Varietà toriche, a.a. 2014/15

Foglio di esercizi n. 4

Toric varieties and morphisms.

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Exercise 1. Let $N = \mathbb{Z}^n$ and let $e_1, \dots, e_n$ be a basis of N. Let $e_0 = -e_1 - e_2 - \dots - e_n$. Now consider the fan Σ whose maximal cones are
$$\sigma_j = \text{cone}\{e_0, \dots, \hat{e}_j, \dots, e_n\},$$
where $j = 0, \dots, n$ and $\hat{e}_j$ means that e_j is not in the list. Show that X_Σ is isomorphic to $\mathbb{P}^n$.

Exercise 2. (*Line bundles on* $\mathbb{P}^n$) Let $Y = (\mathbb{C}^{n+1} - \{0\}) \times \mathbb{C}$, with coordinates $(z_0, \dots, z_n, w)$. Given an integer a, define a $\mathbb{C}^*$ action on Y by
$$\lambda \cdot (z_0, \dots, z_n, w) = (\lambda z_0, \dots, \lambda z_n, \lambda^{-a} w)$$
Show that $L(a) = Y/\mathbb{C}^*$ is a smooth $n+1$-dimensional variety which can be constructed by gluing $n+1$ copies of $\mathbb{C}^{n+1}$ (**Hint:** consider subsets of $L(a)$ given by $U_j = \{[z_0, \dots, z_n, w] \in L(a) \,|z_j \neq 0\}$ for all $j = 0, \dots, n$ and identify U_j with $\mathbb{C}^{n+1}$ by a suitable bijective map $\psi_j = U_j \to \mathbb{C}^{n+1}$, then compute $\psi_k \circ \psi_j^{-1}$). Consider the map $\pi : L(a) \to \mathbb{P}^n$ defined by
$$\pi([z_0, \dots, z_n, w]) = [z_0 : \dots : z_n].$$
Prove that π is well defined (i.e. independent of the representatives) and show that it is a morphism. Show that $\pi^{-1}([z_0 : \dots : z_n]) \cong \mathbb{C}$. The variety $L(a)$ is the total space of the line bundle on $\mathbb{P}^n$ which is often denoted by $\mathcal{O}_{\mathbb{P}^n}(-a)$.

Exercise 3. (*Toric construction of line bundles on* $\mathbb{P}^n$) Let $N = \mathbb{Z}^{n+1}$ and let $e_1, \dots, e_{n+1}$ be a basis of N. Let $e_0 = -e_1 - e_2 - \dots - e_n$. For $a \in \mathbb{Z}$, define Σ_a to be the fan in N whose maximal cones are
$$\begin{aligned} \sigma_0 &= \text{cone}\{e_1, \dots, e_{n+1}\} \\ \sigma_j &= \text{cone}\{e_0 + a e_{n+1}, e_1, \dots, \hat{e}_j, \dots, e_{n+1}\},\ j = 1, \dots, n. \end{aligned}$$
Show that X_{Σ_a} coincides with $L(a)$ defined in the previous exercise. Now let Σ' be the fan for $\mathbb{P}^n$ in $\mathbb{R}^n$ as in Exercise 1. Let $\phi : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ be the map $\phi(m_1, \dots, m_n, m_{n+1}) = (m_1, \dots, m_n)$. Check that ϕ is compatible with Σ_a and Σ' and prove that the induced map $\bar{\phi} : L(a) \to \mathbb{P}^n$ coincides with the map π defined in Exercise 2. **Hint:** try first with $n = 1, 2$.

Exercise 4. (*Hirzebruch surfaces*) Let $Y = (\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^2 - \{0\})$ with coordinates (z_0, z_1, w_0, w_1). Define the following $\mathbb{C}^* \times \mathbb{C}^*$ action on Y:
$$(\lambda_0, \lambda_1) \cdot (z_0, z_1, w_0, w_1) = (\lambda_0 z_0, \lambda_0 z_1, \lambda_1 w_0, \lambda_1 \lambda_0^{-a} w_1)$$
Show that the quotient $F(-a) = Y/(\mathbb{C}^* \times \mathbb{C}^*)$ is a smooth variety which can be obtained by gluing four copies of $\mathbb{C}^2$. (**Hint:** consider subsets of $F(-a)$ given by $U_0 = \{[z_0, z_1, w_0, w_1] \in F(-a) \,|\, z_0 \neq 0, w_0 \neq 0\}$, $U_1 = \{[z_0, z_1, w_0, w_1] \in F(-a) \,|\, z_0 \neq 0, w_1 \neq 0\}$, $U_2 = \{[z_0, z_1, w_0, w_1] \in F(-a) \,|\, z_1 \neq 0, w_0 \neq 0\}$ and $U_3 =$

$\{[z_0, z_1, w_0, w_1] \in F(-a) \,|\, z_1 \neq 0, w_1 \neq 0\}$ and identify them with $\mathbb{C}^2$ by suitable bijective maps $\psi_j = U_j \to \mathbb{C}^2$, then compute $\psi_k \circ \psi_j^{-1}$). Show that the map $\pi : F(-a) \to \mathbb{P}^1$ defined by

$$\pi([z_0, z_1, w_0, w_1]) = [z_0 : z_1]$$

is well defined and is a morphism. Prove that for all $[z_0 : z_1] \in \mathbb{P}^1$, $\pi^{-1}([z_0 : z_1]) \cong \mathbb{P}^1$. The surface $F(-a)$ is called a Hirzerbruch surface and it is an example of a "ruled" surface. The map π is the ruling of $F(-a)$.

Exercise 5. Let $N = \mathbb{Z}^2$. For $a \in \mathbb{Z}$, let Σ_a be the fan in $\mathbb{R}^2$ whose maximal cones are

$$\sigma_1 = \text{cone}\{(1,0),(0,1)\}, \qquad \sigma_2 = \text{cone}\{(1,0),(0,-1)\}$$

$$\sigma_3 = \text{cone}\{(0,-1),(-1,a)\}, \qquad \sigma_4 = \text{cone}\{(-1,a),(0,1)\}$$

Prove that X_{Σ_a} is the Hirzerbruch surface $F(-a)$. Now let Σ' be the fan for $\mathbb{P}^1$ in $\mathbb{R}$ as in Exercise 1 and $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ the map $\phi(n_1, n_2) = n_1$. Check that ϕ is compatible with Σ_a and Σ' and that the induced map $\bar{\phi} : \Sigma_a \to \mathbb{P}^1$ coincides with the morphism π defined in Exercise 4. If we take $\phi(n_1, n_2) = n_2$, is it compatible with Σ_a and Σ'?

Exercise 6. (*Sections of line bundles on* $\mathbb{P}^n$) Let $L(a)$ and the map $\pi : L(a) \to \mathbb{P}^n$ be defined as in Exercise 2. A section of $L(a)$ is a morphism $s : \mathbb{P}^n \to L(a)$ such that $\pi \circ s$ is the identity of $\mathbb{P}^n$. A polynomial $f(z)$, in the coordinates $z = (z_0, \ldots, z_n)$, is homogeneous of degree d if for all $\lambda \in \mathbb{C}$, $f(\lambda z) = \lambda^d f(z)$. One can show that $f(z)$ is homogenous of degree d if all monomials appearing in $f(z)$ are of degree d. Assume a is negative and let $f(z)$ be a homogeneous polynomial of degree $-a$. Prove that the map $s_f : \mathbb{P}^n \to L(a)$ given by

$$s_f([z_0 : \cdots : z_n]) = [z_0, \ldots, z_n, f(z_0, \ldots, z_n)]$$

is well defined (i.e. independent of reppresentatives) and defines a section of $L(a)$.

Exercise 7. (*The blow up of* $\mathbb{C}^2$) If we view $\mathbb{P}^1$ as the space of lines through the origin in $\mathbb{C}^2$, define the tautological line bundle on $\mathbb{P}^1$ as $L = \{(\ell, p) \in \mathbb{P}^1 \times \mathbb{C}^2 \,|\, p \in \ell\}$. There are two maps: $\pi_1 : L \to \mathbb{P}^1$ given by $\pi_1(\ell, p) = \ell$ and $\pi_2 : L \to \mathbb{C}^2$ given by $\pi_2(\ell, p) = p$. The space $L(1)$ is also called the blow up of $\mathbb{C}^2$ at the origin and the map π_2 is called the blow down map. Prove that L is isomorphic to $L(1)$ defined in Exercise 2. (**Hint:** using coordinates $\ell = [z_0 : z_1]$ and $p = (w_1, w_2)$ the condition $p \in \ell$ is equivalent to equation $w_0 z_1 - w_1 z_0 = 0$). Using the description of $L(1)$ given in Exercise 2, prove that the map π_2 defined above is given by $\pi_2([z_0, z_1, w]) = (wz_0, wz_1)$.

Exercise 8. Let Σ_1 be the fan for the line bundle $L(1)$ over $\mathbb{P}^1$ (as defined in Exercise 3 with $n = 1$). Let $\sigma = \text{cone}\{(-1,1),(1,0)\}$. The identity map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is obviously compatible with Σ_1 and σ. Prove that the induced map $\bar{\phi} : L(1) \to \mathbb{C}^2$ coincides with π_2 defined in the previous exercise.

Exercise 9. Let $Y = \mathbb{C}^3 - \{0\}$. For a triple of positive integers (a_0, a_1, a_2) define the following $\mathbb{C}^*$ action on Y:

$$\lambda \cdot (z_0, z_1, z_2) = (\lambda^{a_0} z_0, \lambda^{a_1} z_1, \lambda^{a_2} z_2).$$

It can be shown that $Y/\mathbb{C}^*$ is a variety, which is called a weighted projective space and denoted $\mathbb{P}(a_0, \ldots, a_n)$. Prove this in the case of the triple $(1,1,2)$, following similar methods as in the previous exercises.

Exercise 10. Let $N = \mathbb{Z}^2$ and consider the fan Σ whose cones are

$$\sigma_1 = \text{cone}\{(1,0),(0,1)\}, \;\; \sigma_2 = \text{cone}\{(1,0),(-1,-2)\}, \;\; \sigma_3 = \text{cone}\{(-1,-2),(0,1)\}$$

Prove that X_Σ is the weighted projective space $\mathbb{P}(1,1,2)$.