Continuity of Bilinear Mappings
(L.V., November 2017)

It is well-known that a linear mapping $T\colon X \to Z$ between two normed linear spaces *is continuous if and only if it is continuous at* 0 *if and only if it is bounded, that is, there exists* $C \geq 0$ *such that* $\|Tx\| \leq C\|x\|$ $(x \in X)$.
Moreover, the best such constant C is the operator norm of T, which is given also by

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| \,.$$

We are going to give a short proof of a similar result for bilinear mappings. (Also this is well-known, but maybe less widely known than the theorem about linear mappings.)

In what follows, X, Y, Z will be (real or complex) normed linear spaces. A mapping

$$B\colon X \times Y \to Z$$

is *bilinear* if for every fixed $y \in Y$ and $x \in X$ the mappings

$$B(\cdot, y)\colon X \to Z \quad \text{and} \quad B(x, \cdot)\colon Y \to Z$$

are linear. In other words, a bilinear mapping is a mapping which is linear in each coordinate.

Theorem 0.1. *For a bilinear mapping* $B\colon X \times Y \to Z$ *the following assertions are equivalent:*

(i) B *is continuous;*
(ii) B *is continuous at* $(0,0)$*;*
(iii) B *is* bounded*, that is, there exists* $C \geq 0$ *such that*

$$\|B(x,y)\|_Z \leq C\|x\|_X\|y\|_Y \quad \text{for every } (x,y) \in X \times Y.$$

Moreover, if at least one of the spaces X, Y *is a Banach space then the above properties are equivalent to:*

(iv) B *is* separately continuous*, that is, continuous in each coordinate.*

Proof. The implication $(i) \Rightarrow (ii)$ is trivial.
$(ii) \Rightarrow (iii)$. Suppose that (iii) is false. For each $n \in \mathbb{N}$ there exists $(x_n, y_n) \in X \times Y$ such that $\|B(x_n, y_n)\| > n^2\|x_n\|\|y_n\|$. Since clearly $x_n \neq 0$ and $y_n \neq 0$ $(n \in \mathbb{N})$, we can consider

$$\widetilde{x}_n := \frac{x_n}{n\|x_n\|} \to 0 \quad \text{and} \quad \widetilde{y}_n := \frac{y_n}{n\|y_n\|} \to 0\,.$$

But bilinearity of B now implies that

$$\|B(\widetilde{x}_n, \widetilde{y}_n)\| > n^2 \cdot \frac{1}{n} \cdot \frac{1}{n} = 1 \quad \text{for each } n,$$

and thus (ii) is false.

(iii) $\Rightarrow$ *(i)*. Let (iii) hold, and let $x_n \to x$ in X, and $y_n \to y$ in Y. There exists $M \geq 0$ such that $\|x_n\| \leq M$ and $\|y_n\| \leq M$. Then

$$\begin{aligned}\|B(x_n, y_n) - B(x, y)\| &\leq \|B(x_n, y_n) - B(x_n, y)\| + \|B(x_n, y) - B(x, y)\| \\ &= \|B(x_n, y_n - y)\| + \|B(x_n - x, y)\| \\ &\leq C\|x_n\|\|y_n - y\| + C\|x_n - x\|\|y\| \\ &\leq CM\big(\|x_n - x\| + \|y_n - y\|\big) \to 0\,,\end{aligned}$$

and we are done.

The proof of the second part of our theorem is not elementary since it is based on the following well-known theorem of Functional Analysis:

Uniform Boundedness Principle (Banach–Steinhaus Theorem).
Let X be a Banach space, Z a normed linear space, and let $\mathcal{T}$ be a family of bounded linear operators from X to Z. Assume that the family $\mathcal{T}$ is pointwise bounded, that is, for every fixed $x \in X$ the set $\{Tx : T \in \mathcal{T}\}$ is bounded in Z. Then $\mathcal{T}$ is bounded in the operator norm, that is, $\sup_{T\in\mathcal{T}} \|T\| < +\infty$.

Now we have to prove that (iv) implies any of the (equivalent) conditions (i)–(iii). Let (iv) be satisfied, and let X be a Banach space. For every $y \in Y$ with $\|y\| = 1$, we have the bounded linear operator

$$T_y := B(\cdot, y) \colon X \to Y.$$

Moreover, for every $x \in X$, the linear operator $B(x, \cdot)$ is bounded; denoting $M_x := \|B(x, \cdot)\|$, we obtain

$$\|T_y x\| = \|B(x, y)\| \leq M_x \|y\| = M_x.$$

This means that the family of operators $\mathcal{T} := \{T_y : y \in Y, \|y\| = 1\}$ is pointwise bounded. By the Uniform Boundedness Principle above, $\mathcal{T}$ is bounded in the operator norm, that is, there is a constant C such that $\|T_y\| \leq C$ whenever $y \in Y, \|y\| = 1$. Consequently, for $x \in X$ and $y \in Y$ with $\|y\| = 1$,

$$\|B(x, y)\| = \|T_y x\| \leq C\|x\|.$$

And this easily implies (iii) since for $y \in Y \setminus \{0\}$ we have

$$\|B(x, y)\| = \|B(x, \tfrac{y}{\|y\|})\| \cdot \|y\| \leq C\|x\| \cdot \|y\|.$$

The proof is complete. $\square$

Remark 0.2. It is easy to generalize the above theorem for k-linear mappings as follows. *Let $X_1, \ldots, X_k, Z$ be normed linear spaces, and let $F \colon X_1 \times \cdots \times X_k \to Z$ be a k-linear mapping, i.e., linear in each coordinate. Then the following assertions are equivalent:*

(i) F is continuous;
(ii) F is continuous at $(0, \ldots, 0) \in X_1 \times \cdots \times X_k$;
(iii) F is bounded, that is, $\exists C \geq 0$ s.t.

$$\|F(x_1, \ldots, x_k)\| \leq C\|x_1\| \ldots \|x_k\| \quad \text{for each } x_i \in X_i,\ i = 1, \ldots, k.$$

Moreover, if at least $k-1$ *of the spaces* $X_1, \ldots, X_k$ *are Banach spaces, then the above conditions are also equivalent to:*

(iv) F *is separately continuous, i.e., continuous in each coordinate.*