Category Theory

February 25, 2015

1. Let $\mathbb{C}$ be a category with finite sums. Given a fixed object C in $\mathbb{C}$, for any other object X, let $T(X) = X + C$, $\eta_X : X \to X + C$ be the canonical arrow into the sum and $\mu_X : X + C + C \to X + C$ be $1_X + \nabla_C$, where $\nabla_C = [1_C, 1_C]$ is the codiagonal of C.

   (i) Show that the above assignments give rise to a monad (T, η, μ) on $\mathbb{C}$.

   (ii) Describe the category of the algebras for such a monad in the case $\mathbb{C} = Set$ and $C = 1$, the terminal object.

2. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_A} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle 1_C} & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

   be a morphism of exact sequences in an abelian category $\mathbb{C}$

   Show that α is an isomorphism.

3. Let $\mathbb{C}$ be a small category with pullbacks.

   (i) Show that the collection of all non-empty sieves on objects of $\mathbb{C}$ is a Grothendieck topology

   (ii) Show that in case of $\mathbb{C} = \bullet \rightrightarrows \bullet$ the above assertion is not true.

4. (Just if you want to do it!) Let $\mathbb{C}$ be a category with a subobject classifier $t : 1 \to \Omega$. If A is a suboject of B, and B is a subobject of X, so that A is also a suboject of X, how are related the characteristic arrows $\chi_A : B \to \Omega$ and $\overline{\chi}_A : X \to \Omega$?