CHARACTERISTIC PROPERTIES OF THE GURARIY SPACE.

V. P. Fonf
Ben-Gurion University of the Negev, Isreal

An infinite-dimensional Banach space X is called a Lindenstrauss space if X^* is isometric to $L_1(\mu)$.

A separable Banach space G is called a Gurariy space if given $\varepsilon > 0$ and an isometric embedding $T : L \to G$ of a finite-dimensional normed space L into G, for any finite-dimensional space $M \supset L$ there is an extension $\tilde{T} : M \to G$ with $||\tilde{T}||||\tilde{T}^{-1}|| \leq 1 + \varepsilon$.

The first example of a space G with the property above was given by *Gurariy*.

Also it was proved by *Gurariy* that G has the following property: if $L, M \subset G$ are isometric finite-dimensional subspaces of G and $I : L \to M$ is an isometry then for any $\varepsilon > 0$ there is an extension $\tilde{I} : G \to G$ with $||\tilde{I}||||\tilde{I}^{-1}|| < 1 + \varepsilon$.

It was proved by *Lazar-Lindenstrauss* that a Gurariy space is a Lindenstrauss space and *Lusky* proved that a Gurariy space is isometrically unique. The following 2 properties of the Gurariy space will be important for us:

(M) *Let* $(a_{in})_{i\leq n}$ *be a triangular matrix with vectors* $(a_{1n}, a_{2n}, ..., a_{nn}, 0, 0, ...),\ n = 1, 2, ...,$ *dense in the unit ball of* l_1. *Then the Lindenstrauss space with representing matrix* $(a_{in})_{i\leq n}$ *is the Gurariy space.*

(D) *A separable Lindenstrauss space* X *is the Gurariy space iff* $w^*\mathrm{-cl}\,\mathrm{ext} B_{X^*} = B_{X^*}$.

The initial point of our investigation was the following question: for which pairs $L \subset M$ in the definition of the Gurariy space an extension $\tilde{T}$ may be chosen to be an isometry?

Definition 0.1. *We say that the pair $L \subset M$ of normed spaces has the unique Hahn-Banach extension property (UHB in short) if for any functional $f \in L^*$ there is a unique extension $\hat{f} \in M^*$ with $||\hat{f}|| = ||f||$.*

Note that $x \in S_M$ is a smooth point of S_M iff the pair $L = [x] \subset M$ has UHB.

Theorem 0.2. *Let X be a separable Banach space. TFAE*
(a) $X = G$.
(b) Let $L \subset M, \ \mathrm{codim}_M L = 1,$ be a pair with property (UHB) and let $T : L \to X$ be an isometric embedding of L into X. Then there is an isometric extension $\tilde{T} : M \to X$ of T.

Remark. *The condition UHB in Theorem 0.2 is important. Indeed, let e_1, e_2 be a natural basis of the space $l_1^{(2)}$. Take $L = [e_1]$ and $M = l_1^{(2)}$. Next pick $u_1 \in \mathrm{sm}S_G$ and define $T : L \to G$ by $Te_1 = u_1$. Clearly, T does not have an isometric extension on M.*

Proof of (b)⇒(a).

We will prove (b) $\Rightarrow$

(i) X is a Lindenstrauss space

(ii) $w^* - \mathrm{cl}\,\mathrm{ext} B_{X^*} = B_{X^*}$

X is a Lindenstrauss space:

It is enough to show that

for any finite-dimensional subspace $M \subset X$ and any $\varepsilon > 0$, there is a subspace $N \subset X$ isometric l_∞^n with

$$\min\{d(x, N) : \ x \in M\} < \varepsilon \tag{0.1}$$

We will need the Proposition here:

Proposition 0.3. *Let M be an n-dimensional normed space and $\varepsilon > 0$. Then there is a $2n$-dimensional normed space Z such that*
(i) $M \subset Z$.
(ii) There is a polyhedral subspace $E \subset Z$ with $\theta(M, E) < \varepsilon$.
(iii) There is a chain $M = Y_0 \subset Y_1 \subset Y_2 \subset ... \subset Y_{n-1} \subset Y_n = Z$, such that each pair $Y_{k-1} \subset Y_k$ has UHB and $\mathrm{codim}_{Y_k} Y_{k-1} = 1, \ k = 1, ..., n,$

By using Proposition 0.3 and (b) find a finite-dimensional polyhedral space $Y \subset X$ with $\theta(M, Y) < \varepsilon/2$.

Next:

Definition 0.4. *Let E be a polyhedral finite-dimensional space and* $\mathrm{ext} B_{E^*} = \{\pm h_i\}_{i=1}^n$. *Define* $\psi_E : E \to l_\infty^n$ *as follows*

$$\psi_E x = (h_i(x))_{i=1}^n, \quad x \in E.$$

We call ψ_E *a canonical embedding of* E.

We say that E *is a* fine space *and* B_E *is a* fine polytope *if the pair* $\psi_E(E) \subset l_\infty^n$ *has UHB.*

Proposition 0.5. *Let E be a finite-dimensional polyhedral space and* $\varepsilon > 0$. *Then there are a finite-dimensional polyhedral space* $M,\ M \supset E$, *such that the pair* $E \subset M$ *has UHB, and a fine subspace* $L \subset M$ *with* $\theta(E, L) < \varepsilon$.

Proposition 0.6. *Let* $L \subset M$ *be a pair of finite-dimensional polyhedral spaces with UHB. Then there is a chain*

$$L = L_0 \subset L_1 \subset L_2 \subset ... \subset L_{m-1} \subset L_m = M \tag{0.2}$$

such that for any $k = 0, 1, ..., m-1$, *the pair* $L_k \subset L_{k+1}$ *has UHB and* $\mathrm{codim}_{L_{k+1}} L_k = 1$.

By using Propositions 0.5, 0.6, and (b) we find a fine subspace $L \subset X$ with $\theta(L, Y) < \varepsilon/2$.

Clearly, $\theta(L, M) < \varepsilon$. Finally, by using the definition of a fine space, Proposition 0.6, and (b) we find a subspace $N \subset X$ isometric l_∞^n with (0.1).

So we proved that X is a Lindenstrauss space.

Next we check that $w^* - \mathrm{cl}\,\mathrm{ext}B_{X^*} = B_{X^*}$.

Since X is a separable Lindenstrauss space we have $X = \mathrm{cl}\cup_n X_n,\ X_n = l_\infty^n,\ n = 1, 2, ...$

Clearly, the w^*-topology on B_{X^*} is defined by X_n's.

It is enough to prove that

$\mathrm{cl}\,(\mathrm{ext}B_{X^*}|_{X_n}) = B_{X_n^*}$, for any $n = 1, 2, ...$

Denote $L = X_n = l_\infty^n$.

Let $\{e_i\}_{i=1}^n$ be a natural basis of $l_1^n = L^*$ and $f = \sum_{i=1}^n a_i e_i \in \mathrm{int}B_{L^*},\ \sum_{i=1}^n |a_i| < 1$.

Let $M \supset L$ be l_∞^{n+1} containing L in such a way that if $\{e_i\}_{i=1}^{n+1}$ is a natural basis of $M^* = l_1^{n+1}$ then $e_{n+1}|_L = \sum_{i=1}^n a_i e_i|_L$.

The pair $L \subset M$ has property (UHB).

Let $T : L \to X$ be a natural (isometric) embedding L into X.

By the condition (b) of the theorem there is an isometric extension $\tilde{T} : M \to X$.

By the Krein-Milman theorem there is $e \in \mathrm{ext}B_{X^*}$ with $\tilde{T}^* e = e_{n+1}$.

It is easily seen that $e|_L = f$ which proves that $(\mathrm{ext}B_{X^*})_L = B_{L^*}$.

This completes the $(b) \Rightarrow (a)$.

Corollary 0.7. *Let $L \subset M$ be a pair of finite-dimensional polyhedral spaces (i.e. B_M is a polytope) with UHB. Assume that $T : L \to G$ is an isometry. Then there is an isometric extension $\tilde{T} : M \to G$ of T.*

Proof. Apply Proposition 0.6 and Theorem 0.2, (a)⇒(b) which finish the proof.

Corollary 0.8. $\mathrm{ext} B_G = \emptyset$

Proof. Let $u \in S_G$ and u_1, u_2 be a standard basis of the space $M = l_\infty^2$. If $L = [u_1]$ then the pair $L \subset M$ has (UHB). If $T : L \to G$ is defined by $Tu_1 = u$, then by Theorem 0.2 there is an isometric extension $\tilde{T} : M \to G$. In particular, $||\tilde{T}(u_1 \pm u_2)|| = 1$, which proves that u is not an extreme point of B_G.

Corollary 0.9. *Let Y be a separable smooth Banach space (say $Y = l_2$) and $E \subset Y$ be a finite-dimensional subspace of Y. Assume that $E \subset G$. Then there is a subspace $Z \subset G$ isometric Y with $Z \supset E$.*

Proof. Apply Theorem 0.2 infinitely many times.

Rotations of the Gurariy space:

Theorem 0.10. *For a separable Lindenstrauss space X TFAE:*
(a) Let L_1 and L_2 be 2 isometric polyhedral finite-dimensional subspaces of X such that the pairs $L_1 \subset X$ and $L_2 \subset X$ has UHB, and let $I : L_1 \to L_2$ be an isometry. Then there is a rotation (isometry onto) $\psi : X \to X$ such that $\psi|_{L_1} = I$.
(b) $X = G$.

We only prove $(a) \Rightarrow (b)$.

Proof of Theorem 0.10. (a)⇒(b).

It is enough to prove that $w^* - \mathrm{cl}\,\mathrm{ext} B_{X^*} = B_{X^*}$.
or equivalently:

(d) *If* $X = \mathrm{cl} \cup_n X_n,\ X_n = l_\infty^n, n = 1, 2, ...,$ *then* $\mathrm{cl}\,\mathrm{ext} B_{X^*|_{X_n}} = B_{X_n^*},\ n = 1, 2, ...$

We state a Proposition:

Proposition 0.11. *Let X be a Lindenstrauss space,*

$$X = \mathrm{cl} \cup_n X_n,\ X_n = l_\infty^n,\ \{e_i\}_i \subset \mathrm{ext} B_{X^*},\ e_{n+1|_{X_n}} = \sum_{i=1}^{n} a_{in} e_{i|_{X_n}},\ n = 1, 2, ..., .$$

Let $\{\varepsilon_n\}$ be a sequence of positive numbers with $\sum \varepsilon_n < \infty$. Then there is an increasing sequence $\{E_n\}$ of subspaces of X such that
(1) E_n is isometric l_∞^n and $e_{n+1|_{E_n}} = (1 - \varepsilon_n) \sum_{i=1}^n a_{in} e_{i|_{E_n}},\ n = 1, 2, ...$
(2) $\theta(E_p, X_p) < \sum_{i=p+1}^\infty \varepsilon_i,\ p = 1, 2, ...$ In particular $\mathrm{cl} \cup_n E_n = X$.
(3) Each pair $E_p \subset X$ has UHB.

By Proposition 0.11 we can assume that each pair $X_n \subset X$ has UHB.

A Lemma:

Lemma 0.12. *Let X be a separable Lindenstrauss space. Assume that $X = \mathrm{cl}\cup_{n=1}^{\infty} X_n$, where X_n is an increasing sequence of subspaces such that each X_n is isometric to l_∞^n. Then there is a sequence $\{e_i\}_{i=1}^{\infty} \subset \mathrm{ext}B_{X^*}$ with $w^* - \mathrm{cl}\{\pm e_i\}_{i=1}^{\infty} \supset \mathrm{ext}B_{X^*}$, and such that $\mathrm{ext}B_{X_n^*} = \{\pm e_i|_{X_n}\}_{i=1}^{n},\ n = 1, 2, ...$*

By Lemma 0.12 there is a sequence $\{e_i\}_{i=1}^{\infty} \subset \mathrm{ext}B_{X^*}$ such that $\{\pm e_i|_{X_n}\}_{i=1}^{n} = \mathrm{ext}B_{X_n^*}$, for any n.

Fix an integer p and $\varepsilon > 0$.

Let $\{f_i\}_{i=1}^{q}$ be a finite ε-net in $(1-\varepsilon)B_{X_p^*}$.

Clearly, $f_i = \sum_{j=1}^{p} a_j^i e_i,\ \sum_{j=1}^{p} |a_j^i| \le 1 - \varepsilon.$

Choose a subspace $Y \subset X_{p+q}$, Y isometric to l_∞^p, such that $e_{p+i}|_Y = \sum_{j=1}^{p} a_j^i e_i,\ i = 1, ..., q.$

Another Proposition:

Proposition 0.13. *Let $L \subset M$ be a pair of normed spaces with $L = l_\infty^p$ and $M = l_\infty^q,\ p < q$. Assume that $\{\pm e_i\}_{i=1}^{q} = \mathrm{ext}B_{M^*}$ and $\{\pm e_i\}_{i=1}^{p} = \mathrm{ext}B_{L^*}$. Then*

$L \subset M$ has UHB iff for any $i,\ p+1 \le i \le q$, we have $||e_{i|_L}|| < 1$.

From Proposition 0.13 it follows that $Y \subset X_{p+q}$ has UHB. Since $X_{p+q} \subset X$ has UHB, it follows that $Y \subset X$ has UHB.

Let $I : X_p \to Y$ be a natural isometry of X_p onto Y, i.e.,

$$e_i(Ix) = e_i(x),\ \ x \in X_p,\ \ i = 1, ..., p;\quad e_i(Ix) = f_i(x),\ \ i = p{+}1, ..., p{+}q.$$

By the condition (a) of the theorem there is a rotation $T : X \to X$ such that $T|_{X_p} = I$.

Since T^* is a rotation of X^* it follows that $T^*(\mathrm{ext}B_{X^*}) = \mathrm{ext}B_{X^*}$. In particular, $\{T^*e_{p+i}\}_{i=1}^q \subset \mathrm{ext}B_{X^*}$.

However, $(T^*e_{p+i})|_{X_p} = f_i,\ i = 1, ..., q$.

It follows that $\mathrm{ext}B_{X^*}|_{X_p}$ is an ε-net in $(1 - \varepsilon)B_{X_n^*}$.

Since $\varepsilon > 0$ is arbitrary, it follows that $\mathrm{ext}B_{X^*}|_{X_p}$ is dense in $B_{X_p^*}$.

This finishes the proof of (a)⇒(b).

Extension of finite-dimensional smooth subspaces:

Theorem 0.14. *Let $N \subset G$ be a finite-dimensional smooth subspace of the Gurariy space G. Then there is a smooth subspace $L \subset G$ with $L \supset N$ and $L \neq N$.*

Proof. Put $M = N \oplus \mathbb{R}$ and define in M the norm as follows

$$||(x,t)|| = (||x||^2 + t^2)^{1/2}, \quad x \in N,\ t \in \mathbb{R}.$$

Apply Theorem 0.2 and finish the proof.

Theorem 0.15. *Let X be a separable polyhedral Lindenstrauss space. Then the (Lindenstrauss) space $Y = X \oplus_\infty G$ has the smooth extension property, i.e. for any finite-dimensional smooth subspace $E \subset Y$ there is a finite-dimensional smooth subspace $M \subset Y$ with $M \supset E,\ M \neq E$.*

Theorem 0.16. *Let E be a finite dimensional smooth normed space. Then for every $C(K)$ space with nonseparable dual, there exists an embedding of E in $C(K)$ such that no bigger subspace is smooth.*

Density of smooth subspaces of the Gurariy space.

Theorem 0.17. *For a separable Lindenstrauss space X TFAE:*
(SM) The family $SF(X)$ of all smooth finite-dimensional subspaces of X is θ-dense in the family $F(X)$ of all finite-dimensional subspaces of X.
(G) The space X is the Gurariy space G.