\bigcup_{L \in \mathcal{L}_x} L$, and hence also on $\text{aff}(C)$ by continuity.

Let C be finite-dimensional and (e) holds. Then $\text{ri}(C) \neq \emptyset$ (by the “relative interior theorem”). We can suppose that $0 \in C$ (thus $\text{aff}(C) = \text{span}(C)$). If $x \notin \text{ri}(C)$, the Hahn-Banach separation theorem implies existence of a nonzero functional $\varphi \in [\text{span}(C)]^*$ that attains its supremum over C at x . Obviously, φ is not constant on C . By the Hahn-Banach extension theorem, φ can be extended to an element of X^* . But this contradicts (e).

$(b) \not\Rightarrow (a)$. Let X be an infinite dimensional normed space and $f: X \rightarrow \mathbb{R}$ a discontinuous linear functional. Consider $C = \{y \in X : f(y) > 0\}$ and any $x \in C$.

$(c) \not\Rightarrow (b)$. Let $f: X \rightarrow \mathbb{R}$ be a discontinuous linear functional. Consider the set $C = \{y \in X : f(y) \geq 0\}$ and $x = 0$. Then $\bigcup_{L \in \mathcal{L}_x} L = \text{Ker}(f)$ and $\text{aff}(C) = X$.

$(d) \not\Rightarrow (c)$. Let e_i ($i \in \mathbb{N}$) be the canonical unit elements of c_0 (i.e., $e_i = (0, \dots, 0, 1, 0, 0, \dots)$ where 1 is on the i -th position). Let $X = \text{span}\{e_i\}_{i \in \mathbb{N}}$ (the space of all finitely supported

sequences), equipped with the sup-norm. Define a (discontinuous) linear functional $f: X \rightarrow \mathbb{R}$ by $f(e_i) = i$ ($i \in \mathbb{N}$) and consider

$$C = \{y \in X : 0 \leq f(y) \leq 1, |y(i)| \leq 2^{-i} \forall i \in \mathbb{N}\}, \quad x = 0.$$

It is easy to see that $0 \in C$, $\text{aff}(C) = \text{span}(C) = X$ and $\bigcup_{L \in \mathcal{L}_0} L = \text{Ker}(f)$. Thus (d) holds. To show that (c) does not hold, it suffices to show that f is continuous on C ; indeed, in this case, $\bigcup_{L \in \mathcal{L}_0} (L \cap C) = \text{Ker}(f) \cap C$ is closed in C and hence not dense in C . Let us proceed by contradiction. If $f|_C$ is not continuous, there exist $\varrho > 0$ and a sequence $\{c_n\} \subset C$ converging to some $c \in C$, such that $|f(c_n) - f(c)| > \varrho$ for each n . By definition of C , the sequences $\{c_n(i)\}_{n \in \mathbb{N}}$ ($i \in \mathbb{N}$) and $\{f(c_n)\}_{n \in \mathbb{N}}$ are bounded (in \mathbb{R}) and hence contain convergent subsequences. By a standard diagonal argument and passing to a subsequence, we can suppose that

$$\lim_n c_n(i) = z(i) \in \mathbb{R} \quad (i \in \mathbb{N}), \quad \lim_n f(c_n) = \lambda \in \mathbb{R}.$$

Since $|z_i| \leq 2^{-i}$ ($i \in \mathbb{N}$), the series $\sum_{i \in \mathbb{N}} z(i)e_i$ converges absolutely, and hence its sum is an element z of c_0 . We have

$$\|c_n - z\| = \|\sum_{i \in \mathbb{N}} (c_n(i) - z(i))e_i\| \leq \sum_{i \in \mathbb{N}} |c_n(i) - z(i)|.$$

By the Lebesgue Dominated Convergence Theorem (applied to the counting measure on \mathbb{N}), the last expression tends to 0 as $n \rightarrow \infty$. It follows that $z = c \in X$; in particular, $z(i) = c(i)$ for each i . For each $n \in \mathbb{N}$, denote $N(n) = \max \text{supp}(c_n - c)$. Then we have

$$|f(c_n - c)| = |\sum_{i=1}^{N(n)} (c_n(i) - c(i))f(e_i)| \leq \sum_{i \in \mathbb{N}} i|c_n(i) - c(i)|.$$

Since $i|c_n(i)| \leq i2^{-i}$ and $ic_n(i) \rightarrow ic(i)$ ($i \in \mathbb{N}$), and $\sum_1^\infty i2^{-i} < +\infty$, we can apply again the Lebesgue Dominated Convergence Theorem to conclude that $\lim_n \sum_{i \in \mathbb{N}} i|c_n(i) - c(i)| = 0$. Hence we get $f(c_n) \rightarrow f(c)$, a contradiction.

(e) $\not\Rightarrow$ (d). Let $f: X \rightarrow \mathbb{R}$ be a discontinuous linear functional. Let

$$C = \{y \in X : f(y) > 0\} \cup \{0\}, \quad x = 0.$$

Then $\bigcup_{L \in \mathcal{L}_x} L = \emptyset$, hence (d) is false. If some $x^* \in X^*$ satisfies $x^*|_C \leq 0$, then it must be constant on the level sets of the type $f^{-1}(\lambda)$ with $\lambda > 0$. It follows that $\text{Ker}(x^*)$ contains the dense set $\text{Ker}(f)$; hence $x^* = 0$. This proves (e). \square