Varietà toriche, a.a. 2014-15

Foglio di esercizi n. 1

Varietà affini e morfismi.

Versione del 26 Marzo 2015

Exercise 1. Consider $V \subset \mathbb{C}^3$ given by equation $yz - x^2 = 0$. Consider the action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^3$ given by

$$(t_1, t_2) \cdot (x, y, z) = (t_1 x,\ t_2 y,\ t_2^{-1} t_1^2 z).$$

Prove that this action leaves V fixed, so that it becomes an action on V. For every $n = (n_1, n_2) \in \mathbb{Z}^2$, consider the one parameter subgroup $\lambda_n : t \mapsto (t^{n_1}, t^{n_2})$. Given the point $\bar{z} = (1, 1, 1) \in V$, determine the cone of the values $n \in \mathbb{Z}^2$ such that $\lim_{t \to 0} \lambda_n(t) \cdot \bar{z}$ exists and is finite.

Exercise 2. Consider $V \subset \mathbb{C}^4$ given by equation $wx - yz = 0$. Consider the action of $(\mathbb{C}^*)^3$ on $\mathbb{C}^4$ given by

$$(t_1, t_2, t_3) \cdot (w, x, y, z) = (t_1 w,\ t_2 x,\ t_3 y,\ t_3^{-1} t_1 t_2 z).$$

Prove that this action leaves V fixed. For every $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$, consider the one parameter subgroup $\lambda_n : t \mapsto (t^{n_1}, t^{n_2}, t^{n_3})$. Given the point $\bar{z} = (1, 1, 1, 1) \in V$, determine the cone of the values $n \in \mathbb{Z}^3$ such that $\lim_{t \to 0} \lambda_n(t) \cdot \bar{z}$ exists and is finite.

Esercizio 5. A topological space X is said to be irreducible if it is not the union of two proper closed subsets of X. Given an algebraic set V with Zariski topology, let $U \subseteq V$ be a subset such that: U is irreducible with respect to the subspace topology and $\bar{U} = V$. Prove that V is irreducible.

Esercizio 7. Given a morphism of affine varieties $f : V \to W$, prove that if $f^* : k[W] \to k[V]$ is surjective then the image of f is closed in W (i.e. $f(V)$ is algebraic) and that f induces an isomorphism onto its image $f(V)$.

Exercise 8. Given a morphism of affine varieties $f : V \to W$, prove that if $f^* : k[W] \to k[V]$ is injective, then the image of f is dense in W with respect to the Zariski topology.

Exercise 9. Consider the linear map $\bar{\phi} : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by the matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Write the induced morphism $\phi : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$, prove that ϕ is surjective and that for every $w = (w_1, w_2) \in (\mathbb{C}^*)^2$, $\phi^{-1}(w)$ consists of two points.