

Boundaries and the James theorem

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1. INTRODUCTION

The following theorem is important and well known. All spaces considered here are real normed or Banach spaces. Given a normed space X , we denote by B_X and S_X its closed unit ball and its unit sphere, respectively. If the topology on X is not specified, the topological notions are considered in the norm topology.

Theorem 1.1. *For a Banach space X , the following assertions are equivalent:*

- (i) X is reflexive;
- (ii) B_X is weakly compact;
- (iii) every element of X^* attains its norm.

About the proof. The implications (i) \Leftrightarrow (ii) \Rightarrow (iii) are easy and “elementary” (they only require knowledge of the Banach–Alaoglu and the Goldstine theorems). On the other hand, the remaining implication (iii) \Rightarrow (i), known as the *James theorem*, is a deep and difficult result by R.C. James. \square

From the history. R.C. James obtained the separable version of the above implication (iii) \Rightarrow (i) in 1957 [Ann. Math. 66 (1957), 159–169], and the general version in 1963 [Studia Math. 23 (1963/1964), 205–216]. Finally, in 1964 [Trans. Amer. Math. Soc. 113 (1964) 129–140], he provided the following general result.

Theorem 1.2. *Let C be a bounded closed convex set in a Banach space X . Then C is weakly compact if and only if every element of X^* attains a maximum on C .*

The proof of this last theorem is quite involved. An accessible (though by no means simple) proof, even in a more general setting of complete locally convex topological vector spaces, was given by J.D. Pryce in 1966 [Proc. Amer. Math. Soc. 17 (1966), 148–155].

In last two decades, simpler proofs of the separable version of Theorem 1.2 appeared. We present here the proof by W.B. Moors, published in

W.B. Moors, *An elementary proof of James’ characterization of weak compactness*, Bull. Aust. Math. Soc. **84** (2011), 98–102.

For a different approach by using a Simons’ lemma, see the paragraph 3.11.8 in the book

Fabian, Habala, Hájek, Montesinos and Zizler, *Banach space theory. The basis for linear and nonlinear analysis*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011.

2. BOUNDARIES

Definition 2.1. Let X be a normed space, and $K \subset X^*$ a bounded set. A set $B \subset K$ is a *boundary* or (*James boundary*) for K if every $x \in X$ attains its supremum over K at some point of B , that is,

$$\forall x \in X \exists b^* \in B : b^*(x) = \sup_{x^* \in K} x^*(x).$$

Examples 2.2. Let $K \subset X^*$ be a w^* -compact set (where X is a normed space).

- (a) K is a boundary for itself.
- (b) ∂K is a boundary for K .
- (c) If K is also convex, then $\text{ext } K$ is a boundary for K by Bauer's maximum principle (which is an easy consequence of the Krein–Milman theorem).

The following simple but important fact is left to the reader as an easy exercise on the Hahn–Banach theorem.

Exercise 2.3. Let $K \subset X^*$ be a w^* -compact convex set, and B a boundary for K . Then

$$K = \overline{\text{conv}}^{w^*} B.$$

Let us start with a simple lemma of topological nature. By $\mathcal{U}(0)$ we denote the family of all neighborhoods of 0 (in a topological vector space). It is a well-known fact that every element of $\mathcal{U}(0)$ contains a closed one.

Lemma 2.4. *In a Hausdorff topological vector space, let S , K and K_n ($n \in \mathbb{N}$) be closed sets such that:*

- (a) K is compact,
- (b) $S \cap K = \emptyset$,
- (c) $S \subset \overline{\bigcup_{n \in \mathbb{N}} K_n}$, and
- (d) for every $V \in \mathcal{U}(0)$, $K_n \subset K + V$ for each sufficiently large n .

Then there exists $M \in \mathbb{N}$ such that $S \subset \bigcup_{n \leq M} K_n$.

Proof. By (a),(b), there exists a closed $V \subset \mathcal{U}(0)$ such that $S \cap (K + V) = \emptyset$. Notice that $K + V$ is closed. By (d), there is $M \in \mathbb{N}$ such that $\bigcup_{n > M} K_n \subset K + V$. It follows that

$$\overline{\bigcup_{n > M} K_n} \subset K + V.$$

Using the general equality $\overline{A \cup B} = \overline{A} \cup \overline{B}$, we obtain that

$$S \subset \overline{\bigcup_{n \in \mathbb{N}} K_n} = \bigcup_{n \leq M} K_n \cup \overline{\bigcup_{n > M} K_n},$$

where the last set is disjoint from S . Thus $S \subset \bigcup_{n \leq M} K_n$. \square

Remark 2.5. Let us recall the following two results which will be needed in the sequel.

(a) *Let K be a compact convex set in a locally convex Hausdorff topological vector space, and $A \subset K$. The following assertions are equivalent:*

(i) $K = \overline{\text{conv}} A$;

(ii) $\text{ext } K \subset \overline{A}$.

(The implication (ii) \Rightarrow (i) is the Krein–Milman theorem, while the other one is its “converse” due to Milman.)

(b) *Let X be a Banach space, and $D \subset X^*$ a w^* -closed convex set with nonempty interior. Then the w^* -support points of D are dense in ∂D .*

(This “dual” version of the Bishop–Phelps theorem can be deduced from the standard Bishop–Phelps theorem applied to the pre-polar ${}^{\circ}D \subset X$. Let us remark that Phelps proved a stronger result without the assumption about nonempty interior, but his proof follows a different approach.)

Now we are ready for the basic theorem of this section.

Theorem 2.6. *Let X be a Banach space, and $K \subset X^*$ a w^* -compact convex set. If $\{C_n\}$ is a sequence of w^* -compact convex sets whose union contains a boundary B for K , then*

$$K \subset \overline{\text{conv}} \left(\bigcup_{n \in \mathbb{N}} C_n \right)$$

(closure in the norm topology!).

Proof. We can (and do) suppose that $0 \in B$ and $C_n \subset K$ for each n . Fix an arbitrary $\varepsilon > 0$, and consider the sets

$$K_n := C_n + \frac{\varepsilon}{n} B_{X^*} \quad (n \in \mathbb{N}), \quad D := \overline{\text{conv}}^{w^*} \left(\bigcup_{n \in \mathbb{N}} K_n \right).$$

Notice that the sets K_n and D are w^* -compact since they are contained in $K + \varepsilon B_{X^*}$. Moreover, $B \subset \bigcup_{n \in \mathbb{N}} C_n \subset \bigcup_{n \in \mathbb{N}} \text{int } K_n$, and hence $0 \in B \subset \text{int } D$.

Let $x^* \in D$ be a w^* -support point of D , that is, there exists $x \in X \setminus \{0\}$ such that $x(x^*) = \max x(D) = 1$. Consider the face $F := [x = 1] \cap D$ of D .

Assume for the moment that $F \cap K \neq \emptyset$. Since $K \subset D$, we must have $\max x(K) = 1$. Take $b^* \in B \cap [x = 1]$. But we already know that $b^* \in \text{int } D$, which contradicts the fact that $b^* \in F \subset \partial D$.

Thus we must have $F \cap K = \emptyset$. Since F is an extremal set for D , Remark 2.5(a) implies that

$$\text{ext } F \subset \text{ext } D \subset \overline{\bigcup_{n \in \mathbb{N}} K_n}^{w^*}.$$

Put $S := F \cap \overline{\bigcup_{n \in \mathbb{N}} K_n}^{w^*}$, and notice that $\text{ext } F \subset S \subset \overline{\bigcup_{n \in \mathbb{N}} K_n}^{w^*}$ and $S \cap K = \emptyset$. Moreover, if $V \subset X^*$ is a w^* -neighborhood of 0 then $\frac{\varepsilon}{N} B_{X^*} \subset V$ for some $N \in \mathbb{N}$, and hence $\bigcup_{n > N} K_n \subset K + \frac{\varepsilon}{N} B_{X^*} \subset K + V$. By Lemma 2.4, there exists $M \in \mathbb{N}$ such that

$$\text{ext } F \subset S \subset \bigcup_{n \leq M} K_n.$$

By the Krein–Milman theorem, $x^* \in F = \overline{\text{conv}}^{w^*}(\text{ext } F) \subset \overline{\text{conv}}^{w^*}(\bigcup_{n \leq M} K_n) = \text{conv}(\bigcup_{n \leq M} K_n) \subset \text{conv}(\bigcup_{n \in \mathbb{N}} C_n) + \varepsilon B_{X^*}$.

This shows that all w^* -support points of D are contained in $\text{conv}(\bigcup_{n \in \mathbb{N}} C_n) + \varepsilon B_{X^*}$. By Remark 2.5(b), $\partial D \subset \text{conv}(\bigcup_{n \in \mathbb{N}} C_n) + 2\varepsilon B_{X^*}$. Consequently,

$$K \subset D \subset \text{conv}\left(\bigcup_{n \in \mathbb{N}} C_n\right) + 2\varepsilon B_{X^*},$$

and we are done since $\varepsilon > 0$ was arbitrary. \square

3. APPLICATIONS: THEOREMS BY RAINWATER, RODÉ, AND JAMES

Corollary 3.1. *Let X be a Banach space, $K \subset X^*$ a w^* -compact convex set with a boundary $B \subset K$. Let $\{x_n\} \subset X$ be a bounded sequence, and $x \in X$ such that*

$$b^*(x_n) \rightarrow b^*(x) \quad \text{for each } b^* \in B.$$

Then $x^(x_n) \rightarrow x^*(x)$ for each $x^* \in K$.*

Proof. For simplicity, denote $y_n := x_n - x$. Fix an arbitrary $\varepsilon > 0$, and consider the sets

$$C_n := K \cap \bigcap_{k \geq n} [|y_k| \leq \varepsilon] \quad (n \in \mathbb{N}).$$

These sets form a nondecreasing sequence of w^* -compact convex sets. Notice that $B \subset \bigcup_n C_n$. By Theorem 2.6, the set $C := \bigcup_n C_n = \text{conv}(\bigcup_n C_n)$ is norm-dense in K . Moreover,

$$\limsup_n |y_n(c^*)| \leq \varepsilon \quad \text{for each } c^* \in C.$$

Since C is dense in K , and the sequence $\{y_n\}$ is bounded, we easily obtain that

$$\limsup_n |y_n(x^*)| \leq \varepsilon \quad \text{for each } x^* \in K,$$

and we are done by arbitrariness of $\varepsilon > 0$. \square

The last corollary immediately implies the following useful criterion of weak convergence of a bounded sequence.

Theorem 3.2 (Rainwater 1963). *Let $\{x_n\}$ be a bounded sequence in a Banach space X , and $x \in X$. Assume that $e^*(x_n) \rightarrow e^*(x)$ for each $e^* \in \text{ext } B_{X^*}$. Then $x_n \rightarrow x$ in the weak topology.*

Proof. Apply Corollary 3.1 to $K = B_{X^*}$ and $B = \text{ext } B_{X^*}$. \square

The following theorem was first proved by Rodé (1981) by a completely different method for w^* -compact convex sets. The version stated here was proved by Godefroy (1987) by using a lemma by Simons. The present proof via Theorem 2.6 seems to be “the most elementary” known one.

Theorem 3.3 (Rodé’s theorem (or Rodé–Godefroy theorem)). *Let X be a Banach space, $C \subset X^*$ a closed, convex and bounded set with a separable boundary $B \subset C$. Then $C = \overline{\text{conv}} B$, and the set C is w^* -compact.*

Proof. Define $K = \overline{C}^{w^*}$. Then K is a w^* -compact convex set and B is a boundary for K . Let $\{b_n^*\} \subset B$ be a dense sequence. Given an arbitrary $\varepsilon > 0$, define $C_n := (b_n^* + \varepsilon B_{X^*}) \cap K$ ($n \in \mathbb{N}$), and notice that $B \subset \bigcup C_n$. By Theorem 2.6,

$$\begin{aligned} K &\subset \overline{\text{conv}} \bigcup_n C_n \subset \overline{\text{conv}} (B + \varepsilon B_{X^*}) \\ &\subset \text{conv} (B + \varepsilon B_{X^*}) + \varepsilon B_{X^*} \subset \text{conv } B + 2\varepsilon B_{X^*}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $K \subset \overline{\text{conv}} B \subset C \subset K$, and we are done. \square

Corollary 3.4. *If X is a Banach space such that $\text{ext } B_{X^*}$ is separable, then X^* (and hence also X) is separable.*

Now, we are going to prove the following separable version of Theorem 1.2.

Theorem 3.5 (James). *Let C be a separable, closed, convex and bounded set in a Banach space X . Then C is w -compact if and only if each element of X^* attains a maximum over C .*

Proof. The “only if” part is obvious. To show the “if” part, assume that each element of X^* attains a maximum over C . This means that C , if considered as a subset of X^{**} , is a separable boundary for itself. By Rodé’s theorem, C is w^* -compact in X^{**} , and hence w -compact in X . \square

Corollary 3.6 (James 1957). *A separable Banach space X is reflexive if and only if each element of X^* attains its norm.*