Category Theory

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1. Let $Alg(1)$ be the category whose objects are sets with a unary operation, with no axioms, and morphisms are functions which preserve such an operation. Show that the assignment $F(X) = (X \times \mathbb{N}, \lambda)$, where $\lambda(x, n) = (x, n + 1)$, gives rise to a functor $F : Set \to Alg(1)$ which is a left adjoint. Then describe the free algebras of the associated monad on Set.

2. By definition an object P in an abelian category $\mathbb{C}$ is said to be projective if, for any epimorphism $e : M \to N$ and any arrow $f : P \to N$, there exists an arrow $\tilde{f} : P \to M$ such that $e\tilde{f} = f$

   Show that an object P is projective if and only if in any sequence

   $$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} P \longrightarrow 0$$

   with f monomorphism and g=coker (f), g is a split epimorphism.

3. Let $\mathbb{C}$ be a cartesian closed category, and let T be a terminal object in $\mathbb{C}$. Show that , for any C in $\mathrm{Ob}(\mathbb{C})$, C^T is isomorphic to C.

4. Let $\mathbb{C}$ be a small regular category.

   Show that the collection $T(C) = \{S | S$ sieve on C which contains at least a regular epimorphism with codomain $C\}$ for any C in $\mathbb{C}$ is a Grothendieck topology on $\mathbb{C}$.