Fréchet differentiability of Lipschitz functions and Variational Principle

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Fréchet Differentiability of Lipschitz Functions and Porous Sets in Banach Spaces.

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Theorem (Rademacher)

Every Lipschitz map $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is differentiable a.e.*

For a function f from a Banach space X into a Banach space Y the Gâteaux derivative at $x \in X$ is a bounded linear operator $T\colon X \longrightarrow Y$ such that for every $u \in X$,

$$\lim_{t\to 0} \frac{f(x+tu)-f(x)}{t} = Tu.$$

T is called the Fréchet derivative if the above limit holds uniformly in u in the unit ball.

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Every Lipschitz map from a separable Banach space X into a Banach space with the RNP is Gâteaux differentiable almost everywhere.

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Every Lipschitz function f defined on a nonempty open subset G of an Asplund space has a point of Fréchet differentiability. Moreover, for any $a, b \in G$ for which the segment $[a, b]$ lies entirely in G, and for any $\varepsilon > 0$ there is $x \in G$ at which f is Fréchet differentiable and

$$f(b) - f(a) - \varepsilon < f'(x; b - a) < f(b) - f(a) + \varepsilon.$$

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Definition

- Suppose that (M, d) is a metric space and d_0 a continuous pseudometric on M. We say that M is (d, d_0)-complete if there are functions $\delta_j \colon M^{j+1} \longrightarrow (0, \infty)$ such that every d-Cauchy sequence $(x_j)_{j=0}^{\infty}$ converges to an element of M provided

$$d_0(x_j, x_{j+1}) \leq \delta_j(x_0, \dots, x_j) \quad \text{for each } j \geq 0.$$

- We say that a function $f \colon M \longrightarrow \mathbb{R}$ is (d, d_0)-continuous if

$$\lim_{j\to\infty} f(x_j) = f(x)$$

whenever $x_j \in M$ converge in metric d to x and

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Variational Principle.

Let $f\colon M \longrightarrow \mathbb{R}$ be a function bounded from below on a metric space (M,d). Let d_0 be a continuous pseudometric on M such that M is (d,d_0)-complete, and f is (d,d_0)-continuous. Let $F_j\colon M\times M \longrightarrow [0,\infty)$, $j\geq 0$, be functions (d,d_0)-continuous with $F_j(x,x)=0$ for all $x\in M$ and let $r_j \searrow 0$ be such that

$$\inf_{d(x,y)>r_j} F_j(x,y) > 0.$$

If $x_0 \in M$ and $\varepsilon_j > 0$ such that

$$f(x_0) < \varepsilon_0 + \inf_M f(x)$$

then there is sequence $x_j \to x_\infty \in M$ and d_0-continuous function $\varphi \geq 0$ on M such that the function

$$h(x) = f(x) + \varphi(x) + \sum_{j=0}^{\infty} F_j(x,x_j)$$

attains its minimum at x_∞, $d(x_j,x_\infty) < r_j$, and $F_j(x_\infty,x_j) < \varepsilon_j$.

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An illustrative case.

Theorem

Let f be a Lipschitz and everywhere Gâteaux differentiable function on a Banach space X with separable dual. Then f has a point of Fréchet differentiability.

Fact

Let $\Theta\colon X \longrightarrow \mathbb{R}$ *be Fréchet differentiable,* $\psi\colon X \longrightarrow \mathbb{R}$ *continuous, and* $f\colon X \longrightarrow \mathbb{R}$ *Lipschitz and Gâteaux differentiable. Suppose that the function* $h\colon X \times X \longrightarrow \mathbb{R}$*,*

$$h(x,u) = f'(x;u) + \psi(x) + \Theta(u)$$

attains its minimum at (x_0, u_0)*. Then* f *is Fréchet differentiable at* x_0*.*

Fact

Let $f\colon X \longrightarrow \mathbb{R}$ be Lipschitz and everywhere Gâteaux differentiable. Let $M = X \times X$ be equipped with the metric

$$d\big((x,u),(y,v)\big) = \sqrt{\|x-y\|^2 + \|u-v\|^2}$$

and the continuous pseudometric

$$d_0\big((x,u),(y,v)\big) = \|x-y\|.$$

Then the map $(x,u) \mapsto f'(x;u)$ is (d,d_0)-continuous.