Cones and reflexivity

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Seminario SAA - Milano - 14 maggio 2013

Outline

1. Introduction
   - Our aims
   - Notations and Preliminaries
2. Characterizations of reflexivity
   - Two known results
   - A lemma on weakly compact based cones
   - Characterizations of reflexivity
   - Reflexive Banach lattices
3. Mixed based cones
   - Mixed based cone and nonreflexivity
   - Cones conically isomorphic to ℓ^1_+
4. A different characterization of reflexivity by means of cones

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To study the relationships between the structure of closed convex cones in a Banach space and the structure of the whole space itself.

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More specifically we:

- characterize the reflexivity of a given space through the study of bounded and unbounded bases for closed convex cones;
- introduce the notion of mixed based cone (cone with bounded and unbounded bases) to characterize the nonreflexivity of a Banach space;
- study the properties and the inner structure of the mixed based cones.

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Some Notations.

- Let X be a *real normed space* and let X^* the *norm dual* of X.
- Let $K \subset X$ be a *cone*. Let us suppose that K is *pointed* (i.e. $K \cap (-K) = \{0\}$).
- The *polar cone* of a cone K is the set

$$K^* = \{x^* \in X^* : x^*(k) \geq 0, \forall k \in K\}.$$

- The *strict polar cone* of K is the set

$$K^{*s} = \{x^* \in X^* : x^*(k) > 0, \forall k \in K \setminus \{0\}\}.$$

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Some known facts about polar cone

- The interior of polar and strict polar cone coincide, i.e. $\text{int}K^* = \text{int}K^{*s}$.
- But K^{*s} is not necessarily the interior of K^* (for example, if K is the nonnegative orthant of $\ell^p, 1 < p < \infty$, then $\text{int}K^* = \varnothing$ but K^{*s} is nonempty).
- Moreover the set K^{*s} may be empty. For example, if we consider the space $B([a,b])$ of all bounded functions on the real interval $[a,b]$ endowed with the usual "sup" norm and the standard positive cone

$$K = \{f \in B([a,b]) : f(t) \geq 0 \text{ for all } t \in [a,b]\}$$

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Base for a cone I

Definition

A cone K has a *base* B if B is a convex subset of X such that $0 \notin \mathrm{cl} B$ and

$$\mathrm{cone}\,(B) = \{\lambda b : b \in B, \lambda \geq 0\} = K.$$

- A cone with a base is necessarily *convex* and *pointed*.
- A convex cone K has a base if and only if K^{*s} is nonempty.

If $x^* \in K^{*s}$, the set

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Bases II

- From a standard separation argument between 0 and cl$B$ we obtain that, if K is a convex cone, we can associate to each base B for the cone K a base B_{x^*} defined by a functional $x^* \in K^{*s}$.

- Moreover if B is a bounded base also B_{x^*} is bounded.

Finally we recall a well-known characterization of the existence of a bounded base B_{x^*} for a given cone K (see,e.g. Jameson. Ordered Linear Spaces. Springer Verlag, 1970).

Theorem

B_{x^*} is bounded if and only if $x^* \in \text{int } K^{*s}$.

In the sequel we always suppose that $K^{*s} \neq \emptyset$ for every considered cone K.

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Characterization of reflexivity

- Our aim is to state and prove two new characterizations of reflexivity of a given Banach space X in terms of boundedness of the bases for the closed convex cones of the space X.

To our knowledge, two results in this vein are known in the literature.

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Qiu's Characterization

Theorem

Let X be a Banach space. Then X is reflexive if and only if for each convex cone $P \subset X^*$ admitting a bounded base, the cone

$$P_* = \{x \in X : x^*(x) \leq 0 \text{ for all } x^* \in P\}$$

has nonempty interior.

(see Theorem 1, J. H. Qiu. A cone characterization of reflexive Banach space. J. Math. Anal. Appl. 256 (2001) 39-44)

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Polyrakis' Characterization

Theorem

A Banach space X is reflexive if and only if for any closed cone $K \subset X$ with a bounded base, any strictly positive linear functional of X, continuous on K, attains maximum on the base B_{y^*} for every $y^* \in K^{*s}$.

(see Theorem 11, Y. A. Polyrakis. Demand functions and reflexivity. J. Math. Anal. Appl. 338 (2008) 695-704).
The proof of this theorem is based on:

- if part: characterization of non reflexivity given by D. and V. Mil'man (D.P. Mil'man, V.D. Mil'man, Some properties of non-reflexive Banach spaces, Math. Sb. 65 (1964) 486-497)
- only if part: Polyrakis Dichotomy Theorem.

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Polyrakis Dichotomy Theorem

We recall the mentioned dichotomy theorem since it will be useful in the sequel.

Polyrakis Dichotomy Theorem

If X is a normed space, K a closed cone of X so that the positive part $B_X^+ = B_X \bigcap K$ of the closed unit ball B_X of X is weakly compact, we have: either the base B_{x^*} is bounded for every $x^* \in K^{*s}$ or B_{x^*} is unbounded for every $x^* \in K^{*s}$.

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Lemma

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Let X be a Banach space and let $K \subset X$ be a closed convex cone such that $K^{*s} = \emptyset$. If $K^{*s} = int\ K^{*s}$ then B_{x^*} is weakly compact for every $x^* \in K^{*s}$.

- The proof is based on the James' characterization of weakly compact sets as sets where every continuous linear functionals attains its infimum.
- This lemma and PDT imply a characterization of the coincidence between the interior of the polar cone K^* and the strict polar cone K^{*s} in a Banach space: $K^{*s} = int\ K^{*s}$ if and only if there exists $x^* \in K^{*s}$ such that B_{x^*} is weakly compact.

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First characterization of reflexivity

Theorem

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First characterization of reflexivity

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First characterization of reflexivity - proof

X reflexive $\Longrightarrow$ there exists a closed convex cone K in X such that $int\ K \neq \varnothing$ and $K^{*s} = \text{int } K^{*s}$.

Proof.

Let us consider the closed convex cone

$$K_x = \text{ cone}(x + B_X)$$

where B_X is the unit ball of X and use Polyrakis Dichotomy Theorem. □

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There exists a closed convex cone K in X such that *int* $K \neq \varnothing$ and $K^{*s} = \text{int } K^{*s} \Longrightarrow X$ is reflexive.

Proof.

By Lemma there exists a weakly compact base $B_{y^*} = \{k \in K : y^*(k) = 1\}$.

- We can find a point $x_0 \in K$ such that $\|x_0\| > 2$ and the set $G = x_0 + B_X \subset K$.
- We show that G is weakly compact showing that there exists $\alpha > 0$ such that

$$G \subset \bigcup_{0 \leq \beta \leq \alpha} \beta B_{y^*} = \{\beta b : 0 \leq \beta \leq \alpha, b \in B_{y^*}\}.$$

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Second characterization of reflexivity

Theorem

Let X be a Banach space. Then X is reflexive if and only if for every closed convex cone K in X such that int $K^{*s} \neq \varnothing$ we have $K^{*s} = \text{int } K^{*s}$.

- The *first* characterization of reflexivity is based on the *existence of only one cone K* satisfying assumptions involving both *the structure of the cone K* itself and *its polar*.
- On the other hand, the *second* characterization of reflexivity of the space X involves only the *structure of the polars* but it concerns *all the closed convex cones* in X *with a bounded base*.

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A reformulation

- The previous Theorem means that a Banach space X is reflexive if and only if every closed and convex cone K is such that $K^{*s} = \text{int } K^{*s}$ or $\text{int } K^{*s} = \varnothing$.
- Hence the theorem can be reformulated in the following way:

The Banach space X is reflexive if and only if each closed convex cone K (with a base) in X is such that either a bounded base for K exists (hence all the bases are bounded) or every base for K is unbounded.

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Second characterization of reflexivity - proof

X reflexive $\Longrightarrow$ for every closed convex cone K in X such that *int* $K^{*s} \neq \varnothing$ we have $K^{*s} = \textit{int } K^{*s}$.

Proof.

Let $x^* \in \text{int } K^{*s} \neq \varnothing$.

- The set B_{x^*} is weakly compact base for the cone K.
- Therefore $K^{*s} = \text{int } K^{*s}$ by Polyrakis Dichotomy Theorem.

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Second characterization of reflexivity - proof

X reflexive $\Longrightarrow$ for every closed convex cone K in X such that *int* $K^{*s} \neq \varnothing$ we have $K^{*s} =$ *int* K^{*s}.

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For every closed convex cone K in X such that $int\ K^{*s} \neq \varnothing$ we have $K^{*s} = int\ K^{*s} \Longrightarrow X$ reflexive.

Proof.

- Let us consider the closed convex cone

$$K = \text{ cone } (x + B_X).$$

- int $K \neq \varnothing$ and int $K^{*s} \neq \varnothing$.
- Apply the first characterization of reflexivity.

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For every closed convex cone K in X such that $int\ K^{*s} \neq \emptyset$ we have $K^{*s} = int\ K^{*s} \Longrightarrow X$ reflexive.

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   - A lemma on weakly compact based cones
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Reflexive Banach Lattices

Our last result allow us to say that in a reflexive space there are only two types of closed convex cone (with base):

- those with bounded bases only,
- those with unbounded bases only.

- Now, let us consider a *reflexive Banach lattice* X endowed with the order relation $\leq$. Let us denote by $X_+ = \{x \in X : x \geq 0\}$ its *lattice cone*. What can we say about a lattice cone with respect to our classification?

- It is known that the lattice cone $X_+$of an infinite dimensional reflexive Banach lattice has only unbounded base.

Here, we give a theorem that extends this result.

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A result on Banach lattices

Theorem

Let X be an infinite dimensional Banach lattice and let X_+ its lattice cone. If there exists $x^* \in (X_+)^{*s}$ such that the base B_{x^*} of X_+ is bounded then X contains a sublattice isomorphic to ℓ^1.

It is easy to see that from this result we can obtain the mentioned known result.

Corollary

Let X be a reflexive Banach lattice and let X_+ its lattice cone. Then the cone X_+ has not a bounded base.

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Some notations

Let X be a Banach lattice with lattice cone X_+. We recall the following notations.

- $\sup\{y,z\} = y \vee z, \quad \inf\{y,z\} = y \wedge z,$
- $x^+ = \sup\{x,0\}, \quad x^- = \inf\{-x,0\}, \quad |x| = x^+ + x^-,$

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The proof

Proof.

- It is known that there exists a sequence $\{x_n\} \subset X_+ \setminus \{0\}$ of disjoint vectors, i.e., $|x_n| \wedge |x_m| = 0$ for every $n, m \in \mathbb{N}$.
- Without loss of generality, we can always suppose that $\{x_n\} \subset S_X$.
- Now, by contradiction let us suppose that X does not contain any sublattice isomorphic to ℓ^1. Hence, a known result (see,e.g, the book of Meyer-Nierberg) implies that $x_n \rightharpoonup 0$.
- Since the base B_{x^*} is bounded, every weakly converging to 0 sequence in X_+ is norm converging (Kountzakis & Polyrakis,JMAA 2006), a contradiction.

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A characterizazion of nonreflexivity

The second characterization of reflexivity gives us the related characterization of nonreflexive space.

Theorem

A Banach space X is nonreflexive if and only if there exists a closed convex cone $K \subset X$ such that int $K^{*s} \neq \varnothing$ and int $K^{*s} \neq K^{*s}$.

- Since a closed convex cone K such that int $K^{*s} \neq \varnothing$ and int $K^{*s} \neq K^{*s}$ has simultaneously a bounded base and an unbounded base we say that such a cone is a *mixed based cone*.
- Now we give two examples of mixed based cones that will play an important role in the sequel: the nonnegative orthant of ℓ^1 and the c_0-summing cone.

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The nonnegative orthant of ℓ^1

Example

Let ℓ^1 and ℓ^∞ the classical Banach spaces endowed with their usual norms.

- The nonnegative orthant of the space ℓ^1 is

$$\ell^1_+ = \{x = (x_i) \in \ell^1 : x_i \geq 0 \text{ for every } i\}.$$

-

$$\left(\ell^1_+\right)^{*s} = \{x^* = (x_i^*) \in \ell^\infty : x_i^* > 0 \text{ for every } i\}.$$

- Let $x^* \in \left(\ell^1_+\right)^{*s}$, then the base B_{x^*} for the cone ℓ^1_+
  - is bounded for every $x^* \in \left(\ell^1_+\right)^{*s}$ such that $x_i^* \geq \alpha > 0$,
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The c_0-summing cone

Example

Let c_0 the spaces of real sequences $x = (x_i)$ converging to 0 endowed with the usual norm.

- Let $\{b_n\}$ be the summing basis of c_0, ($b_n = \sum_{k=1}^n e_k$ where $\{e_n\}$ is the standard Schauder basis of c_0.
- The c_0-summing cone is the cone

$$K_{summ}(c_0) = \left\{ \sum_{k=1}^{\infty} \lambda_k b_k \in c_0 : \lambda_k \geq 0 \text{ for each } k \right\}.$$

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The c_0-summing cone

Example

- We have

$$(K_{summ}(c_0))^* = \left\{ y^* = (y_i^*) \in \ell^1 : \sum_{i=1}^{n} y_i^* \geq 0 \text{ for each } n \right\},$$

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- Moreover, $int\,(K_{summ}(c_0))^* \neq \varnothing$ and

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- Let $y^* \in (K_{summ}(c_0))^{*s}$, the base B_{y^*} for the cone $K_{summ}(c_0)$ is bounded whenever $\sum_{i=1}^{\infty} y_i^* > 0$.
- We remark that the cone $K_{summ}(c_0)$ has also an unbounded base.
- Indeed, let us consider the base B_{z^*} where

$$z^* = \left(1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \ldots\right).$$

It easy to observe that $z^* \in (K_{summ}(c_0))^{*s} \setminus int\,(K_{summ}(c_0))^*$.

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Conical isomorphism between cones

The first example mentioned above is especially relevant because we show that the notion of mixed based cone is strictly linked with the notion of cone conically isomorphic to the cone ℓ^1_+.

Definition

Let X and Y be two normed spaces. The cone $P \subset X$ is said to be *conically isomorphic* to the cone $K \subset Y$ if there exists a map $T : P \to K$ such that

- T is additive and positively homogeneous,
- T is one-to-one map T of P onto K,
- T and T^{-1} are continuous in the induced topologies.

Then we also say that T is a *conical isomorphism* of P onto K.

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Cones conically isomorphic to ℓ^1_+

- We focus our attention on the class of cones conically isomorphic to the positive cone ℓ^1_+ of the classical space of sequence ℓ^1.
- The properties of this class is widely studied (see, e.g. the papers by Polyrakis).
- We recall also that the idea to consider the cones conically isomorphic to the positive cone of ℓ^1 amounts to the famous characterization of nonreflexivity given by D.P. Mil'man and V.D. Mil'man (1964).

Theorem (Mil'man & Mil'man)

A Banach space X is nonreflexive if and only if the cone ℓ^1_+ is embeddable in X by a conical isomorphism.

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- We recall also that the idea to consider the cones conically isomorphic to the positive cone of ℓ^1 amounts to the famous characterization of nonreflexivity given by D.P. Mil'man and V.D. Mil'man (1964).

Theorem (Mil'man & Mil'man)

A Banach space X is nonreflexive if and only if the cone ℓ^1_+ is embeddable in X by a conical isomorphism.

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Some known facts about cones conically isomorphic to ℓ^1_+

- Let X be a Banach space and $Q \subset X$ be a closed convex cone conically isomorphic to the cone ℓ^1_+.
- We denote by T the conical isomorphism of ℓ^1_+ onto Q.
- We recall that ℓ^1 is a Banach lattice, whose lattice cone is ℓ^1_+. Hence $\ell^1 = \ell^1_+ - \ell^1_+$ and

$$x = x^+ - x^- = \sup\{x, 0\} - \sup\{-x, 0\} \quad \text{for every } x \in \ell^1.$$

- Then the conical isomorphism T can be extended to a one-to-one linear operator of ℓ^1 onto the subspace $Q - Q$ by taking

$$T(x) = T(x^+) - T(x^-).$$

- Moreover, it is proved that the considered extension of T is continuos on the whole space ℓ^1. (Use the continuity of T and $\|x\| = \|x^+\| - \|x^-\|$ for every $x \in \ell^1$).

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Every cone conically isomorphic to ℓ^1_+ is mixed based

Theorem

Let X be a Banach space and $Q \subset X$ be a closed convex cone conically isomorphic to the cone ℓ^1_+ then Q is a mixed based cone such that $\mathrm{int}\, Q = \varnothing$.

Proof.

(Sketch) Let T be the continuous extension of the isomorphism between ℓ^1_+ and Q.

- Since ℓ^1_+ has a bounded base there exists $q^* \in \mathrm{int} Q^*$.
- ℓ^1_+ has also an unbounded base U. By the continuity of T^{-1} at 0, we have that $T(U)$ is an closed unbounded base for Q.
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Now we prove a sort of the converse of the previous Theorem.

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Let X be a Banach space. If is a closed mixed based cone $K \subset X$ then there exists a conical isomorphism of ℓ^1_+ onto a cone $Q \subseteq K$. Moreover only three cases occur:

(i) *ℓ^1 embeds in X (Q is conically isomorphic to ℓ^1_+),*

(ii) *c_0 embeds in X (Q is conically isomorphic to $K_{summ}(c_0)$),*

(iii) $Q = \{q \in X : q = \sum_{i=1}^{+\infty} \alpha_i q_i, \quad \alpha_i \in \mathbb{R}, \alpha_i \geq 0$ *for each* $i\}$ *where $\{q_n\} \subset X$ is a strongly summing sequence.*

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Strongly summing sequence

Definition

Let $\{x_n\}$ be a sequence in a Banach space X. $\{x_n\}$ is a *strongly summing* sequence if $\{x_n\}$ is a weak Cauchy basic sequence such that whenever the sequence of real numbers $\{\gamma_n\}$ satisfy

$$\sup_j \left\| \sum_{n=1}^{j} \gamma_n x_n \right\| < \infty$$

then $\sum_{n=1}^{\infty} \gamma_n$ converges.

- This notion was introduced by H.P. Rosenthal in 1994, to obtain his well known subsequence principle characterizing c_0.
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Examples of the three cases

- (i) the nonnegative orthant ℓ^1_+ of ℓ^1,
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It remains to find an example where the item (iii) of the Theorem occurs.

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Example

Let J be the space of all real sequences $x = (x_n)$ such that $\lim_{n\to\infty} x_n = 0$ endowed with the norm

$$\|x\|_J = \sup \left(\frac{1}{2} \sum_{i=0}^{n} \left(x_{p_{i+1}} - x_{p_i}\right)^2 \right)^{\frac{1}{2}}$$

where $x_0 = 0$ and the sup is taken over all choices of n and all positive integers $0 = p_0 < p_1 < \ldots < p_{n+1}$.

- The space J is the famous Banach space known as James space.
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- The set $\{e_n^*\}$ of biorthogonal functionals associated to $\{e_n\}$ is a basis for J^*.
- The second dual of J is given by $J^{**} = \imath(J) \oplus [\mathbf{1}]$ where $\imath$ is the canonical injection of J into J^{**} and $[\mathbf{1}]$ is closed linear span of the functional given by $\mathbf{1}(e_n^*) = 1$ for every n.
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$$(K_{J^*})^* = \{x^{**} \in J^{**} : x^{**} = (\gamma_n), \quad \gamma_n \in \mathbb{R}, \gamma_n \geq 0 \text{ for each } n\}.$$

- The cone K_{J^*} is an example of the situation that occurs in the point (iii) of our theorem.
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Two corollaries (embedded subspaces)

Corollary

If X is a weakly complete Banach space and X contains a mixed based cone K then ℓ^1 embeds in X.

Corollary

Let X be a Banach space such that X^* is weakly complete. If X contains a mixed based cone then X contains either c_0 or ℓ^1.

Proof.

By a result of Rosenthal, if X contains a strongly summing sequence, then X^* is not a weakly complete space. The case (iii) in our theorem does not occur ... □

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The key point of the proof of our result is a application of both these theorems due to H.P. Rosenthal.

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Every bounded sequence in a Banach space has either a weak Cauchy subsequence or a subsequence equivalent to the standard basis of ℓ^1.

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- Let $\{y_n\} \subset B_{y^*}$ be such that $\|y_n\| \to \infty$.
- there exist $\{\lambda_n\} \subset \mathbb{R}_+$such that $\lambda_n \to +\infty$, $\{x_n\} \subset B_{x^*}$ and ω,Ω such that

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Sketch of the proof - II

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- We show that $\left\{x_{n_j}\right\}$ is not a trivial weak-Cauchy sequence.
- Using Rosenthal's c_0-theorem we have again two cases:
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- We show that in both cases we have a conical isomorphism of ℓ^1_+ onto the cone generated by the subsequence $\left\{x_{n_{j_s}}\right\}$.

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Almost Final Remark

- The Mil'man's characterization of nonreflexivity can be proved in a new way combining our results.
- Our approach also allows us to give a deep insight in the structure of a cone conically isomorphic to ℓ^1_+ and to the cones in a nonreflexive setting.

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Reflexive cones

We introduce a new class of cones. Let X be a Banach space and let us denote by B_X by the closed unit ball of X.

Definition

A cone $P \subset X$ is *reflexive* when the set

$$B_X^+ = B_X \cap P$$

is weakly compact.

- A reflexive cone is always closed.
- If X is reflexive, every closed cone of X is reflexive.

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Reflexive cones and mixed based cones

The notions of reflexive and mixed based cone are related. Indeed from Polyrakis Dichotomy Theorem it follows:

Theorem

Any reflexive cone is not a mixed based cone.

- The converse of this theorem does not hold (see,e.g., c_0^+).

Nevertheless a characterization in terms of the absence of a cone conically isomorphic to ℓ^1_+ can be formulated.

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A closed cone $P \subset X$ is reflexive if and only if P does not contain a cone conically isomorphic to ℓ^1_+.

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Characterization of reflexive cones

A different characterization of reflexive cones can be provided in the spirit of the definition of reflexivity for a whole space.

Given a Banach space X, let $J_X : X \to X^{**}$ the natural embedding of X in X^{**}.

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A closed cone P of a Banach space X is reflexive if and only if

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Characterization of reflexivity

The previous result allow us to prove a new characterization of the reflexivity of a Banach space based on the notion of reflexive cones.

Theorem

A Banach space X is reflexive if and only if there exists a closed cone $P \subset X$ such that the cones P and P^* are reflexive.

The following decomposition property of reflexive cones plays a key role in the proof of the previous theorem.

- If P is a reflexive cone then

$$P^{***} = J_{X^*} \oplus (J_X(X))^{\perp}$$

where $(J_X(X))^{\perp} = \{x^{***} \in X^{***} : x^{***}(x^{**}) \, \forall \, x^{**} \in J_X(X)\}$.

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Some References

The talk is mainly based on the two following papers:

- E. Casini, E. Miglierina, Cones with bounded and unbounded bases and reflexivity, *Nonlinear Analysis* **72** (2010), 2356-2366.
- E. Casini, E. Miglierina, I.A. Polyrakis, F. Xanthos, Reflexive cones, *Positivity* (published online November 11, 2012).