Weak topologies

Weak-type topologies on vector spaces. Let $X$ be a vector space with the algebraic dual $X^\sharp$. Let $Y \subset X^\sharp$ be a subspace. We want to define a topology $\sigma$ on $X$ in order to make continuous all elements of $Y$. Fix $x_0 \in X$. If $\sigma$ is such a topology, then the sets of the form
\[ V_{\varepsilon;g}^{x_0} := \{ x \in X : |g(x) - g(x_0)| < \varepsilon \} = \{ x \in X : |g(x - x_0)| < \varepsilon \} \quad (\varepsilon > 0, g \in Y) \]
are open neighborhoods of $x_0$. But this family is not a basis of $\sigma$-neighborhoods of $x_0$, since the intersection of two of its members does not necessarily contain another member of the family. This is the reason why we instead consider the sets of the form
\[ (1) \quad V_{\varepsilon;g_1,\ldots,g_n}^{x_0} := \{ x \in X : |g_i(x - x_0)| < \varepsilon, i = 1,\ldots,n \} \quad (\varepsilon > 0, n \in \mathbb{N}, g_i \in Y) . \]
It is easy to see that the intersection $V_{\varepsilon;g_1,\ldots,g_n}^{x_0} \cap V_{\varepsilon';h_1,\ldots,h_m}^{x_0}$ of two of such sets contains $V_{\varepsilon'';g_1,\ldots,g_n,h_1,\ldots,h_m}^{x_0}$ where $\varepsilon'' = \min\{\varepsilon, \varepsilon'\}$.

Theorem 0.1. Let $X$ be a vector space, and $Y \subset X^\sharp$ a subspace which separates the points of $X$ (that is, a so-called total subspace).

1. There exists a (unique) topology on $X$ such that, for each $x_0 \in X$, the sets 
\[ (1) \quad V_{\varepsilon;g_1,\ldots,g_n}^{x_0} := \{ x \in X : |g_i(x - x_0)| < \varepsilon, i = 1,\ldots,n \} \quad (\varepsilon > 0, n \in \mathbb{N}, g_i \in Y) . \]
form a basis of neighborhoods of $x_0$. This topology, denoted by $\sigma(X,Y)$, is called the weak topology determined by $Y$.

2. $\sigma(X,Y)$ is the weakest topology on $X$ that makes continuous all elements of $Y$.

3. $(X, \sigma(X,Y))$ is a Hausdorff locally convex t.v.s. for which the sets
\[ V_{\varepsilon;g_1,\ldots,g_n}^{x_0} = \{ x : |g_i(x)| < \varepsilon, i = 1,\ldots,n \} \quad (\varepsilon > 0, g_i \in Y) \]
form a basis of $\mathcal{U}(0)$, consisting of open symmetric convex sets.

4. $(X, \sigma(X,Y))^\star = Y$.

Proof. We only sketch the proof of the first three parts. A set $A \subset X$ will be defined open if, for each $a \in A$, there exist $\varepsilon > 0$, $n \in \mathbb{N}$ and $g_1,\ldots,g_n \in Y$ such that $V_{\varepsilon;g_1,\ldots,g_n}^{a} \subset A$. It is elementary to show that the family of such open sets satisfies all axioms for topology. This topology is Hausdorff since $Y$ separates the points of $X$. Moreover, the discussion before Theorem 0.1 implies 2.

To show that the mappings $(x, y) \mapsto x + y$ and $(x, t) \mapsto tx$ are continuous is an easy exercise.

Let us show 4. The inclusion $(X, \sigma(X,Y))^\star \supset Y$ is clear. To prove the reverse inclusion, we need the following algebraic lemma.

Lemma 0.2. Let $X$ be a vector space. Let $g, f_1,\ldots,f_n \in X^\sharp$ be such that
\[ \bigcap_{i=1}^{n} \text{Ker}(f_i) \subset \text{Ker}(g). \]
Then $g \in \text{span}\{f_1,\ldots,f_n\}$.
Proof. Let us define a linear mapping $T: X \to \mathbb{R}^n$ by

$$ Tx = (f_1(x), \ldots, f_n(x)). $$

Now, we have $\ker(T) \subseteq \ker(g)$, and this easily implies that $g$ is constant on each set $T^{-1}(y)$ ($y \in T(X)$). Let $P: \mathbb{R}^n \to T(X)$ be the orthogonal projection. Define a functional $h: \mathbb{R}^n \to \mathbb{R}$ by

$$ h(\xi) = g(T^{-1}(P\xi)). $$

Then $h$ is well-defined and linear, that is $h \in (\mathbb{R}^n)^\ast$. Thus there exists $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$ h(\xi) = \langle \xi, \alpha \rangle $$

($\xi \in \mathbb{R}^n$).

Then, for each $x \in X$,

$$ \sum_{i=1}^n \alpha_i f_i(x) = \langle Tx, \alpha \rangle = h(Tx) = g(T^{-1}(PTx)) $$

$$ = g(T^{-1}(Tx)) = g(x + \ker(T)) = g(x). $$

□

Now, let $\ell$ be a $\sigma(X,Y)$-continuous linear functional on $X$. There exists a neighborhood of 0 such that $|\ell| \leq 1$ on this neighborhood. This neighborhood contains some basic neighborhood $V_{g_1, \ldots, g_n}$ which, in its turn, contains the subspace $L = \bigcap_{i=1}^n \ker(g_i)$. Since $\ell$ is bounded on $L$, we must have $\ell|_L = 0$; in other words, $\bigcap_{i=1}^n \ker(g_i) \subseteq \ker(\ell)$. By Lemma 0.2, $\ell \in \text{span}\{g_1, \ldots, g_n\} \subseteq Y$. □

The topology $\sigma(X,Y)$ and the product topology on $\mathbb{R}^Y$. Let us recall that the product topology on an arbitrary Cartesian product $P = \Pi_{i \in I} T_i$ of topological spaces is the weakest topology on $P$ for which the canonical projections

$$ \pi_k: P \to T_k, \ \pi_k((x_i)_{i \in I}) = x_k, $$

are all continuous. An open basis of the product topology consists of the sets

$$ A = \Pi_{i \in I} A_i \quad \text{where} \quad A_i \subseteq T_i \quad \text{is open (}i \in I), \text{ and} \ A_i \neq T_i \quad \text{for at most finitely many} \ i\text{'s}. $$

The famous Tychonoff Product Theorem asserts that a Cartesian product of compact topological spaces is compact in the product topology.

A particular case of a product space is $T^I = \Pi_{i \in I} T$, the space of all functions $I \to T$. The corresponding product topology on $T^I$ is the topology of the pointwise convergence.

Now, given a vector space $X$ and a total subspace $Y \subseteq X^2$, we can see every $x \in X$ as a function $x: Y \to \mathbb{R}$, $x(y) = g(x)$. With this identification, $X$ can be viewed as a subset of $\mathbb{R}^Y$, and, moreover, the topology $\sigma(X,Y)$ coincides on $X$ with the product topology of $\mathbb{R}^Y$ (this follows easily from definitions). Thus $(X, \sigma(X,Y))$ is a topological subspace of $\mathbb{R}^Y$, and $\sigma(X,Y)$ is the topology of the pointwise convergence on $Y$. 
The weak topology of a normed space. Let $X$ be a normed space. Since $X^*$ (which is a subspace of $X^t$) separates points by the Hahn-Banach theorem, the topology $\sigma(X,X^*)$ makes $X$ a Hausdorff locally convex t.v.s. This topology is called the weak topology of $X$, and is denoted by $w$ or $w_X$.

Let us list some of its basic properties. (By $\tau_{\|\cdot\|}$ we denote the norm topology.)

(a) A basis of $\mathcal{U}(0)$ in the $w$-topology consists of the sets

$$V_{\varepsilon,y_1^{*},...,y_n^{*}} = \{x \in X : |y_i^{*}(x)| < \varepsilon, \ i = 1, \ldots, n\} \quad (\varepsilon > 0, \ n \in \mathbb{N}, \ y_i^{*} \in X^*) .$$

(b) $(X, w)^* = X^*$ (see Theorem 0.1).

(c) $w \leq \tau_{\|\cdot\|}$ (since the basic neighborhoods from (a) are clearly $\tau_{\|\cdot\|}$-open).

(d) $w = \tau_{\|\cdot\|}$ if and only if dim$(X) < \infty$. (Indeed, if $X$ is infinite-dimensional, then each basic neighborhood $V_{\varepsilon,y_1^{*},...,y_n^{*}}$ contains the nontrivial subspace $\bigcap_{i=1}^{n} \text{Ker}(y_i^{*})$ which is not contained in the open unit ball. If $X$ is finite-dimensional, then $w = \tau_{\|\cdot\|}$ since all vector topologies on $X$ coincide.)

(e) $x_n \overset{w}{\to} x$ if and only if $y^*(x_n) \to y^*(x)$ for each $y^* \in X^*$. (This holds since $w$ is the topology of the pointwise convergence on $X^*$.)

The weak* topology of a dual of a normed space. Let $X$ be a normed space. Then $X$ can be considered as a subspace of $X^{**}$ (by the canonical isometric immersion) and hence also as a subspace of $(X^*)^t$. Moreover, $X$ clearly separates the points of $X^*$. Thus the topology $\sigma(X^*,X)$ is a Hausdorff locally convex vector topology on $X^*$. This topology is called the weak* topology of $X^*$ and is denoted by $w^*$ or $w^*_X$.

Let us remark that the $w^*$ topology of $X^*$ does not depend only on the normed space $X^*$ but also on its predual $X$. There exist examples of distinct Banach spaces $X_1$ and $X_2$, for which $X_1^*$ and $X_2^*$ are isometrically isomorphic but $(X_1^*, w^*)$ and $(X_2^*, w^*)$ are not isomorphic as topological vector spaces. It can also happen that, for a normed space $X$, the topologies $\sigma(X^*,X)$ and $\sigma(X^*,\overline{X})$ (where $\overline{X}$ is the completion of $X$) are different.

Let us list some of the basic properties of the $w^*$-topology.

(a) A basis of $\mathcal{U}(0)$ in the $w^*$-topology consists of the sets

$$W_{\varepsilon,y_1^{*},...,y_n^{*}} = \{x^{*} \in X^* : |x^{*}(y_i)| < \varepsilon, \ i = 1, \ldots, n\} \quad (\varepsilon > 0, \ n \in \mathbb{N}, \ y_i \in X).$$

(b) $(X^*, w^*)^* = X$.

(c) $w^* \leq w_{X^*} \leq \tau_{\|\cdot\|}$.

(d) $w^* = \tau_{\|\cdot\|}$ if and only if $X$ is finite-dimensional (from the same reason as for the weak topology).

(e) $w^* = w_{X^*}$ if and only if $X$ is reflexive. (If $X$ is reflexive, then $w^* = \sigma(X^*,X) = \sigma(X^{**},X^*) = w_{X^*}$. If $w^* = w_{X^*}$, then $X = (X^*, w^*)^* = (X^*, w_{X^*})^* = X^{**}$, that is, $X$ is reflexive.)

(f) $x_n^{*} \overset{w^*}{\to} x^{*}$ if and only if $x_n^{*}(y) \to x^{*}(y)$ for each $y \in Y$. 

Some results concerning weak topologies.

**Theorem 0.3.** Let $X$ be a normed space, $\{x_n\} \subset X$, and $\{x_n^*\} \subset X^*$.

(a) If $\{x_n\}$ is $w$-convergent, then $\{x_n\}$ is bounded.

(b) If $X$ is a Banach space and $\{x_n^*\}$ is $w^*$-convergent, then $\{x_n^*\}$ is bounded.

**Proof.** (a) The points $x_n$ can be viewed as continuous linear functionals on the Banach space $X^*$. Since they are pointwise bounded, the Banach-Steinhaus Uniform Boundedness Principle implies that their norms are equi-bounded. (b) The functionals $x_n^*$ are pointwise bounded on the Banach space $X$. Use again the Uniform Boundedness Principle. □

The following proposition is simple but useful. (The simple arrow “$\rightarrow$” always denotes the convergence in the norm topology.)

**Proposition 0.4.** Let $X$ be a normed space, $\{x_n\} \subset X$, and $\{x_n^*\} \subset X^*$, $x_0 \in X$, $x_0^* \in X^*$. Assume that at least one of the following conditions hold:

(a) $x_n \rightarrow x_0$, $x_n^*(x_0) \rightarrow x_0^*(x_0)$, and $\{x_n^*\}$ is bounded;

(b) $x_n \rightarrow x_0$, $x_n^* \xrightarrow{w} x_0^*$, and either $X$ is a Banach space or $\{x_n^*\}$ is bounded;

(c) $x_n^* \rightarrow x_0^*$ and $x_0^*(x_n) \rightarrow x_0^*(x_0)$, and $\{x_n\}$ is bounded;

(d) $x_n \rightarrow x_0^*$ and $x_n^* \xrightarrow{w} x_0^*$.

Then $x_n^*(x_n) \rightarrow x_0^*(x_0)$.

**Proof.** (a) Put $M := \sup_n \|x_n^*\|$ and observe that $\|x_n^*(x_n - x_0)\| \leq \|x_n^*\| \|x_n - x_0\| \leq M \|x_n - x_0\| \rightarrow 0$. Thus $x_n^*(x_n) = x_n^*(x_n - x_0) + x_n^*(x_0) \rightarrow x_0^*(x_0)$.

(b) follows from (a) since $\{x_n^*\}$ is bounded by Theorem 0.3. (c) and (d) are analogous to (a) and (b), respectively. □

**Remark 0.5.** Notice that $x_n \xrightarrow{w} x_0$ and $x_n^* \xrightarrow{w^*} x_0^*$ is not enough to conclude that $x_n^*(x_n) \rightarrow x_0^*(x_0)$. Consider, for instance, the space $X = c_0$ whose dual is (isometrically isomorphic to) $\ell_1$. Let $e_n \in c_0$ and $e_n^* \in \ell_1$ be the canonical unit vectors, that is, $e_n(i) = e_n^*(i) = \delta_{ni}$. Then $e_n \xrightarrow{w} 0$ and $e_n^* \xrightarrow{w^*} 0$, but $e_n^*(e_n) = 1$ for each $n$. The same example works for $X = \ell_2$.

**Theorem 0.6.** Let $X$ be a normed space.

(a) Every closed convex set $C \subset X$ is $w$-closed.

(b) Every closed ball $B \subset X^*$ is $w^*$-closed.

**Proof.** (a) Let $x \in X \setminus C$. By the H-B Strong Separation Theorem, there exists $y^* \in Y^*$ such that $y^*(x) > \sup y^*(C) =: \sigma$. Observe that the set $\{y^* > \sigma\}$ is a weakly open neighborhood of $x$, disjoint from $C$. This proves that the complement of $C$ is weakly open.

(b) By continuity of algebraic operations, it suffices to show that the closed unit dual ball $B_{X^*}$ is $w^*$-closed. Let $x^* \in X^* \setminus B_{X^*}$, that is $\|x^*\| > 1$. By the definition of the norm of $x^*$, there exists $y_0 \in S_X$ such that $x^*(y_0) > 1$. Then the set $W := \{y^* : y^*(y_0) > 1\}$ is a $w^*$-open neighborhood of $x^*$. Moreover, each of its elements satisfies $\|y^*\| = \|y^*\| \|y_0\| \geq y^*(y_0) > 1$, that is $W \cap B_{X^*} = \emptyset$. □
Theorem 0.7 (Goldstine). Let $X$ be a normed linear space. Then $B_X$ is dense in $(B_{X^{**}}, w^*)$.

Proof. If not, there exists $x_0^{**} \in B_{X^{**}} \setminus C$ where $C$ is the $w^*$-closure of $B_X$ in $X^{**}$. By the H-B Strong Separation Theorem, applied to the locally convex space $(X^{**}, w^*)$, and Theorem 0.1(part 4), there exists $y^* \in Y^*$ with $x_0^{**}(y^*) > \sup y^*(C)$. But then we obtain
\[ \|y^*\| = \sup y^*(B_X) \leq \sup y^*(C) < x^{**}(y^*) \leq \|x^{**}\| \|y^*\| \leq \|y^*\|, \]
a contradiction. \qed

Theorem 0.8 (Banach-Alaoglu). Let $X$ be a normed space. Then every closed ball $B \subset X^*$ is $w^*$-compact.

Sketch of the proof. It suffices to show that $(B_{X^*}, w^*)$ is compact. The elements of $B_{X^*}$ can be considered as the functions $x^*: B_X \to [-1, 1]$. In this identification, the $w^*$-topology corresponds to the product topology on $[-1, 1]^{B_X}$ (which is Hausdorff), and $B_{X^*}$ corresponds to the set $A_0$ of all affine elements of $[-1, 1]^{B_X}$ which are null at the origin. By the Tychonoff Product Theorem, $[-1, 1]^{B_X}$ is compact in the product topology. Moreover, it is an easy exercise to show that $A_0$ is compact in $[-1, 1]^{B_X}$. The proof is complete since $A_0$ is compact and homeomorphic to $(B_{X^*}, w^*)$. \qed

Corollary 0.9. Let $X$ be a normed space. Then every infinite bounded set $E \subset X^*$ has a $w^*$-accumulation point (that is, a point $x^*$ whose each $w^*$-neighborhood intersects $E \setminus \{x^*\}$).

Remark 0.10. In general, it is not true that every bounded sequence would contain a $w^*$-convergent subsequence. However, if $X$ is separable, it is known that $(B_{X^*}, w^*)$ is metrizable, and hence each bounded sequence in the dual of a separable normed space admits a $w^*$-convergent subsequence.

Recall that a normed space $X$ is reflexive if $X = X^{**}$ (when we identify $X$ with its canonical image in $X^{**}$). Since each dual space is complete, every reflexive normed space is necessarily a Banach space.

Theorem 0.11. A normed space $X$ is reflexive if and only if $B_X$ is $w$-compact. (In this case, all closed balls in $X$ are $w$-compact.)

Proof. If $X$ is reflexive, then $(B_X, w) = (B_X, \sigma(X, X^*)) = (B_{X^{**}}, \sigma(X^{**}, X)) = (B_{X^{**}}, w^*)$ and the last topological space is compact.

Let $B_X$ be $w$-compact. This is equivalent to say that $B_X$ is $w^*$-compact in $X^{**}$. Thus $B_X$ is $w^*$-closed in $X^{**}$. By Theorem 0.7, $B_{X^{**}} = B_X$. But this immediately implies that $X^{**} = X$. \qed

Corollary 0.12. Let $X$ be a reflexive Banach space.

1. Each bounded infinite set $E \subset X$ has a $w$-accumulation point.
2. Each bounded $w$-closed set $E \subset X$ is $w$-compact.
3. Each bounded (norm-)closed convex set $C \subset X$ is $w$-compact.
**Proof.** The first two statements follow from Theorem 0.11 since $E$ is contained in some closed ball. The third statement follows from Theorem 0.6(a).

**Remark 0.13.** $(B_X, w)$ is not necessarily metrizable (it is known that this happens if and only if $X^*$ is separable). However, by a famous Eberlein–Šmulian theorem (see Theorem 0.14), every bounded sequence in a reflexive Banach space has a $w$-convergent subsequence.

Three more theorems. Let us state without proofs three important results concerning weak topologies. In all of them, the assumption that $X$ is a Banach space is essential.

**Theorem 0.14** (Eberlein–Šmulian). Let $E$ be a bounded set in a Banach space $X$. Then $E^w$ is $w$-compact if and only if each bounded sequence in $E$ admits a $w$-convergent (in $X$) subsequence.

**Theorem 0.15** (James). Let $C$ be a (nonempty) bounded closed convex set in a Banach space $X$. Then $C$ is $w$-compact if and only if each functional $x^* \in X^*$ attains its supremum over $C$ at some point of $C$.

**Corollary 0.16** (James’ characterization of reflexivity). A Banach space $X$ is reflexive if and only if each element of $X^*$ attains its norm. (This follows from Theorem 0.11 and the James theorem.)

**Theorem 0.17** (Krein–Šmulian). Let $X$ be a Banach space. A convex set $C \subseteq X^*$ is $w^*$-closed if and only if the intersection of $C$ with every closed ball (in $X^*$) is $w^*$-closed.