

E, L $\text{Ext}(E, L) = \frac{\text{classi di } L}{\text{di estensioni}}$

$$0 \rightarrow L \xrightarrow{f} R \xrightarrow{g} E \rightarrow 0$$

$$\downarrow f' \quad \downarrow h' \quad \downarrow g' \quad \downarrow \Omega'$$

$$0 \rightarrow L \oplus L \xrightarrow{f \oplus f'} M \oplus M' \xrightarrow{g \times g'} E \otimes E \rightarrow 0$$

$$\parallel \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \Delta = \langle 1, 1 \rangle$$

$$0 \rightarrow L \oplus L \longrightarrow \Delta^!(M \oplus M') \longrightarrow E \rightarrow 0$$

$$\nabla = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \parallel$$

$$0 \rightarrow L \xrightarrow{\Gamma} \nabla_! \Delta^!(M \oplus M') \longrightarrow E \rightarrow 0$$

$M + M'$ CORR A DI BAEK
di Ω e Ω'

Poiché $M \oplus M' \cong M' \oplus M$

CORR A DI BAEK È CORR STATUA

Le classi $\underline{o} \in \text{Ext}(E, L)$ è
quelle delle estensioni spettanti.

$$0 \rightarrow L \xrightarrow{i_L} L \oplus E \xrightarrow{\pi_E} E \rightarrow 0$$

Rapp. come nco

$$\underline{\pi}: 0 \rightarrow L \xrightarrow{f} \Pi \xrightarrow{g} E \rightarrow 0$$

$$\begin{array}{ccc} & ? & \rightarrow E \\ & \downarrow & \downarrow \Delta \\ 0 \rightarrow L \oplus L & \rightarrow \Pi \oplus L \oplus E & \xrightarrow{g \times \pi_E} E \oplus E \rightarrow 0 \\ & (m, l, e) & \xrightarrow{(g(m), e)} (g(m), e) \end{array}$$

$$\text{f.b.} = \left\{ (m, l, e, e') \mid \begin{array}{l} (g \times \pi_E)(m, l, e) = \Delta(e') \\ (g(m), e) = (e', e) \\ e = e' \\ g(m) = e \end{array} \right\}$$

$$\cong \{ (m, e) \} = \Pi \oplus L$$

$$\begin{array}{ccc} L \oplus L & \xrightarrow{f \times 1} & \Pi \oplus L \xrightarrow{(m, e) \mapsto g(m)} E \\ \parallel & & \downarrow \Delta \\ L \oplus L & \xrightarrow{f \times i_L} & \Pi \oplus L \oplus E \xrightarrow{g \times \pi_E} E \oplus E \end{array}$$

$$0 \rightarrow L \oplus L \xrightarrow{f \times 1} R \oplus L \longrightarrow E \rightarrow 0 \quad (3)$$

(1) ↴ ① ↵ ↴ (f)

$$0 \rightarrow L \xrightarrow{f} R \xrightarrow{g} E \rightarrow 0$$

TS ① è vero perché detto

$$\begin{array}{ccc} L \oplus L & \xrightarrow{f \times 1} & R \oplus L \\ (1) \downarrow & & \downarrow (f) \\ L & \xrightarrow{f} & R \\ & \searrow \beta & \swarrow \alpha \\ & & N \end{array}$$

$$\text{Se } (f \times 1) \circ = \beta (1) \quad f = \alpha \text{ in}$$

$$f(u) := \alpha(u, 0)$$

EX Provare che tale f è
quello richiesto

Data

$$0 \longrightarrow L \xrightarrow{f} N \xrightarrow{g} E \longrightarrow 0$$

M

$$0 \longrightarrow L \xrightarrow{f} N \xrightarrow{g} E \longrightarrow 0$$

-M

$f \downarrow$ $\downarrow -1$ $\downarrow -g$

Calcolo M + (-M):

$$\begin{array}{ccc}
 0 & \xrightarrow{\quad f \times (-f) \quad} & P \\
 \parallel & & \downarrow \\
 0 & \xrightarrow{\quad f \times (-f) \quad} & N \oplus N
 \end{array}$$

$P \xrightarrow{\pi_{(m, \bar{m})}} g(m) = g(\bar{m})$

 $\downarrow \Delta$

 $\xrightarrow{\quad g \times g \quad} E \oplus E \longrightarrow 0$

$$P = \{(m, \bar{m}) \in N \oplus N \mid g(m) = g(\bar{m})\}$$

$$(m, \bar{m}) \in P \iff g(m - \bar{m}) = 0 \iff$$

$$(m - \bar{m}) \in \ker g = \ker f \iff \exists! \bar{e}$$

$$f(\bar{e}) = m - \bar{m}$$

(5)

$$\begin{array}{ccccc}
 0 & \longrightarrow & L \oplus L & \xrightarrow{f \times (-f)} & P \xrightarrow{\pi} E \\
 & & \downarrow (1) & \nearrow \textcircled{1} \downarrow \varphi & \downarrow \alpha \\
 0 & \longrightarrow & L & \xrightarrow{i_L} & L \oplus E \xrightarrow{\pi_E} E \\
 & & & & \downarrow \beta \\
 & & & & N
 \end{array}$$

$$\varphi(u, \bar{u}) = (\bar{e}, g(u) = g(\bar{u}))$$

$$f(\bar{e}) = u - \bar{u}$$

① \bar{e} een feest:

def $\alpha \in P$ t.c.

$$\alpha(f \times (-f)) = \beta(1)$$

$$g(l, e) = \beta(l) + \alpha(u, u) \quad \text{stove } u \in \mathbb{F} \text{ t.c. } g(u) = e$$

Ex Prüfen die f ist bee definierte
e die obige proprietà und
der push out.

Rechtsseitig $f \in E$, H_L

$\text{Ext}(E, \mathbb{Z}) \in \underline{\text{Ab}}$

(6)

Fiss, E:

$$\text{Ext}(E, -) : A\text{-mod} \longrightarrow \underline{\text{Ab}}$$

$$L \longmapsto \begin{matrix} \text{Ext}(E, L) \\ \alpha! \downarrow \end{matrix}$$

$$L' \longmapsto \begin{matrix} \text{Ext}(E, L') \\ \alpha'_! \downarrow \end{matrix}$$

$$0 \longrightarrow L \longrightarrow R \longrightarrow E \longrightarrow 0 : \square$$

$$a \downarrow \quad \downarrow \quad //$$

$$0 \longrightarrow L' \longrightarrow R' \longrightarrow E \longrightarrow 0 : \alpha'_! \frac{R}{A}$$

$$\text{Ext}(E, L')$$

offre alors une autre constante
(EX)

soit $\text{Ext}(E, L)$ possède de la
forme $a \cdot A$: $\forall a \in A$ ($a \cdot$ n'est pas
nécessairement un entier)

$$0 \longrightarrow L \longrightarrow R \longrightarrow E \longrightarrow 0 : \square$$

$$a \cdot \downarrow \quad r \downarrow \quad //$$

$$0 \longrightarrow L \longrightarrow R' \longrightarrow E \longrightarrow 0$$

$$a \cdot M = a'_! \underline{M}$$

$\text{Ext}(E, \cdot)$ ist eng mit A -mod [7]

$\text{Ext}(E, -) : A\text{-mod} \longrightarrow A\text{-mod}$

es additive abel. KPO

$$\begin{matrix} L \\ \downarrow \\ \alpha \downarrow \downarrow \alpha' \\ L' \end{matrix}$$

$\text{Ext}(E, L)$

$$\alpha_! \downarrow \downarrow \alpha'_!$$

$\text{Ext}(E, L')$

$$(\alpha + \alpha')_! = \alpha_! + \alpha'_!$$

coextensioen

$\text{Ext}(-, L) : (A\text{-mod})^{\text{op}} \longrightarrow A\text{-mod}$

$$\begin{matrix} E \\ \uparrow \gamma \\ E' \end{matrix}$$

$\longleftrightarrow \text{Ext}(E, L)$

$$\downarrow \gamma'!$$

$\text{Ext}(E', L)$

$$\begin{array}{ccccccc} f^* \underline{H} : & L & \longrightarrow & E' & \longrightarrow & E' & \longrightarrow 0 \\ & \parallel & & \downarrow \gamma & & \downarrow \gamma' & \\ H : 0 & \longrightarrow & R & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

18

$$\begin{array}{ccccccc}
 \text{Partitions} & \xrightarrow{\text{do}} & n' & \longrightarrow & E \\
 & \downarrow & & & \downarrow h & & \\
 0 \longrightarrow L & \xrightarrow{f} & n & \xrightarrow{g} & N & \longrightarrow & 0 : \underline{M}
 \end{array}$$

E A -module fusione:

Applicazioni flusso $(E, -)$

$$0 \longrightarrow \text{Hom}(E, L) \longrightarrow \text{Hom}(E, N) \xrightarrow{h} \text{Hom}(E, N)$$

~~Ext~~

~~Hom~~ $\text{Ext}(E, L)$

$$\partial(h) = h^! \underline{M} \quad \partial \in \text{co morphisms}$$

a A -modul'

∂ dà luogo a una app. effetto

debole in $\text{Hom}(E, N)$:

T.S.: per $\partial = \text{Hom}(g \cdot (-))$ sottomodul'
 $\text{Hom}(E, N)$

$\text{free}(g \cdot (-)) \subseteq \text{free } \partial = \{ h : E \rightarrow N \mid$

\exists spezzante}

$$u : E \rightarrow N$$

t.c. esiste $v : E \rightarrow D$

$$\begin{array}{ccc} E & \xrightarrow{u} & N \\ & \searrow v & \nearrow g \\ & M & \end{array}$$

$$g \circ u = u$$

$$\begin{array}{ccccc} E & \xrightarrow{1} & & & \\ \downarrow s & & & & \\ u(E) & \xrightarrow{g'} & E & & \\ \downarrow v & \nearrow u' & \downarrow u & & \\ D & \xrightarrow{g} & N & \longrightarrow & \end{array}$$

perché per unico p.b. \exists s.t.c.

$g' \circ s = 1 \Rightarrow g'$ è unico spezzante

u' D spezzante

Viceversa

$\ker \partial \subseteq \text{free}(g \cdot (-))$

$$\begin{array}{ccccccc} L & \xrightarrow{i_L} & L \oplus E & \xleftarrow{\pi_E} & E & & \\ \parallel & & \downarrow i'_L & & & & \\ & & v & & & & \\ & & \swarrow & & \downarrow g & & \\ & & & & g \circ v = h & & \end{array}$$

$$0 \longrightarrow L \longrightarrow D \xrightarrow{g} N \longrightarrow 0 \quad \square$$

COSTRUZIONE CON 2 mezzi separati "lungo" (D)

$$0 \rightarrow \text{Hoee}(E, L) \rightarrow \dots \rightarrow \text{Hoee}(E, N)$$



$$\text{Ext}(E, L) \xrightarrow{f_!} \text{Ext}(E, M) \xrightarrow{g_!} \text{Ext}(E, N)$$

Effetto lungo (Th 3.4)

"Homology" di S. MacLane pag 74

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$$

~~modo~~ ~~co~~ controvariente

Applica Hoee(-, E)

$$0 \rightarrow \text{Hoee}(N, E) \rightarrow \text{Hoee}(M, E) \xrightarrow{\alpha^*} \text{Hoee}(L, E)$$



$$\text{Ext}(N, E) \xrightarrow{g^*} \text{Ext}(M, E) \xrightarrow{f^*} \text{Ext}(L, E)$$

$$\partial \alpha = \alpha_1 \sqcap$$

Effetto lungo

Hoee - Ext contrav.

(Th 3.2 pag 73)

$G \in Ab$

$$\mathrm{Ext}(Z, G) = 0 \quad Z \text{ is projective} \quad (1)$$

over Z -module.

$$\mathrm{Ext}(\mathbb{Z}_n, G) = ? \quad \# 4$$

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{\cdot q} \mathbb{Z}_n \rightarrow 0$$

$$\text{Affine Hull}(-, G) : G \xrightarrow{\sim n} G$$

$$0 \rightarrow \text{Hom}(Z_n, G) \rightarrow \text{Hom}(Z, G) \rightarrow H(Z, G)$$

$$\text{Ext}(\mathbb{Z}_n, G) \longrightarrow \text{Ext}(\mathbb{Z}, G) \xrightarrow{\cong} \text{Ext}(\mathbb{Z}, G)$$

L'esattezza si dice che è obiettiva

$$e \in \text{Ext}(Z_n, G) \cong G/\mu G$$

- $\otimes N \rightarrow \text{Hom}(N, -)$

- $\otimes N : A\text{-mod} \rightarrow A\text{-mod}$

$$\begin{array}{ccc} M & & P \otimes N \\ \downarrow f & & \downarrow f \otimes 1_N \\ M' & & M' \otimes N \end{array}$$

stabilo a ds

$$R' \xrightarrow{f} R \xrightarrow{g} R'' \longrightarrow 0$$

stabilo
a d

$$R' \otimes N \xrightarrow{f \otimes 1} R \otimes N \xrightarrow{g \otimes 1} R'' \otimes N \longrightarrow 0$$

Non è in generale stabile a sin

$$R \xrightarrow{f} P \quad \text{noetherian rings}$$

$$R \otimes N \xrightarrow{f \otimes 1} P \otimes N \quad \text{noe +}\newline \text{ewins}$$

Def E è un A-modulo

P(AUTO (FCAT) \Leftrightarrow - $\otimes E$ è
elle finire eselt)

E è fatto $\Leftrightarrow \forall f: N \rightarrow P$ (13)

esatta $f \otimes f: N \otimes E \rightarrow P \otimes E$

iniettiva

OSS

$$\begin{array}{ccc} 0 & \longrightarrow & L \xrightarrow{f} M \\ & \cong \uparrow & \downarrow \cong \\ 0 & \longrightarrow & L \otimes A \xrightarrow{f \otimes 1_A} M \otimes A \end{array}$$

am
esatta
uso

$\Rightarrow f \otimes 1_A$ è iniettiva

$\Rightarrow A$ è fatto

$- \otimes N$ preserva i "collietti":

$$\left(\bigoplus_{i \in I} R_i \right) \otimes N = \bigoplus_{i \in I} R_i \otimes N$$

$$N \otimes \left(\bigoplus_{i \in I} R_i \right) = \dots$$

Proposizione $(R_i)_{i \in I}$ R -moduli

$\bigoplus_{i \in I} R_i$ è fatto $\Leftrightarrow \bigvee_i R_i$ è fatto