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COVERING THE SPHERE AND THE BALL IN BANACH SPACES

Notations: X (real) Banach space

$B(\bar{x}, r) = \{x \in X ; \|x - \bar{x}\| \leq r\}$ (ball)

$B_X = B(\theta, 1)$ $\qquad S_X = \partial B_X$

I shall discuss some results of this type:

Cover S_X, or B_X, by means of (sets, or) balls, not too big; not too many

Particular cases:

- The centers of the balls must be on S_X
- We look for a covering of S_X not covering θ

extensions: • "small" coverings

• thickness (parameter)

finite versus infinite dimensional

A simple example $\cancel{X}=(\mathbb{R}^2, \| \ \|)$

Let B_1 B_2 B_3 cover S_X;

radii ≤ 1. Then they cover B_X

Proved first in $\mathbb{R}_2^2$ (MOLNAR 1958)

Then for smooth, strictly convex norms

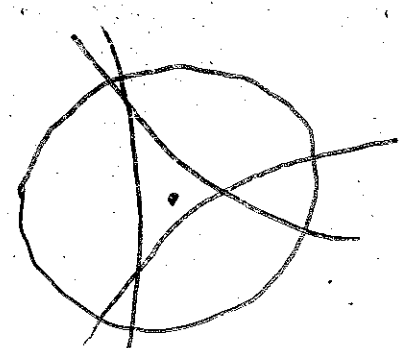
(ASPLUND - GRUNBAUM 1961)

Discussion for the general case:

MARTINI - SPIROVA 2007

Easy to see: 1 is the

"critical value"



Of course, in general, covering S_X is different from covering B_X; but

$$\forall X: \quad S_X \subset \bigcup_{i=1}^{n} B(x_i, \alpha_i);$$

$$x_i \in S \quad \text{and } \alpha_i \geq 1 \ \forall i$$

$$\Longrightarrow B_X \subset \bigcup_{i=1}^{n} B(x_i, \alpha_i)$$

Also: if $\dim(X) < \infty$ (and only if) for any $\varepsilon > 0$ we can find $x_1 \dots x_n$ such that $S_X \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$.

THE FOLLOWING CAN BE STUDIED:
what is the minimal number $f(\varepsilon)$ such that this can be done with $n = f(\varepsilon)$?
Of course $f(\varepsilon)$ [also] depends on the dimension of X, and on the norm used

"folklore" results:

If $\dim(X)=\infty$ and $S_X \subset \bigcup_{i=1}^{n} B(x_i, \alpha_i)$,
then $|\theta|$, but also $B_X \subset$ ....

Related questions (still $\dim(X)=\infty$):

does a finite covering of A by means of balls, or by means of closed convex sets, also cover $\overline{co}(A)$?

=

– BALL COVERING (– BC)

The following notion was considered, for $\dim(X)=\infty$, starting from ~ 2005 (L. X. CHENG). Say that S_X has a countable – BC if $x_1, \dots x_n \dots$ exist s.t.

$$S \subset \bigcup_{i=1}^{\infty} B(x_i, \alpha_i) \quad ; \quad \theta \notin \bigcup_{i=1}^{\infty} B(x_i, \alpha_i)$$

APPARENTLY THE SITUATION CONCERNING THIS PROBLEM IS ALREADY CLEAR, NOTWITH-STANDING SOME MISTAKES IN THE LITERATURE (CONTRIBUTIONS BY L.X. CHENG, V. FONF, C. ZANCO) + ...

MANY RECENT RESULTS IN GOOD JOURNALS

Some results: S_X has a countable BC

$$\Downarrow \qquad \not\Uparrow \quad \text{(some renorming)}$$

X^* is w^*-sep

A PARAMETER

If $\dim(X) = \infty$, if a finite family of balls cover S_X, then at least one of them must contain a pair $(-x, x)$, $x \in S$.

(LIUSTERNIK - SCHNIREL'MAN 1930; BORSUK 1933)

$\Rightarrow$ if the centers of balls are on S_X, then the radius of one of them is ≥ 1

Define $T(X) = \inf\{\varepsilon > 0 ;$ we can cover S_X with finitely many balls centered on S_X, with radii $\leq \varepsilon\}$ (equivalently: we can cover B_X ...)

WHITLEY 1968

Properties for $\dim(X) = \infty$: $(\dim(X) < \infty \Leftrightarrow T(X) = 0)$

- $1 \leq T(X) \leq 2$ always
- $T(X) = 1$ if $X = \ell_\infty, c, c_0$
- $T(X) = 2$ if $X = \ell_1, C[0,1]$ $L_1[0,1]$
- $T(\ell_p) = 2^{1/p}$ $1 < p < \infty$ $(T(H) = \sqrt[2]{2})$
- $T(X) = 1 \Rightarrow X$ not UNS $\not\Leftarrow$
- $T(X) = 2 \Rightarrow X$ not UNS $\not\Leftarrow$
- $T(\cancel{X}) = \sqrt{2} \not\Rightarrow X$ Hilbert

$(T(c_0 \oplus_2 \ell_1) = T(\mathbb{R} \oplus_2 \ell_\infty) = \sqrt{2}$

- upper bounds can be given by using $\delta(\varepsilon)$

$T(X) = 2$ well characterized

($\cong$ space containing isomorphically ℓ_1)

Pb. $T(X) = 1 \iff ???$

$T(X^{**}) \leq T(X)$; $= ?$

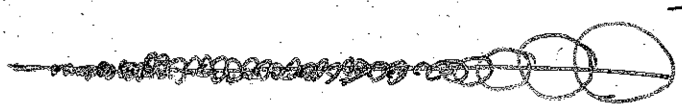
SMALL COVERING (by balls).

We say that $A \subset X$ is <u>small</u> if

$\forall \varepsilon > 0$ there is a sequence of balls

$B(x_i, \alpha_i)$ s.t. - $A \subset \bigcup_{i=1}^{\infty} B(x_i, \alpha_i)$

- $\alpha_i \searrow 0$



- $\alpha_1 \leq \varepsilon$

More generally, we can define the

REMARK $X \subset Y$: we can have both

$T(X) > T(Y)$ $\quad X = \ell^2 \quad Y = \mathbb{R} \oplus_\infty \ell^2$

$\sqrt{2} \qquad 1: \begin{matrix} 1\,0\,0\,0\ldots \\ -1\,0\,0\,0\ldots \end{matrix}$

or

$T(X) < T(Y)$ $\quad X = c_0 \quad Y = c_0 \oplus_1 \ell_2$

$1 \qquad \sqrt{2}$

In many classical spaces (c_0, ℓ_p, ℓ_∞, L_1...) the most "economic" covering (concerning size) of S or B, by a finite number of balls centered on S can in fact be done by two balls. More precisely, we have:

$$T(X) = j(X) = \inf_{x \in S} \left[\sup_{y \in S} \inf \{ \|x-y\|, \|x+y\| \} \right]$$

But this is not always true in general:

$$\sqrt{2} = T(X) < j(X) = 3/2 \quad \text{for } X = \mathbb{R} \oplus_1 \ell_2$$

PROBLEMS. Covering S by $B(x,r)$, $B(y,r)$ $(x, y \in S)$ implies the existence of a similar covering by using a pair $\bar{x}$, $-\bar{x}$?

smallness of A in this way: L8

$sm(A) = \inf \{ \varepsilon > 0$; there exist a sequence $B(x_i, \alpha_i)$ of balls s.t. $A \subset \bigcup_{i=1}^{\infty} B(x_i, \alpha_i)$; $\alpha_i \leq \varepsilon \; \forall i$; $\lim_{i \to \infty} \alpha_i = 0 \}$

MANY RESULTS ON THESE NOTIONS ~ 1967-1971 (DUDA, CONNETT, KÖRNER ...)

MORE SISTEMATICALLY PUBLISHED IN 1988 (CASCALES DE REYNA); LATER, AGAIN, IN 2005-06: BEHRENDS (+ KADETS) + SOME POLISH MATHEMATICIANS

FACTS: $sm(S_X) = sm(B_X) = T(X)$

(NOT VERY SIMPLE) CASTILLO + P.

LEMMA NEEDED: AN ε-SMALL COVERING OF S_X, ALSO COVERS B_X

SCHRAMM 1988 $f(d) \leq (\sqrt{3/2} + \varepsilon)^d$ $\forall \varepsilon, \forall d$

again: BOURGAIN-LINDENSTRAUSS

KAHN-KALAI 1992 $f(d) \geq (1.2)^{\sqrt{d}}$ $\forall d >>$

simple paper

$f(d) > d+1$ $d = 1325$

$+ \forall d > 2014$

1993 - - $d = 946 \binom{44}{2}$

combinatorial arguments

$d=4$? ? ? $f(d) = ??$ (≤ 9)

HINRICHS RICHTER $f(d) > d+1$ if $d = 298$

Related problems:

Covering by homothetic sets

KADETS: Covering by bodies with "small" interior

Borsuk PROBLEM

Let $X = E^d$. Set

$f(d) = \inf \{ n \in \mathbb{N} ;$ for every $A \subset E^d$, $\delta(A) = 1$, there is a covering with $\leq n$ sets A_i, $i = 1 \ldots n$, $\delta(A_i) < 1 \ \forall i \}$

Borsuk 1933: $f(2) \leq 3$ ($f(d) \geq d+1$ for every d: think at the simplex)

Conjecture: $f(d) = d+1 \quad \forall d$ (B. PB) ※

$f(3) = 4$: Eggleston 1955 (then also GRUNBAUM 1957, HEPPES 1957)

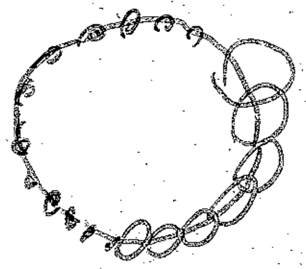
※ is true with some additional assumptions:

A convex body centrally symmetric 1981
A " " smooth 1988
A " " with many reg points 1995
A - - - of revolution DEKSTER with bills

LASSAK 1982 $f(d) \leq 2^{d-1} + 1$

Finite covers give a measure of noncompactness; ε-small covering give another kind of measure

(- any σ-compact set is small, but not only; countable unions of small sets are small ...)



THANK YOU!