Advanced arguments in Analytic Number Theory

Fourier Analysis in locally compact abelian groups

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These are my personal notes of the course in *Advanced Arguments in Analytical Number Theory* in A.Y. 2017–'18. In this edition the course deals mainly with topological groups, Haar measure, integration, Pontryagin duality, Fourier analysis, classification of locally compact fields, and construction of Adèles–Idèles and (maybe) Tate thesis.
It is strongly based on D. Ramakrishnan and R. J. Valenza book *Fourier analysis on number fields* cited in bibliography, from which I borrow the main organization and the proofs of all main results.
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I am the unique responsible for any possible error in these notes.

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Contents

Notation

- Let (X, τ) be a topological space.
  - Let $U \subseteq X$. Then $\mathring{U}$ denotes the open part of U and $\overline{U}$ its closure.
- $\mathcal{C}(X) := \{f \colon X \to \mathbb{C},\ f \text{ is continuous}\}$.
- $\mathcal{C}_c(X) := \{f \colon X \to \mathbb{C}, \operatorname{supp}(f) \text{ is compact}\}$ (the set of continuous and compactly supported functions).
- $\| \cdot \|_{\infty,X}$ the sup norm on X, i.e., $\|f\|_{\infty,X} := \sup_{x\in X} |f(x)|$.
- $\mathcal{C}_c^+(X) := \{f \colon X \to \mathbb{R}^+,\ f \text{ is continuous },\ \operatorname{supp}(f) \text{ is compact}\}$ (the set of nonnegative continuous and compactly supported functions).
- $\mathcal{C}_b(X) := \{f \in \mathcal{C}(X),\ \operatorname{supp}(f) \text{ is compact}\}$ (the set of continuous and bounded functions).
- $\mathcal{C}_0(X) := \{f \in \mathcal{C}(X),\ f \text{ “goes to 0 to the } \infty\text{”}\}$ (actually, let $X' := X \cup \{\infty\}$ be the one point compactification of X, then $f \in \mathcal{C}_0(X)$ iff it can be extended as continuous function on X' with $f(\infty) := 0$).

CHAPTER 1

Topological groups

1.1. Preliminary facts

Let X be a topological space, and let τ be its topology. Let $x \in X$. A *neighborhood* of x is any set U containing an open set V such that $x \in V \subseteq U$. Thus, U is not assumed to be an open set in itself, but $x \in \mathring{U}$.

Convergence in topological spaces is introduced using the machinery of nets (see [K], p.62): $\lim x_\alpha = y$ means that (x_α) is a directed set (i.e., a map $A \to X$, where A is the set of indexes supporting a notion of $\geq$ which is transitive and reflexive, and such that for every couple α, $\beta \in A$, there is $\gamma \in A$ with $\gamma \geq \alpha$ and $\gamma \geq \beta$), such that for every open neighborhood U of y, the (x_α) net is eventually (in the sense of nets, i.e. using the $\geq$ notion) in U.
This notion allows to prove that given two topological spaces X and Y, a map $f\colon X \to Y$ is continuous (i.e. the preimage $f^{-1}(V)$ is an open set in X for every open set V in Y) if and only if $\lim f(x_\alpha) = f(x)$ for every net (x_α) converging to an element x (see [K] p. 86). When the topology satisfies the first axiom of countability (i.e., each point has a countable base of neighborhoods) the mechanism of nets can be substituted by sequences.

We also need a local version of compactness: we adopt the following definition. A topological space (X, τ) is *locally compact* when every point $x \in X$ has a compact neighborhood, i.e. for every x there is a compact K such that $\{x\} \subseteq \mathring{K} \subseteq K$.

1.1.1. Product topology. Here we collect several facts about product topology: proofs are mainly omitted but these results are standard and all of them can be found for example in [K].

Let X_α for every $\alpha \in I$ (any cardinality admitted here) be a topological set. Then in $X := \prod_{\alpha\in I} X_\alpha$ we can introduce the product topology, which is the coarsest topology such all the projection maps $\pi_\beta\colon X \to X_\beta$, $\pi_\beta((x_\alpha)_{\alpha\in I}) := x_\beta$, are continuous.

A *cylindric open set* is a set of type $\prod_{\alpha\in I} U_\alpha$ where each U_α is an open subset of X_α and $U_\alpha = X_\alpha$ with only finitely many exceptions. The set of cylinders is a base for the product topology (see [K], p. 90).

When the set of indexes I is finite, all sets of type $\prod_{\alpha=1}^{N} U_\alpha$ with U_α open in X_α are cylinders, thus the product topology coincides with the topology generated by the products of open sets. However, if I is *not* finite then not every product of open sets is a cylinder, and in fact in this case the product topology is strictly coarser than the topology generated by product of open sets.

Note that every open set $U \neq \emptyset$ in X contains a cylinder: this is a consequence of the fact that cylinders are a base for that topology (see [K], p. 46).

The projections π_α are always open maps, i.e., $\pi_\alpha(U)$ is open in X_α whenever U is open in X: this is evident for cylinders and the general claim follows by the fact that cylinders are a base for the topology.

Proposition 1.1 *If each X_α is T_2 (i.e., Hausdorff), then X is T_2.*

Proof. (see [K], p. 92). Take $x, y \in X$, $x \neq y$. Then $x_\alpha \neq y_\alpha$ for some index α. Let U_α, V_α be open sets separating x_α and y_α in X_α (they exist, because X_α is T_2). Take $U := \pi_\alpha^{-1}(U_\alpha)$ and $V := \pi_\alpha^{-1}(V_\alpha)$. They are open sets in X which separate x and y. ■

Proposition 1.2 *Let Y be another topological space, and let $f : Y \to X$. Then f is continuous if and only if $\pi_\alpha \circ f : Y \to X_\alpha$ is continuous for each $\alpha \in I$.*

Proof. See [K], p. 91. ■

The following proposition states the key feature of the product topology.

Proposition 1.3 (Tychonoff's theorem) *if each X_α is compact, then X is compact with respect to the product topology.*

Proof. See [K] p. 143. ■

Unfortunately the product topology behaves badly with respect the local compactness, because in general the product of locally compact sets is *not* locally compact.
For example (see [SS] p. 122), consider $X_\alpha = \mathbb{Z}$ for all $\alpha \in I$, $I = \mathbb{N}$ (hence countably many copies of $\mathbb{Z}$), and in X_α (i.e. in $\mathbb{Z}$) set the discrete topology. Choose any $(x_\alpha)_\alpha \in X$. Let U be any open set in X containing $(x_\alpha)_\alpha$ (so it is not empty). Then it is the union of cylinders (because cylinders are a base for the product topology). In particular there is an index $\bar{\alpha}$ such that $\pi_{\bar{\alpha}}(U) = \mathbb{Z}$ (because every cylinder is eventually equal to the base set). If X is locally compact, then there is a compact K such that $\{(x_\alpha)_\alpha\} \subseteq \mathring{K} \subseteq K$. The previous argument applied to $\mathring{K}$ shows that there exists an index $\bar{\alpha}$ such that $\pi_{\bar{\alpha}}(\mathring{K}) = \mathbb{Z}$. Hence $\mathbb{Z} = \pi_{\bar{\alpha}}(\mathring{K}) \subseteq \pi_{\bar{\alpha}}(K)$, but this is impossible, because $\pi_{\bar{\alpha}}(K)$ is a compact (because $\pi_{\bar{\alpha}}$ is continuous) in the discrete topology, and hence it is finite.

However, it is possible to prove that if the locally compact X_α spaces are actually compact with only finitely many exceptions, then X is locally compact, and viceversa (see [K] Th. 19, p. 147).

1.2. Topological groups

Let G be a *group*, hence a set with a binary operation that we denote as a product $\cdot$, having good properties: identity, associativity, and inverse. Note that we do not assume commutativity.
Suppose that G has also a *topology*, so we have a notion of open/closed sets. By definition the set G becomes a *topological group* when these structures interact well. The correct definition is the following: G is a topological group G when the product $\rho\colon G \times G \to G$, $\rho(a,b) := a \cdot b$ and the inverse $\iota\colon G \to G$, $\iota(a) := a^{-1}$, both are continuous maps (in $G \times G$ we set the product topology).

Let $a \in G$, and let $\mathcal{L}_a, \mathcal{R}_a\colon G \to G$ be defined as $\mathcal{L}_a(x) := ax$ (left multiplication by a), $\mathcal{R}_a(x) := xa^{-1}$ (right multiplication by a^{-1}). Note that definitions are such that $\mathcal{L}_a\mathcal{L}_b = \mathcal{L}_{ab}$ and $\mathcal{R}_a\mathcal{R}_b = \mathcal{R}_{ab}$. The maps $\mathcal{L}_a$ and $\mathcal{R}_a$ are homeomorphisms acting transitively on G, because for every couple x, y of elements in G there is a such that $\mathcal{L}_a(x) = y$ (set $a := yx^{-1}$). Thus G is a *homogeneous space* with respect to the family $\{\mathcal{L}_a\colon a \in G\}$ (and $\{\mathcal{R}_a\colon a \in G\}$, as well). This property is extremely important, since it allows to transfer properties which are true locally in the identity e to the neighborhood of each other point: in other words, all points behave the same from the point of view of the topology.

Here we collect some basic properties.

Proposition 1.4 *Let G be a topological group. Then*

1. *every neighborhood U of e contains an open neighborhood V still containing e and such that $VV \subseteq U$. If necessary, V can be chosen symmetric, i.e. such that $V = V^{-1}$.*
2. *Let U be an open set in G, and $W \subseteq G$ be any subset. Then UW and WU are open sets.*
3. *The product map $\rho\colon G \times G \to G$ is an open map.*
4. *Let $H \subseteq G$ be a subgroup, then also the closure $\overline{H}$ is a subgroup.*
5. *Let $H \subseteq G$ be an open subgroup, then H is also closed.*
6. *Let K_1, K_2 be compact sets in G, then K_1K_2 is compact.*
7. *Let K be compact and F be closed sets in G, then FK and KF are closed.*

Proof.

1. By hypothesis $e \in \mathring{U}$. Let $\rho\colon G \times G \to G$ be the product. The set $\rho^{-1}(\mathring{U})$ is open in $G \times G$ (because ρ is continuous), and contains (e,e). Hence there are open sets $U_1, U_2 \subseteq G$ such that $(e,e) \in U_1 \times U_2 \subseteq \rho^{-1}(\mathring{U})$ (because the product of open sets is a base for the product topology). Let $V' := U_1 \cap U_2$, an open subset of G. Then $e \in V'$, and $V'V' = \rho(V' \times V') \subseteq \rho(U_1 \times U_2) \subseteq \rho(\rho^{-1}(\mathring{U})) \subseteq \mathring{U} \subseteq U$.
The inclusion of e in V' implies $V' \subseteq V'V'$, which is in U, thus $V' \subseteq U$. Finally, take $V := V' \cap V'^{-1}$ in order to produce the symmetric open set.
2. Note that $WU = \cup_{w \in W}\mathcal{L}_w(U)$, hence it is open because each $\mathcal{L}_w(U)$ is open. The same for $UW = \cup_{w \in W}\mathcal{R}_{w^{-1}}(U)$.
3. Every open set U in $G \times G$ may be written as $\cup_\alpha U_\alpha \times U'_\alpha$, for suitable open sets U_α, U'_α in G (because the sets $U_\alpha \times U'_\alpha$ are a base for the product topology). Then
$$\rho(U) = \rho\big(\cup_\alpha (U_\alpha \times U'_\alpha)\big) = \cup_\alpha \rho(U_\alpha \times U'_\alpha) = \cup_\alpha (U_\alpha U'_\alpha)$$
and each $U_\alpha U'_\alpha$ is open, by [2.].
4. Easy, because the product and the inverse are continuous maps.
5. We decompose G as disjoint union of lateral classes G/H. Thus
$$H^c = \bigcup_{\substack{a \in G/H \\ a \neq e}} aH$$

which is open, because each aH is open by [2.].

6. The product $K_1 \times K_2$ is compact in $G \times G$, and the product map $\rho\colon G \times G \to G$ is continuous. Thus $K_1K_2 = \rho(K_1 \times K_2)$ is compact.
7. It is sufficient to prove the claim for FK (for the other case consider that $KF = (F^{-1}K^{-1})^{-1}$ and that the inverse map ι is a homeomorphism). Let z in the closure of the product FK. Then there is a net of the form $x_a y_a$ converging to z, with x_a in F and y_a in K. The compactness of K ensures that, passing to a subnet if necessary, the net $\{y_a\}$ converges to an element y, say.
   Let U be a neighborhood of e, and let V be a symmetric neighborhood of e such that $VV \subseteq U$ (it exists by [1.]). Then the nets $\{z^{-1}x_a y_a\}$ and $\{y^{-1}y_a\}$ are both eventually in V, whence the product
   $$z^{-1}x_a y = (z^{-1}x_a y_a)(y^{-1}y_a)^{-1}$$
   is eventually in $VV^{-1} = VV \subseteq U$. This proves that $z^{-1}x_a y$ tends to e, i.e. that x_a tends to zy^{-1}.
   By hypothesis x_a is in F which is closed, thus zy^{-1} is in F as well. Then the equality $z = (zy^{-1})y$ shows that $z \in FK$.

■

Remark. 1.1 Note that claims [6.] and [7.] in previous proposition do not imply that compact×compact is closed: this happens because we are not assuming T_2 property, so that in this general setting a compact set is not necessarily a closed set. □

Remark. 1.2 Note that ρ is not necessarily a closed map. For example, let $G = (\mathbb{R}, +)$ (i.e., $\mathbb{R}$ with the sum as binary operation), and take $F := \{(x, x^{-1} - x) \in \mathbb{R} \times \mathbb{R} \colon x > 0\}$ is a closed subset in $\mathbb{R} \times \mathbb{R}$, but $\rho(F) = \{x^{-1} \colon x > 0\} = (0, +\infty)$ is not closed. □

Remark. 1.3 A similar example shows that in general the product of closed sets is not closed. Take $G = (\mathbb{R}^2, +)$ (i.e., $\mathbb{R}^2$ with the vector sum as operation), and take $F := \{(x, x^{-1}) \in \mathbb{R}^2 \colon x \neq 0\}$ in $\mathbb{R}^2$. Then $\rho(F, F) = \{(x+y, x^{-1}+y^{-1}) \colon x, y \neq 0\} = \{(a, b) \colon ab < 0\} \cup \{(a, b) \colon ab \geq 4\} \cup \{(0, 0)\}$, which is not closed. □

The following result shows how the algebraic structure interacts with the separation properties.

Proposition 1.5 *Let G be a topological group. Then the following properties are equivalent:*

1. *$\{e\}$ is a closed set.*
2. *Each element in G is a closed set.*
3. *G is T_1.*
4. *G is T_2.*

Proof.

1. $\Longrightarrow$ 2. In fact, let $x \in G$. Then $\{x\} = \mathcal{L}_x(\{e\})$ and $\mathcal{L}_x$ is a homeomorphism and $\{e\}$ is closed by assumption. Thus $\{x\}$ is closed, too.
2. $\Longrightarrow$ 3. Trivial.
3. $\Longrightarrow$ 4. Let $x, y \in G$, $x \neq y$. By hypothesis G is T_1, thus there is a neighborhood U of e separating xy^{-1} and e. Let V be an open and symmetric set still containing e and such that $VV \subseteq U$ (it exists by Proposition 1.4[1.]). Then Vx and Vy are open sets containing x and y respectively, and without common elements, because if $vx = v'y$ then $e = v'^{-1}vxy^{-1} \in V^{-1}Vxy^{-1} = VVxy^{-1} \subseteq Uxy^{-1}$, which is impossible.
4. $\Longrightarrow$ 1. Trivial.

■

Let H be a subgroup of the topological group G. In the quotient space $G/H := \{aH : a \in G\}$ (i.e. the set of left cosets of H in G) we take the *quotient topology*, which is the strongest topology (i.e., largest topology) such that the canonical projection $\pi\colon G \to G/H$ is continuous. In other words, a subset W of G/H is open if and only if $\pi^{-1}(W)$ is open in G. The following proposition shows how the topological properties of H affect the quotient topology.

Proposition 1.6 *Let G be a topological group, and let H be a subgroup. Then:*

1. *the quotient space G/H is homogeneous under G.*
2. *The canonical projection $\pi\colon G \to G/H$ is an open map.*
3. *If H is a compact subgroup, then the canonical projection $\pi\colon G \to G/H$ is a closed map.*
4. *G/H is T_1 if and only if H is closed.*
5. *G/H is discrete if and only if H is open. When G is compact, G/H is finite if and only if H is open.*
6. *If H is normal, then G/H is a topological group with respect to the quotient topology and the induced operation.*
7. *Let H be the closure of $\{e\}$ in G. Then H is normal and G/H is Hausdorff.*

Exercise. 1.1 Let G be a topological group.

1. let $S \subseteq G$ be the maximal connected component containing e. Prove that S is a closed subgroup of G.
2. let S as before. Suppose that S has a finite index in G. Prove that S is also an open subgroup of G.
3. Suppose that G is connected and let $H \subseteq G$ be an open subgroup. Prove that $H = G$.
4. In general (i.e. without assuming that G is connected), let S as before and let $H \subseteq G$ be an open subgroup. Prove that $S \subseteq H$, with equality if and only if H is connected.

Proof.

1. For every couple of cosets xH, yH, the left multiplication by the element yx^{-1} acts on G/H and sends xH to yH, in other words, $\mathcal{L}_{yx^{-1}}(xH) = yH$ (but note that this is an abuse of notation, because by definition $\mathcal{L}_{yx^{-1}}\colon G \to G$, not $G/H \to G/H$). This map is also continuous. In fact, for every set $U \subseteq G/H$, $(\mathcal{L}_a)^{-1}(U) = \{a^{-1}bH : bH \in U\}$, implying $\pi^{-1}((\mathcal{L}_a)^{-1}(U)) = (\mathcal{L}_a)^{-1}(\pi^{-1}(U))$. Thus, if U is open in G/H, then $\pi^{-1}(U)$ is open in G (by definition of quotient topology), hence $(\mathcal{L}_a)^{-1}(\pi^{-1}(U))$ is open in G (because $\mathcal{L}_a$ is continuous), hence $\pi^{-1}((\mathcal{L}_a)^{-1}(U))$ is open in G, hence $(\mathcal{L}_a)^{-1}(U)$ is open in G/H (by definition of quotient topology).
2. Let U be an open set in G. Then $\pi(U)$ is open in G/H if and only if $\pi^{-1}(\pi(U))$ is open in G. This is true, since $\pi^{-1}(\pi(U)) = UH$ and UH is open by Proposition 1.4[2.].
3. Let F be a closed set in G. Then $\pi(F)$ is closed in G/H if and only if $\pi^{-1}(\pi(F))$ is closed in G. This is true, since $\pi^{-1}(\pi(F)) = FH$ and FH is closed by Proposition 1.4[7.].
4. Imitate the proof of Proposition 1.5: this is possible since G/H is homogeneous with respect to G. It follows that G/H is T_1 if and only if the trivial class in G/H is closed, and by the definition of quotient topology this happens if and only if H is closed in G.
5. By definition, G/H is discrete if and only if the points in G/H are open, and by homogeneity this happens if and only if the trivial class is open, and this happens (by the definition of the quotient topology) if and only if H is open.
   When G is compact, then G/H is compact as well (because π is continuous). Thus H

is open if and only if G/H is discrete (by previous step), and a compact is discrete if and only if it is finite.

6. When H is normal, the product
$$\rho_{G/H}\colon G/H \times G/H \to G/H, \qquad \rho_{G/H}((aH, bH)) := (aH)(bH) := abH$$
is well defined and the diagram
$$\begin{array}{ccc} G \times G & \xrightarrow{\pi\times\pi} & G/H \times G/H \\ \Big\downarrow{\scriptstyle \rho_G} & & \Big\downarrow{\scriptstyle \rho_{G/H}} \\ G & \xrightarrow{\qquad\pi\qquad} & G/H \end{array}$$
commutes. It shows that $\rho_{G/H}^{-1}(U) = (\pi \times \pi)(\rho_G^{-1}(\pi^{-1}(U)))$ for every set U in G/H. When U is open, then $\pi^{-1}(U)$ is open in G (because π is continuous), $\rho_G^{-1}(\pi^{-1}(U))$ is open (because ρ_G is continuous), and $(\pi \times \pi)(\rho_G^{-1}(\pi^{-1}(U)))$ is open (because π is an open map). This proves that $\rho_{G/H}$ is continuous. A similar argument proves the same for the inverse map.
7. $\{e\}$ is a subgroup in G, thus also its closure H is a group (by Proposition 1.4[4.]). Each conjugate group aHa^{-1} of H is closed (because it is $\mathcal{L}_a\mathcal{R}_a(H)$) and contains e. Thus H is contained in each conjugate group (because H is the intersection of all closed set containing e). In particular H is normal.
H is closed, hence G/H is T_1 (by step 4), hence it is T_2 (by Proposition 1.5).

■

Remark. 1.4 Note that [7.] is a generalization of the implication [1.] $\Longrightarrow$ [4.] of Proposition 1.5. □

Let G be a topological group, and let $D(G, S)$ be the set of functions $f\colon G \to S$, where S is $\mathbb{R}$ or $\mathbb{C}$ (or any other Banach space). Then G acts on D both via the *left* action
$$L_h\colon D \to D, \qquad L_h(f)(g) := f(h^{-1}g),$$
and the *right* action
$$R_h\colon D \to D, \qquad R_h(f)(g) := f(gh).$$
Note that the definitions are similar to the ones of $\mathcal{L}_h$ and $\mathcal{R}_h$, but the details are different (L_h modifies a function f via a left multiplication by h^{-1}, not by h as for $\mathcal{L}_h$). This is a bit strange, but it is necessary in order to preserve the validity of identities $L_aL_b = L_{ab}$ and $R_aR_b = R_{ab}$. In fact, in this way
$$\begin{aligned}(L_aL_bf)(g) &= (L_a(L_bf))(g) = (L_bf)(a^{-1}g) = f(b^{-1}a^{-1}g) \\ &= f((ab)^{-1}g) = (L_{ab}f)(g),\end{aligned}$$
and an analogous computation shows that $R_aR_b = R_{ab}$.
A function f is *left-uniformly* continuous on G when for every $\epsilon > 0$, there exists a neighborhood V of e such that
$$h \in V \implies \|L_hf - f\|_{\infty,G} := \sup_{g\in G} \|(L_hf)(g) - f(g)\|_S = \sup_{g\in G} \|f(h^{-1}g) - f(g)\|_S \le \epsilon.$$
For the right-uniformity the definition is similar.
Let $\mathcal{C}(G, S)$ be the set of continuous maps, and $\mathcal{C}_c(G, S)$ be the subset of continuous maps having compact support. The following property is probably highly expected, but it is extremely important for our purposes and a proof is welcome.

Proposition 1.7 *Let G be a topological group, and let $\mathcal{C}_c(G, S)$ be the set of continuous maps with compact support. Then each map in $\mathcal{C}_c(G, S)$ is both left and right uniformly continuous.*

Proof. We prove the left uniformity, the proof of the right uniformity being similar. Let K be the support of f. It is a compact, by assumption. Set a value for $\epsilon > 0$. f is by hypothesis a continuous map, hence for every g in K there is an open neighborhood U_g of e such that

$$h \in U_g \quad \Longrightarrow \quad \|f(hg) - f(g)\|_S \leq \epsilon.$$

Let V_g be the open and symmetric subset of U_g such that $e \in V_g$, and $V_g V_g \subseteq U_g$ (once again, it exists by Proposition 1.4[1.]). The family $\{V_g g\}_{g\in K}$ is an open covering of K, from which we can produce a finite covering: $g_1, \dots, g_N$ and sets $V_1, \dots, V_N$ (where $V_i := V_{g_i}$ for every $i = 1, \dots, N$).
Let $V := \cap_{i=1}^N V_i$, which contains e, is open, symmetric, and satisfies $V \subseteq VV \subseteq V_i V_i \subseteq U_i$ for every i. Assume $h \in V$. Then, when $g \in K$, there is an index i such that $g \in V_i g_i$, and we have

$$\|(L_h f)(g) - f(g)\|_S = \|f(h^{-1}g) - f(g)\|_S \leq \|f(h^{-1}g) - f(g_i)\|_S + \|f(g_i) - f(g)\|_S.$$

The inclusions $h^{-1}g = (h^{-1})(gg_i^{-1})g_i \in V(gg_i^{-1})g_i$ (because V is symmetric) $\in VV_i g_i$ (because $g \in V_i g_i$) $\in U_i g_i$ (because $V \subseteq V_i$ and $V_i^2 \subseteq U_i$) prove that $\|f(h^{-1}g) - f(g_i)\|_S \leq \epsilon$. In similar way, the inclusion $g = (gg_i^{-1})g_i \in V_i g_i \in U_i g_i$, shows that also $\|f(g_i) - f(g)\|_S \leq \epsilon$. This proves that

$$g \in K \quad \Longrightarrow \quad \|(L_h f)(g) - f(g)\|_S \leq 2\epsilon.$$

Suppose $g \notin K$. If also $h^{-1}g \notin K$ then $f(g) = f(h^{-1}g) = 0$, and the bound $\|f(h^{-1}g) - f(g)\|_S = 0 \leq 2\epsilon$ is trivial.
On the contrary, suppose $h^{-1}g \in K$. Let i be the index such that $h^{-1}g \in V_i g_i$. Then

$$\|(L_h f)(g) - f(g)\|_S = \|f(h^{-1}g) - f(g)\|_S \leq \|f(h^{-1}g) - f(g_i)\|_S + \|f(g_i) - f(g)\|_S.$$

The inclusion $h^{-1}g \in V_i g_i \subseteq U_i g_i$ shows that $\|f(h^{-1}g) - f(g_i)\|_S \leq \epsilon$, and the inclusions $g = h(h^{-1}gg_i^{-1})g_i \in V(h^{-1}gg_i^{-1})g_i$ (because $h \in V$) $\in VV_i g_i$ (because $h^{-1}gg_i^{-1} \in V_i$) $\in U_i g_i$ (because $VV_i \subseteq V_i V_i \subseteq U_i$) prove that also $\|f(g_i) - f(g)\|_S \leq \epsilon$.
Thus, in any case we have proved that $h \in V$ implies that $\|(L_h f)(g) - f(g)\|_S \leq 2\epsilon$. ■

1.2.1. Locally compact groups. A topological group G which is locally compact, and T_2 (Hausdorff) is called *locally compact group.*

Note that a topological group which is also locally compact is *not* necessarily a locally compact group: the separation property T_2 (Hausdorff) is assumed for a locally compact group. The following property is quite distinctive of these groups.

Proposition 1.8 *Let G be a locally compact group. Let H be a subgroup which is locally compact. Then H is closed.*

Proof. We prove that $\overline{H} \subseteq H$. Let $x \in \overline{H}$. Also $\overline{H}$ is a group (Prop. 1.4[4]), so $x^{-1} \in \overline{H}$ as well. By hypothesis H is locally compact in the induced topology, thus there exists a neighborhood K of e which is compact in H. Then K is compact also in G. Hence K is closed both in G and in H, because G and hence H are Hausdorff spaces. In particular, there exists F a closed (in G) neighborhood of e such that $K = F \cap H$.
We know that there exists $V \subseteq G$ such that $e \in V$, and $VV \subseteq F$. The sets Vx^{-1} and H intersect nontrivially, because $x^{-1} \in \overline{H}$ and Vx^{-1} is a neighborhood of x^{-1}: let $y \in Vx^{-1} \cap H$.
We prove now that $xy \in H$. In fact, let W be any neighborhood of xy. Then $y^{-1}W$ is a neighborhood of x, hence $y^{-1}W \cap xV$ is again a neighborhood of x, and there exists $z \in y^{-1}W \cap xV \cap H$ (because $x \in \overline{H}$). Then $yz \in W$ (evident), and $yz \in H$ (because y and z are in H which is a group). Moreover, the fact that $y \in Vx^{-1}$ with the inclusion $z \in xV$ implies that $yz \in Vx^{-1} \cdot xV = VV \subseteq F$.
Thus $yz \in W \cap (F \cap H)$, in particular $W \cap (F \cap H) \neq \emptyset$.
This happens for every neighborhood W of xy, and $F \cap H$ is closed (because it is K which

is closed in G), hence $xy \in F \cap H$. In particular $xy \in H$.
Now we can conclude the proof, because $x = xy \cdot y^{-1}$ and $xy \in H$ and $y^{-1} \in H$, hence also $x \in H$. ■

1.2.2. Haar measure. A collection $\mathfrak{M}$ of subsets of a set X is called *σ-algebra* when

i. $X \in \mathfrak{M}$,
ii. if $E \in \mathfrak{M}$, then $E^c \in \mathfrak{M}$ where E^c denotes the complement of E in X,
iii. suppose $E_n \in \mathfrak{M}$ for all $n \in \mathbb{N}$, then $E := \bigcup_{n\in\mathbb{N}} E_n \in \mathfrak{M}$.

(thus $\emptyset \in \mathfrak{M}$ and $\bigcap_{n\in\mathbb{N}} E_n \in \mathfrak{M}$).
A *measured* space X is the triplet $(X, \mathfrak{M}, \mu)$ where $\mathfrak{M}$ is a σ-algebra and μ is a *measure*, i.e. a map: $\mathfrak{M} \to [0, +\infty]$ ($+\infty$ admitted here) such that

$$\mu\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) = \sum_{n\in\mathbb{N}} \mu(E_n)$$

whenever the sets E_n are pairwise disjoint.
In particular, a *Borel-measure* is any such structure when X is locally compact and Hausdorff topological space, and $\mathfrak{M}$ contains the Borel sets. Essentially, we have a Borel measure when we have both a topology and a measure, and the measure interacts well with the topology.
Assume that X is locally compact and Hausdorff, and that μ is a Borel measure. Then

i. μ is called *outer regular* in a set $E \in \mathfrak{M}$ when
$$\mu(E) = \inf\{\mu(U) \colon E \subseteq U,\ U \text{ open}\};$$
ii. μ is called *inner regular* in a set $E \in \mathfrak{M}$ when
$$\mu(E) = \sup\{\mu(K) \colon K \subseteq E,\ K \text{ compact}\}.$$

A *Radon-measure* is a Borel measure (hence X is locally compact and Hausdorff) which is finite on compact sets, is outer regular on all Borel sets, and is inner regular on all open sets.
Essentially, a Radon measure is a Borel measure in which the measure of all Borel sets depends only on measures of compact sets. Or, in other words, a measure which is completely known on Borel sets when it is known on compact sets.

Let G be a topological group and let μ be a Borel measure on G. We say that it is *left invariant* when

$$\mu(gE) = \mu(E)$$

for every Borel subset $E \subseteq G$, and all $g \in G$, and *right invariant* when

$$\mu(Eg) = \mu(E)$$

for every Borel subset E and all $g \in G$.

Finally, let G be a locally compact topological group (hence it is also Hausdorff, by definition). A *left (resp. right) Haar measure* on G is a nonzero Radon measure that is left (resp. right) invariant.

Example. 1.1 $G := (\mathbb{R}^k, +)$, i.e. $\mathbb{R}^k$ with the vector sum. The Lebesgue measure $\mathrm{d}\mu := \mathrm{d}x$ is both a left and a right Haar measure.

Example. 1.2 $G := (\mathbb{R}^+, \times)$, i.e. $(0, +\infty)$ with the usual product. The measure $\mathrm{d}^*\mu := \mathrm{d}x/x$, where $\mathrm{d}x$ is the Lebesgue measure, is both a left and a right Haar measure.

Example. 1.3 $G := (\mathrm{GL}(n, \mathbb{R}), \cdot)$, i.e. the set of invertible matrices with real entries and the 'rows-times-columns' product. We introduce in G the topology that we obtain identifying $\mathrm{GL}(n, \mathbb{R})$ with a subset of $\mathbb{R}^{n^2}$. The measure $\mathrm{d}\mu := \prod_{i,j=1}^n \mathrm{d}x_{i,j}/|\det(M)|^n$, where each $\mathrm{d}x_{i,j}$ is the Lebesgue measure in the $x_{i,j}$ space, and $M := [x_{i,j}]_{i,j=1}^n$, is both a left and a right Haar measure.

Example. 1.4 Let

$$G := \left\{ \begin{pmatrix} \sqrt{b} & a/\sqrt{b} \\ 0 & 1/\sqrt{b} \end{pmatrix} : a \in \mathbb{R},\ b \in \mathbb{R}^+ \right\},$$

which is a subgroup of $\mathrm{GL}(n, \mathbb{R})$ since

$$\begin{pmatrix} \sqrt{b} & a/\sqrt{b} \\ 0 & 1/\sqrt{b} \end{pmatrix} \begin{pmatrix} \sqrt{b'} & a'/\sqrt{b'} \\ 0 & 1/\sqrt{b'} \end{pmatrix} = \begin{pmatrix} \sqrt{bb'} & (a + a'b)/\sqrt{bb'} \\ 0 & 1/\sqrt{bb'} \end{pmatrix}.$$

The formula shows that G is isomorphic to the set $\{(a, b) \in \mathbb{R} \times \mathbb{R}^+\}$, with $(a, b) \cdot (a', b') := (a + a'b, bb')$. Note that G is not abelian and is locally compact with respect to the usual topology of $\mathbb{R} \times \mathbb{R}^+$. Then

$$\mu_L(E) := \int_E \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \qquad \text{and} \qquad \mu_R(E) := \int_E \frac{\mathrm{d}x\,\mathrm{d}y}{y}$$

are left (respectively right)-Haar measures.

Proof. Let $g = \begin{pmatrix} \sqrt{b} & a/\sqrt{b} \\ 0 & 1/\sqrt{b} \end{pmatrix}$. Suppose (u, v) varies in $E \subseteq G$ and let $(x, y) \in gE$. Then

$$\begin{cases} x = a + ub \\ y = vb \end{cases}$$

thus the jacobian of the transform is $J = \left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right)$, with $\det J = b^2$. Then

$$\mu_L(gE) = \int_{gE} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_E b^2 \frac{\mathrm{d}u\,\mathrm{d}v}{(bv)^2} = \int_E \frac{\mathrm{d}u\,\mathrm{d}v}{v^2} = \mu_L(E).$$

On the other hands, suppose (u, v) varies in $E \subseteq G$ and let $(x, y) \in Eg$. Then

$$\begin{cases} x = u + va \\ y = vb \end{cases}$$

thus the jacobian of the transform is $J = \left(\begin{smallmatrix} 1 & a \\ 0 & b \end{smallmatrix}\right)$, with $\det J = b$. Then

$$\mu_R(Eg) = \int_{Eg} \frac{\mathrm{d}x\,\mathrm{d}y}{y} = \int_E b \frac{\mathrm{d}u\,\mathrm{d}v}{bv} = \int_E \frac{\mathrm{d}u\,\mathrm{d}v}{v} = \mu_R(E).$$

In this case μ_L and μ_R are different. ∎

Example. 1.5 This is a generalization of the previous example. Let

$$G := \{(g, w) \colon g \in \mathrm{GL}(n, \mathbb{R}),\ w \in \mathbb{R}^n\},$$

with

$$(g_1, w_1) \cdot (g_2, w_2) := (g_1 g_2, g_1 w_2 + w_1).$$

This is a group, with $(1, 0)$ as identity and $(g, w)^{-1} := (g^{-1}, -g^{-1}w)$ as inverse. It is the group of affine transformations in $\mathbb{R}^n$ (take the action $(g, w) \colon \mathbb{R}^n \to \mathbb{R}^n$, $(g, w)(a) := ga + w$). We introduce in G the topology that we obtain identifying $\mathrm{GL}(n, \mathbb{R})$ with a subset of $\mathbb{R}^{n^2}$, so that G becomes a subset of $\mathbb{R}^{n^2} \times \mathbb{R}^n$. The measures

$$\mu_L(E) := \int_E \frac{\prod_{i,j=1}^n \mathrm{d}x_{i,j} \cdot \prod_{i=1}^n \mathrm{d}y_i}{|\det(M)|^{n+1}} \quad \text{and} \quad \mu_R(E) := \int_E \frac{\prod_{i,j=1}^n \mathrm{d}x_{i,j} \cdot \prod_{i=1}^n \mathrm{d}y_i}{|\det(M)|^n},$$

where each $\mathrm{d}x_{i,j}$ and each $\mathrm{d}y_i$ is the Lebesgue measure in the corresponding space and $M := [x_{i,j}]_{i,j=1}^n$, are left (respectively right)-Haar measures. Also in this case the measures μ_L and μ_R are different.

Example. 1.6 Let G be the group of nonzero quaternions, i.e.

$$G := \{a + b\underline{i} + c\underline{j} + d\underline{k} \colon (a,b,c,d) \neq (0,0,0,0) \in \mathbb{R}^4\},$$

with $\underline{i}^2 = \underline{j}^2 = \underline{k}^2 = \underline{i}\underline{j}\underline{k} = -1$. G may be represented via 4×4 matrices,

$$a + b\underline{i} + c\underline{j} + d\underline{k} \qquad \mapsto \qquad M := \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix},$$

and the product of elements in G corresponds to the product (in the same order) of the representing matrices. A (tedious) computation shows that

$$\det(M) = (a^2 + b^2 + c^2 + d^2)^2 = \|a + b\underline{i} + c\underline{j} + d\underline{k}\|^4.$$

The identification of G with the open subset $\mathbb{R}^4 \backslash \{(0,0,0,0)\}$ allows one to consider G as a locally compact topological group. The measure

$$\mu(E) := \int_E \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}u}{(x^2+y^2+z^2+u^2)^2} = \int_E \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}u}{\|x + y\underline{i} + z\underline{j} + u\underline{k}\|^4}$$

is a doubly invariant Haar measure (i.e. it is both left and right invariant). Here each $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}u$ is the Lebesgue measure in $\mathbb{R}^4$.

Let

$$\mathcal{C}_c^+(G) := \{f \colon G \to \mathbb{R}^+, \operatorname{supp}(f) \text{ is compact, } \|f\|_\infty > 0\}$$

the set of nonnegative, not identically zero, continuous and compactly supported functions.

Proposition 1.9 *Let G be a locally compact group. Then:*

- *Let μ_L be a left-Haar measure. Let μ_R be the measure defined by $\mu_R(E) := \mu_L(E^{-1})$ for every $E \in \mathfrak{M}$. Then μ_R is a right-invariant Haar measure.*
- *Let μ be a Radon measure. It is left-Haar on G if and only if*

$$\int_G L_g f \,\mathrm{d}\mu = \int_G f \,\mathrm{d}\mu$$

  for every $f \in \mathcal{C}_c^+(G)$ and every $g \in G$.
- *Let μ be a left-Haar measure. Then μ is positive on open and nonempty sets, and $\int_G f \,\mathrm{d}\mu > 0$ for all $f \in \mathcal{C}_c^+(G)$.*
- *Let μ be a left-Haar measure. Then $\mu(G) < +\infty$ if and only if G is compact.*

Proof.

- The map $\iota \colon G \to G$ is a homeomorphism, thus μ_R is a Radon measure. It is right invariant because $\mu_R(Eg) = \mu_L((Eg)^{-1}) = \mu_L(g^{-1}E^{-1}) = \mu_L(E^{-1}) = \mu_R(E)$.
- Let μ be left invariant. Then the equality of integrals $\int_G L_g f \,\mathrm{d}\mu = \int_G f \,\mathrm{d}\mu$ holds true for all positive simple functions f (i.e., positive linear combinations of characteristic functions of measurable sets), and it is true also for functions in $\mathcal{C}_c^+(G)$ by density arguments[1].

[1] The usual argument runs as follows. Let $\phi_n \colon [0,+\infty) \to [0,+\infty)$, with

$$\phi_n(x) := \min\Big(2^n, \frac{\lfloor 2^n x \rfloor}{2^n}\Big).$$

Then, for every compact $K \subseteq [0,+\infty)$, there is N such that $\sup_{x\in K} |\phi_n(x) - x| \leq 2^{-n}$ when $n \geq N$. Let $f \in \mathcal{C}_c^+(G)$, and set $f_n := \phi_n \circ f$. The range of f is a compact in $[0,+\infty)$ (because f is continuous and its support is a compact). Hence $\|f_n - f\|_\infty \leq 2^{-n}$ as soon as n is large enough. Each f_n is a simple function (because ϕ_n is left-continuous and f is continuous). This proves that each function in $\mathcal{C}_c^+(G)$ is the uniform limit of simple functions. Moreover, the support of f_n is the one of f and is a compact, therefore f is the limit of f_n also in L^1 norm.

On the contrary, suppose $\int_G L_g f \,\mathrm{d}\mu = \int_G f \,\mathrm{d}\mu$ for all $f \in \mathcal{C}_c^+(G)$. Let U be any open set. The equality $\mu(gU) = \mu(U)$ for every $g \in G$ follows immediately as soon as we prove that

$$\mu(U) = \sup\Big\{ \int_G f \,\mathrm{d}\mu \colon f \in \mathcal{C}_c^+(G),\ \mathrm{supp}(f) \subseteq U,\ \|f\|_\infty = 1 \Big\}.$$

Once the the invariance is proved for all open set, then the invariance for all Borel sets follows by the assumed outer regularity of the measure.
Thus we have to prove the previous formula for $\mu(U)$. Let ℓ be the value of the sup. For every such f we have $\int_G f \,\mathrm{d}\mu \le \int_G \chi_{\mathrm{supp}(f)} \,\mathrm{d}\mu$ (because $0 \le f(g) \le 1$ by hypothesis) $= \mu(\mathrm{supp}(f)) \le \mu(U)$. This proves that $\ell \le \mu(U)$. On the other hand, suppose that $\mu(U) < +\infty$. μ is inner regular, hence for every $\epsilon > 0$ there is a compact $K \subseteq U$ such that $\mu(K) \ge \mu(U) - \epsilon$. By Urysohn's Lemma (see [R], p. 39: note that local compactness and T_2 separation are used here) there is function $f \in \mathcal{C}_c^+(G)$ such that $\chi_K(g) \le f(g) \le \chi_U(g)$. Thus $\|f\|_\infty = 1$ and

$$\mu(U) \le \mu(K) + \epsilon = \int_G \chi_K \,\mathrm{d}\mu + \epsilon \le \int_G f \,\mathrm{d}\mu + \epsilon.$$

This proves that $\mu(U) \le \ell + \epsilon$. ϵ being arbitrary, we deduce that $\mu(U) \le \ell$, which proves the equality $\mu(U) = \ell$ in this case. For the case $\mu(U) = +\infty$ the proof is similar.

- μ is not trivial (i.e. identically zero), hence by inner regularity there is a compact K with $\mu(K) > 0$. Let U be a nonempty open set. Without loss of generality we can assume that $e \in U$ (otherwise set $x \in U$ and consider $x^{-1}U$: by left invariance, $\mu(U) = \mu(x^{-1}U)$ and hence the first one is positive if and only if the second is positive). The set $\bigcup_{a\in K} aU$ is a covering of K, which by compactness we can reduce to a finite covering $\bigcup_{i=1}^N a_j U$. Hence $\mu(K) \le \mu(\bigcup_{i=1}^N a_j U)$ (inclusion) $\le \sum_{i=1}^N \mu(a_j U)$ (subadditivity) $= \mu(U)N$ (because μ is left-invariant). By hypothesis $\mu(K) > 0$, hence $\mu(U) \ge \mu(K)/N > 0$, which sets the first claim.
  Let $f \in \mathcal{C}_c^+(G)$, f not identically zero; fix any δ in the range of f, and take $U := \{g \in G \colon f(g) > \delta/2\}$. Then U is open and nonempty, and
  $$\int_G f \,\mathrm{d}\mu \ge \int_U \frac{\delta}{2} \,\mathrm{d}\mu \ge \frac{\delta}{2}\mu(U) > 0.$$
- If G is compact, then $\mu(G) < +\infty$ for all Radon measures. Assume that G is not compact. Let K be any compact neighborhood of e (it exists, because G is locally compact), and let V be a symmetric neighborhood of e such that $e \in V \subseteq VV \subseteq K$ (we know that it exists). We define a countable family of points $(g_n)_{n\in\mathbb{N}}$ in the following way. We set $g_0 := e$, and we proceed by induction. Suppose we have already found $g_0, \dots, g_{n-1}$, then we pick any g_n in $G \setminus \bigcup_{i=0}^{n-1} g_i K$: g_n exists because $\bigcup_{i=0}^{n-1} g_i K$ is compact and G is *not* compact by hypothesis.
  Note that $g_i V$ and $g_j V$ do not intersect for $i \ne j$. In fact, assume $i < j$, then $g_i V \cap g_j V \ne \emptyset$ if and only if $g_j \in g_i V V^{-1} = g_i V V \subseteq g_i K$, which contradicts the rule we have used for the selection of g_j. Hence
  $$\mu(G) \ge \mu\Big(\bigcup_{n\in\mathbb{N}} g_n V\Big) = \sum_{n\in\mathbb{N}} \mu(g_n V) = \sum_{n\in\mathbb{N}} \mu(V) = \infty.$$
  The last step comes from $\mu(V) > 0$, which is true because V is a nonempty open set.

■

The main result of this section is the following claim. It is due to Haar, with important simplifications for its proof by von Newman, H. Cartan and Weil.

Theorem 1.1 *Let G be a locally compact group. Then G admits a left-Haar measure. Moreover, the measure is unique up to a moltiplicative factor.*

According to the spirit of Proposition 1.9, we prove the theorem by proving the existence of a linear functional on $\mathcal{C}_c(G)$ which is left-invariant. The construction needs some preparatory tools which are collected here; the *covering number* is *the* great idea allowing Haar to prove his result. The introduction of the product space and the use of Tychonoff's theorem is a simplification due to Weil, while H. Cartan proposed a different (and longer) approach avoiding the need of the choice axiom which is implicit in Tychonoff theorem. For this alternative approach see [DS], Ch. 6.

Let $f, \varphi \in \mathcal{C}_c^+(G)$. Recall that the definition of $\mathcal{C}_c^+(G)$ ensures that φ is not identically 0, so that $\|\varphi\|_\infty$ is positive. Let $U := \{g \in G \colon |\varphi(g)| > \|\varphi\|_\infty/2\}$. U is a nonempty open set. The support of f is a compact, and $\bigcup_{s\in G} sU$ equals G and hence covers $\operatorname{supp}(f)$. Therefore finitely many s_j, $j = 1, \ldots, N$ suffice to have $\operatorname{supp}(f) \subseteq \bigcup_{j=1}^N s_jU$. This inclusion implies that

$$f \leq \frac{2\|f\|_\infty}{\|\varphi\|_\infty} \sum_{j=1}^N L_{s_j}\varphi$$

because if s belongs to the support of f, then it is in some s_jU, and hence $s_j^{-1}s \in U$, so that $(L_{s_j}\varphi)(s) > \|\varphi\|_\infty/2$. This suggests the introduction of the following object:

$$(f : \varphi) := \inf\Big\{\sum_{j=1}^N c_j \colon 0 < c_1, \ldots, c_N, \text{ with } f \leq \sum_{j=1}^N c_j L_{s_j}\varphi \text{ for some } s_1, \ldots, s_N \in G\Big\}.$$

It is called *Haar covering number* of f with respect to φ. Note that its value is never zero, because f is not identically zero (see also Proposition 1.10[5.]).

Proposition 1.10 *The Haar covering number satisfies the following properties:*

1. $(f : \varphi) = (L_s f : \varphi)$ *for every* $s \in G$ *(left-invariant).*
2. $(f_1 + f_2 : \varphi) \leq (f_1 : \varphi) + (f_2 : \varphi)$ *(subadditive).*
3. $(cf : \varphi) = c(f : \varphi)$ *for all* $c > 0$ *(multiplicative by constants).*
4. $f_1 \leq f_2$ *implies* $(f_1 : \varphi) \leq (f_2 : \varphi)$ *(monotone).*
5. $(f : \varphi) \geq \frac{\|f\|_\infty}{\|\varphi\|_\infty}$ *(bounded from below).*
6. $(f : \varphi) \leq (f : f_0)(f_0 : \varphi)$ *for every* $f_0 \in \mathcal{C}_c^+(G)$ *(multiplically transitive).*

Proof.

1. This is easy, since L_s is a linear map and a morphism, so that

$$f \leq \sum_{j=1}^N c_j L_{s_j}\varphi \qquad \Longleftrightarrow \qquad L_s f \leq \sum_{j=1}^N c_j L_{ss_j}\varphi,$$

showing that the same set of c_j's serves for f and for $L_s f$.

2,3,4. Evident.

5. Let $f \leq \sum_{i=1}^N c_j L_{s_j}\varphi$. Then $\|L_{s_j}\varphi\|_\infty = \|\varphi\|_\infty$, thus $f \leq \|\varphi\|_\infty \sum_{i=1}^N c_j$. Therefore $\|f\|_\infty \leq \|\varphi\|_\infty \sum_{i=1}^N c_j$, i.e. $\sum_{i=1}^N c_j \geq \|f\|_\infty/\|\varphi\|_\infty$, which proves the claim since the lower bound is independent of the c_j's.
6. Let $f \leq \sum_{i=1}^M c_i L_{s_i} f_0$ and $f_0 \leq \sum_{j=1}^N c_j' L_{s_j'}\varphi$, then

$$f \leq \sum_{i=1}^M c_i L_{s_i}\Big(\sum_{j=1}^N c_j L_{s_j'}\varphi\Big) \leq \sum_{i=1}^M \sum_{j=1}^N c_i c_j L_{s_i s_j'}\varphi$$

proving that $(f : \varphi) \leq \big(\sum_{i=1}^M c_i\big)\big(\sum_{j=1}^N c_j'\big)$. The claim follows since the right hand side is as close to $(f : f_0)(f_0 : \varphi)$ as we want.

■

The Haar covering number gives a kind of functional on $\mathcal{C}_c^+(G)$, but it is not a true functional since there are cases where the subadditivity proved in Proposition 1.10[2.] is a proper inequality. The idea is that we can overcome this difficulty changing φ in a suitable net where the support becomes smaller and smaller: in this way φ behaves as a kind of 'Dirac-delta' that in the limit produces a real functional. Unfortunately this idea cannot be applied as it is, because in this way we only produce a functional which is additive because it equals ∞ in every function! To control this phenomenon we apply the idea to a *quotient* of indexes, which in the limit will stay bounded. Thus, we fix f_0 and φ in $\mathcal{C}_c^+(G)$, not identically zero, and set

$$I_\varphi(f) := \frac{(f : \varphi)}{(f_0 : \varphi)}$$

for every $f \in \mathcal{C}_c^+(G)$. By Proposition 1.10[6.], we deduce that

$$(f_0 : f)^{-1} \leq I_\varphi(f) \leq (f : f_0). \tag{1.1}$$

This formula is (one of) the key features of this object: it shows how to bound $I_\varphi(f)$ both from below and from above, with numbers *which do not depend on* φ: the existence of this *double* bound is the key property showing that we are on the good path toward the construction of the functional.

The following proposition shows a kind of control from below for its values, which substitutes the additivity and will produce the full additivity when we will run in a suitable limit process.

Proposition 1.11 *Let $f_1, f_2 \in \mathcal{C}_c^+(G)$ and $\epsilon > 0$. Then there is a neighborhood U of e such that*

$$I_\varphi(f_1) + I_\varphi(f_2) \leq I_\varphi(f_1 + f_2) + \epsilon$$

for every φ with $\operatorname{supp}(\varphi) \subseteq U$.

Proof. By Urysohn's Lemma there is a function $g \in \mathcal{C}_c^+(G)$ that takes value 1 on $\operatorname{supp}(f_1 + f_2) = \operatorname{supp}(f_1) \cup \operatorname{supp}(f_2)$. Let $\delta > 0$ and set $h := f_1 + f_2 + \delta g$, and let $F_1 := f_1/h$, $F_2 := f_2/h$, with the convention that $F_i = 0$ outside $\operatorname{supp}(f_i)$, for $i = 1, 2$. Each F_i is in $\mathcal{C}_c^+(G)$ (because g equals 1 on $\operatorname{supp}(f_1 + f_2)$, so that f_i and h have no common zeros), and their sum

$$F_1 + F_2 = \frac{f_1 + f_2}{f_1 + f_2 + \delta g}$$

approaches 1 from below when $\delta \to 0$. By uniform continuity, there is a neighborhood U of e such that $|F_i(s) - F_i(t)| \leq \delta$ when $t^{-1}s \in U$.
Assume that $\operatorname{supp}(\varphi) \subseteq U$ and that

$$h \leq \sum_{j=1}^{N} c_j L_{s_j} \varphi.$$

Then

$$f_i(s) = h(s)F_i(s) \leq \sum_{j=1}^{N} c_j L_{s_j}\varphi(s)F_i(s) = \sum_{j=1}^{N} c_j \varphi(s_j^{-1}s)F_i(s).$$

By assumption φ is supported on U, hence this is

$$\leq \sum_{j=1}^{N} c_j \varphi(s_j^{-1}s)(F_i(s_j) + \delta) \qquad (i = 1, 2),$$

and it follows that

$$(f_i : \varphi) \leq \sum_{j=1}^{N} c_j (F_i(s_j) + \delta) \qquad (i = 1, 2).$$

Since $F_1 + F_2 \leq 1$, this inequality implies that

$$(f_1 : \varphi) + (f_2 : \varphi) \leq (1 + 2\delta) \sum_{j=1}^{N} c_j.$$

But $\sum_{j=1}^{N} c_j$ can be made arbitrarily close to $(h : \varphi)$, and therefore by definition of I_φ we get

$$I_\varphi(f_1) + I_\varphi(f_2) \leq (1 + 2\delta) I_\varphi(h).$$

Since $h = f_1 + f_2 + \delta g$, By Proposition 1.10[2.] this is bounded by

$$\begin{aligned} I_\varphi(f_1) + I_\varphi(f_2) &\leq (1 + 2\delta)(I_\varphi(f_1 + f_2) + \delta I_\varphi(g)) \\ &= I_\varphi(f_1 + f_2) + \delta(2 I_\varphi(f_1 + f_2) + (1 + 2\delta) I_\varphi(g)). \end{aligned}$$

Equation (1.1) shows that all terms appearing to the right may be bounded independently of φ, so they may be made smaller than ϵ with a suitable choice of δ. ■

Existence of Haar measure. Let X be the compact topological set defined by

$$X := \prod_{f \in \mathcal{C}_c^+(G)} [(f_0 : f)^{-1}, (f : f_0)].$$

By (1.1) each function I_φ may be considered as an element in X.
For every compact neighborhood U of e let K_U be the *closure* in X of the set

$$\{I_\varphi \colon \operatorname{supp}(\varphi) \subseteq U\}.$$

The collection $\{K_U\}_U$ satisfies the finite intersection property, i.e. every finite intersection is not empty. This happens because

$$\bigcap_{j=1}^{N} K_{U_j} \supseteq K_{\bigcap_{j=1}^{N} U_j}$$

and a nonzero function g in $\mathcal{C}_c^+(G)$ supported in $\bigcap_{j=1}^{N} U_j$ exists by Urysohn's Lemma. Since X is compact, this implies that $\bigcap_U K_U$ is not empty[2]. Let I be an element in this intersection. We will prove that I is the functional we are looking for.
Since I is in the intersection of all K_U, it follows that every open neighborhood if I intersects each of the sets $\{I_\varphi \colon \operatorname{supp}(\varphi) \subseteq U\}$ nontrivially. In particular this happens for open cylindric sets so that:

> for every open neighborhood U of e and for every finite set of functions $f_j \in \mathcal{C}_c^+(G)$, $j = 1, \ldots, N$ and every $\epsilon > 0$, there exists a function $\varphi \in \mathcal{C}_c^+(G)$ with $\operatorname{supp}(\varphi) \subseteq U$ such that $|I(f_j) - I_\varphi(f_j)| \leq \epsilon$ for $j = 1, \ldots, N$.

So, given $f \in \mathcal{C}_c^+(G)$ and a fixed $s \in G$, we can have both $|I(f) - I_\varphi(f)| \leq \epsilon$ and $|I(L_s f) - I_\varphi(L_s f)| \leq \epsilon$. Recalling Proposition 1.10[1.] implying that $I_\varphi(L_s f) = I_\varphi(f)$, we get

$$\begin{aligned} |I(L_s f) - I(f)| &\leq |I(L_s f) - I_\varphi(L_s f)| + |I_\varphi(L_s f) - I(f)| \\ &= |I(L_s f) - I_\varphi(L_s f)| + |I_\varphi(f) - I(f)| \leq 2\epsilon. \end{aligned}$$

The arbitrariness of ϵ yields the equality $I(L_s f) = I(f)$ (i.e. I is left-invariant).

[2]Recall that the finite intersection property for X is equivalent to the compactness of X. In fact, let X be a compact set. The $\bigcap_U K_U = \emptyset$ if and only if $\bigcup_U K_U^c = X$. Each K_U^c is open and X is compact, hence a finite set is sufficient to cover X, i.e., $\bigcup_{j=1}^{N} K_{U_j}^c = X$, and this happens if and only if $\bigcap_{j=1}^{N} K_{U_j} = \emptyset$. On the other hand, suppose that X has the finite intersection property. Let $\bigcup_\alpha V_\alpha = X$ an open covering of X. Then $\bigcap_\alpha V_\alpha^c = \emptyset$. The there is some finite subset $\bigcap_{j=1}^{N} V_{\alpha_j}^c = \emptyset$ (by the finite intersection property), i.e. $\bigcup_{j=1}^{N} V_{\alpha_j} = X$.

In similar way, given $f \in \mathcal{C}_c^+(G)$ and $c > 0$, we can have both $|I(f) - I_\varphi(f)| \leq \epsilon$ and $|I(cf) - I_\varphi(cf)| \leq \epsilon$. Multiplying the first relation by c (and recalling Proposition 1.10[3.]), we get

$$|I(cf) - cI(f)| \leq |I(cf) - I_\varphi(cf)| + |I_\varphi(cf) - cI(f)| \leq (1 + c)\epsilon.$$

The arbitrariness of ϵ yields the equality $I(cf) = cI(f)$ (i.e. I is multiplicative).

In similar way, given f_1, f_2 and $f_1 + f_2$, we can have $|I(f_1) - I_\varphi(f_1)| \leq \epsilon$, $|I(f_2) - I_\varphi(f_2)| \leq \epsilon$, and $|I(f_1 + f_2) - I_\varphi(f_1 + f_2)| \leq \epsilon$. Recalling Proposition 1.10[2.], we get

$$\begin{aligned} I(f_1 + f_2) &\leq I_\varphi(f_1 + f_2) + \epsilon \leq I_\varphi(f_1) + I_\varphi(f_2) + \epsilon \\ &\leq I(f_1) + I(f_2) + 3\epsilon. \end{aligned}$$

The arbitrariness of ϵ proves that I is subadditive.

Actually, it is additive. In fact, let $\epsilon > 0$. by Proposition 1.11, there is an open neighborhood U of e such that

$$I_\varphi(f_1) + I_\varphi(f_2) \leq I_\varphi(f_1 + f_2) + \epsilon$$

for every φ with $\mathrm{supp}(\varphi) \subseteq U$. Moreover, we choose U also such that all φ give $I_\varphi(f_1)$, $I_\varphi(f_2)$ and $I_\varphi(f_1 + f_2)$ within ϵ of $I(f_1)$, $I(f_2)$ and $I(f_1 + f_2)$, respectively. Thus, the previous inequality gives

$$I(f_1) + I(f_2) \leq I(f_1 + f_2) + 4\epsilon.$$

Since ϵ is arbitrary, this means that

$$I(f_1) + I(f_2) \leq I(f_1 + f_2)$$

which with the subadditivity already proved, gives the full additivity of I.

Finally, we extend I to a full linear functional on $\mathcal{C}_c(G)$ by setting $I(f) := I(f^+) - I(f^-)$.

Uniqueness of Haar measure. Suppose we have two left-Haar measures μ and ν on G. We prove that the quotient

$$\frac{I(f)}{J(f)} := \frac{\int_G f \, \mathrm{d}\mu}{\int_G f \, \mathrm{d}\nu},$$

which is well defined for $f \in \mathcal{C}_c^+(G)$, is actually independent on f. This implies immediately that $\mu(K)/\nu(K)$ is independent on K for every compact set K with nonempty $\mathring{K}$, and this implies that the measures (which are Radon measures by definition of Haar measure) differ only by a constant nonzero factor.

We prove the independence of $I(f)/J(f)$ of f by proving that for every couple $f, g \in \mathcal{C}_c^+(G)$ and every $\epsilon > 0$ we can produce a new function $h \in \mathcal{C}_c^+(G)$ such that $I(f)/J(f)$ and $I(g)/J(g)$ are both ϵ-close to $I(h)/J(h)$. Thus, let $f, g \in \mathcal{C}_c^+(G)$ be arbitrarily chosen, but fixed. Let K be a compact neighborhood of e (which exists, because G is locally compact). Then K contains an open neighborhood of e which is symmetric (by Proposition 1.4[1.]) and whose closure K_0 is a compact subset of K (K_0 is closed in a compact K, hence it is compact), and which is also symmetric (because it is the closure of a symmetric set). Let

$$K_f := \mathrm{supp}\, f \cdot K_0 \cup K_0 \cdot \mathrm{supp}\, f, \qquad K_g := \mathrm{supp}\, g \cdot K_0 \cup K_0 \cdot \mathrm{supp}\, g.$$

They are compact sets (recall that the product of two compacts sets is compact). For $t \in K_0$, let $\gamma_t f$ and $\gamma_t g$ be the new functions

$$\begin{aligned} \gamma_t f(s) &:= f(st) - f(ts) = (R_t f)(s) - (L_{t^{-1}} f)(s), \\ \gamma_t g(s) &:= g(st) - g(ts) = (R_t g)(s) - (L_{t^{-1}} g)(s). \end{aligned}$$

They are supported in K_f and K_g respectively, because K_0 is symmetric. Moreover, both $\gamma_t f$ and $\gamma_t g$ vanish identically whenever t belongs to the center of G, and in particular for $t = e$. Let $\epsilon > 0$. By left and right uniform continuity, K_0 contains an open neighborhood U_0 of e such that for all $s \in G$ and all $t \in U_0$ both $|\gamma_t f(s)|$ and $|\gamma_t g(s)|$ are $\leq \epsilon$ (note that the uniformity here is exploited in the parameter t). In turn, U_0 contains an open

and symmetric neighborhood U_1 of e, whose closure K_1 is symmetric, compact, and which is contained in K_0. Moreover, by continuity $|\gamma_t f(s)| \leq \epsilon$ and $|\gamma_t g(s)| \leq \epsilon$ for $s \in G$ and $t \in K_1$. In other words, for $t \in K_1$ the left and the right t-translation have on f and g approximatively the same effect.
Now the construction of h. Since e is in the open part of K_1, there exists a second compact neighborhood K_2 of e with $K_2 \subseteq \mathring{K}_1$.

Proof. Let V be an open and symmetric neighborhood of e such that $VV \subseteq \mathring{K}_1$ (it exists, by Proposition 1.4[1.]). The set $\overline{V}$ is compact, because $V \subseteq VV \subseteq \mathring{K}_1 \subseteq K_1$, so that $\overline{V}$ is a closed subset of K_1. Moreover, take $g \in \overline{V}$. Then gV is a neighborhood of g and its intersection $gV \cap V$ is not empty (because $g \in \overline{V}$). Thus there exists $v_1, v_2 \in V$ such that $gv_1 = v_2$. Hence $g = v_2 v_1^{-1} \in VV^{-1} = VV \subseteq \mathring{K}_1$. This proves that $\overline{V} \subseteq \mathring{K}_1$, so that the claim holds with $K_2 := \overline{V}$. ■

From Urysohn's lemma ([R], p. 39), there is a continuous function $\tilde{h}\colon G \to \mathbb{R}^+$ which is 1 in K_2 and 0 outside K_1. Let $h\colon G \to \mathbb{R}^+$, $h(s) := \tilde{h}(s) + \tilde{h}(s^{-1})$. Note that $h \in \mathcal{C}_c^+(G)$, it is supported on K_1, and it is an even function (i.e., $h(s) = h(s^{-1})$).
Now some computations.

$$\begin{aligned} I(f)J(h) &= \int_G \int_G f(s)h(t)\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t \\ &= \int_G \int_G f(ts)h(t)\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t \quad (s \mapsto ts \text{ in the inner integral}) \end{aligned}$$

(by left-invariance of the μ measure). Moreover,

$$\begin{aligned} I(h)J(f) &= \int_G \int_G h(s)f(t)\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t \\ &= \int_G \int_G h(t^{-1}s)f(t)\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t \quad (s \mapsto t^{-1}s \text{ in the inner integral}) \\ &= \int_G \int_G h(s^{-1}t)f(t)\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t \quad (\text{using the parity of } h). \end{aligned}$$

Now we consider the double integral as a unique integral in $G \times G$ with respect to the product measure $\mathrm{d}\mu_s \otimes \mathrm{d}\nu_t$. The classical theorems of Tonelli-Fubini apply here, since $G \times G$ is locally compact, the measures are Radon measures, and the functions we are integrating are continuous and compactly supported in $G \times G$. As a consequence we can exchange the order of integrals, without affecting its value. This yields

$$\begin{aligned} &= \int_G \int_G h(s^{-1}t)f(t)\,\mathrm{d}\nu_t\,\mathrm{d}\mu_s \\ &= \int_G \int_G h(t)f(st)\,\mathrm{d}\nu_t\,\mathrm{d}\mu_s \quad (t \mapsto st \text{ in the inner integral}) \\ &= \int_G \int_G h(t)f(st)\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t \quad (\text{exchange the integrals}). \end{aligned}$$

Hence their difference is

$$\begin{aligned} |I(h)J(f) - I(f)J(h)| &= \Big| \int_G \int_G h(t)[f(st) - f(ts)]\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t \Big| \\ &\leq \int_G \int_G h(t)|\gamma_t f(s)|\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t. \end{aligned}$$

The function h is supported on K_1 and $\gamma_t f$ on K_f, hence we can introduce the characteristic functions of these sets, without affecting the value of the integral which becomes

$$= \int_G \int_G h(t)\chi_{K_1}(t)\chi_{K_f}(s)|\gamma_t f(s)|\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t.$$

We know that $|\gamma_t f(s)| \leq \epsilon$ when t is in K_1, thus this is

$$\begin{aligned} &\leq \epsilon \int_G \int_G h(t)\chi_{K_1}(t)\chi_{K_f}(s)\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t \\ &= \epsilon \int_G \int_G h(t)\chi_{K_f}(s)\,\mathrm{d}\mu_s\,\mathrm{d}\nu_t \\ &= \epsilon\mu(K_f)\int_G h(t)\,\mathrm{d}\nu_t = \epsilon\mu(K_f)J(h). \end{aligned}$$

Dividing by $J(h)J(f)$ we get:

$$\Big|\frac{I(h)}{J(h)} - \frac{I(f)}{J(f)}\Big| \leq \epsilon\frac{\mu(K_f)}{J(f)}.$$

A similar computation with g and h gives

$$\Big|\frac{I(h)}{J(h)} - \frac{I(g)}{J(g)}\Big| \leq \epsilon\frac{\mu(K_g)}{J(g)}.$$

Thus

$$\Big|\frac{I(f)}{J(f)} - \frac{I(g)}{J(g)}\Big| \leq \epsilon\Big[\frac{\mu(K_f)}{J(f)} + \frac{\mu(K_g)}{J(g)}\Big],$$

showing that $\frac{I(f)}{J(f)} = \frac{I(g)}{J(g)}$, because ϵ is arbitrary.

1.3. Modular function

Let G be a locally compact group, and let μ be (one determination of) its right-Haar measure with respect to a σ-algebra $\mathfrak{M}$. Pick any $g \in G$, and let $\mu_g \colon \mathfrak{M} \to \mathbb{R}$, $\mu_g(E) := \mu(gE)$. Then μ_g is a Radon measure which is right invariant, so it is a right Haar measure. Thus, by the theorem characterizing the Haar-measures, there is a number $\Delta(g)$ such that

$$\mu_g(E) = \mu(gE) = \Delta(g)\mu(E), \qquad \forall E \in \mathfrak{M}.$$

The function $\Delta\colon G \to \mathbb{R}$ associating $g \mapsto \Delta(g)$ is the so called *modular* function. Its definition immediately shows that

$$\Delta(gg')\mu(E) = \mu(gg'E) = \mu_{gg'}(E) = \Delta(g)\mu(g'E) = \Delta(g)\Delta(g')\mu(E).$$

There always exists a set E with $0 < \mu(E) < +\infty$: for example take a nonempty open with a compact closure, which always exists in Hausdorff locally compact spaces. Thus we can remove $\mu(E)$ from the previous computation, getting

$$\Delta(gg') = \Delta(g)\Delta(g'),$$

which proves that the modular function is a morphism of G into the group $\mathbb{R}^+$. In particular $\Delta(g) \neq 0$ for every g, $\Delta(e) = 1$ and $\Delta(g^{-1}) = \Delta(g)^{-1}$.

Note that Δ is trivial when G is commutative, because in that case the left and the right Haar measures coincide.

We want to prove that Δ is well related also with the topological structure of G: in fact, it is a continuous map.

Lemma 1.1 *Let K be a compact set, let U be an open set, and assume $K \subseteq U$. Then there exists an open and symmetric neighborhood V of e such that $VK \subseteq U$.*

Proof. For every $g \in K$, let $W_g := Ug^{-1}$. It is an open set which contains e (because $g \in K \subseteq U$). Let V_g be an open and symmetric neighborhood of e such that $V_gV_g \subseteq W_g$. Then $\bigcup_{g\in K} V_g g$ is an open covering of K. By compactness there exists a finite sub covering $K \subseteq \bigcup_{j=1}^N V_{g_j}g_j$. Let $V := \bigcap_{j=1} V_{g_j}$, which is an open and symmetric neighborhood of e. Then $VK \subseteq \bigcup_{j=1}^N VV_{g_j}g_j \subseteq \bigcup_{j=1}^N V_{g_j}V_{g_j}g_j \subseteq \bigcup_{j=1}^N W_{g_j}g_j = U$. ■

Proposition 1.12 *The modular function is a continuous map.*

Proof. It is a morphism, thus it is sufficient to prove that it is continuous in e, i.e. that for every $\epsilon > 0$ there is an open neighborhood V of e such that $|\Delta(g) - 1| \leq \epsilon$.

Let K be any compact with $\mathring{K} \neq \emptyset$; it exists because G is locally compact and Hausdorff. Fix $\epsilon > 0$. The Haar measure is outer regular, hence there is an open set U such that $K \subseteq U$ and

$$\mu(K) \leq \mu(U) \leq \mu(K)(1+\epsilon).$$

Let V be an open and symmetric neighborhood of e such that $VK \subseteq U$ (it exists by Lemma 1.1). Let $g \in V$. Then

$$\Delta(g) = \frac{\mu(gK)}{\mu(K)} \leq \frac{\mu(U)}{\mu(K)} \leq 1+\epsilon.$$

On the other hand, by construction also g^{-1} is in V, hence

$$\Delta(g) = \frac{1}{\Delta(g^{-1})} \geq \frac{1}{1+\epsilon} \geq 1-\epsilon.$$

Therefore we have proved that $|\Delta(g) - 1| \leq \epsilon$ for every $g \in V$. ■

Proposition 1.13 *Let G be compact group. Then Δ is trivial, so that the right-Haar measure is also left invariant.*

Proof. The modular function is continuous, hence $\Delta(G)$ is a compact subgroup of $\mathbb{R}^+$. But $\{1\}$ is the unique compact subgroup in $\mathbb{R}^+$, so the claim follows. ■

Note that Δ is trivial on G', the subgroup generated by the commutators of G, because its image is in $\mathbb{R}^+$ which is abelian. Since Δ is continuous, it is trivial also on $\overline{G'}$, the closure of G'. Thus Δ can be considered as a morphism from the quotient $G/\overline{G'}$ into $\mathbb{R}^+$, setting $\Delta' \colon G/\overline{G'} \to \mathbb{R}^+$ with $\Delta'(g\overline{G'}) := \Delta(g)$.

$$\begin{array}{ccc} G & \xrightarrow{\ \Delta\ } & \mathbb{R}^+ \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \Delta'} & \\ G/\overline{G'} & & \end{array}$$

The map Δ' remains continuous, since the projection $G \to G/\overline{G'}$ is an open map and the diagram commutes by construction.

Let μ be the right-Haar measure. For every $f \in \mathcal{C}_c(G)$ the function

$$g \mapsto f(g)\Delta(g^{-1}) = f(g)/\Delta(g)$$

is again in $\mathcal{C}_c(G)$, because the modular function is nonzero and continuous. Hence the number

$$I_\ell(f) := \int_G \frac{f(g)}{\Delta(g)} \, \mathrm{d}\mu(g)$$

is well defined, and actually defines a functional on $\mathcal{C}_c(G)$. It is *left* invariant, i.e. $I_\ell(L_s f) = I_\ell(f)$ for every $s \in G$ and every $f \in \mathcal{C}_c(G)$. In fact,

$$I_\ell(L_s f) = \int_G \frac{L_s f(x)}{\Delta(x)} \, \mathrm{d}\mu(x) = \int_G \frac{f(s^{-1}x)}{\Delta(x)} \, \mathrm{d}\mu(x).$$

The change of variable $x \to sy$ modifies the measure $\mathrm{d}\mu(x)$ into $\Delta(s)\, \mathrm{d}\mu(y)$. Thus

$$I_\ell(L_s f) = \int_G \frac{f(y)}{\Delta(s)\Delta(y)} \Delta(s) \, \mathrm{d}\mu(y) = I_\ell(f).$$

Moreover, the functional I_ℓ is evidently positive on $\mathcal{C}_c^+(G)$, hence it is associated with the unique (up to positive multiplies) left-Haar measure. From its definition (as integral) we deduce that $\mathrm{d}\mu_L(g) := \Delta(g^{-1})\, \mathrm{d}\mu(g)$ is a left-Haar measure. In particular, the formula shows that μ_L is absolutely continuous with respect to the right-Haar measure. Since we know that also the map $E \to \mu(E^{-1})$ is a left invariant measure, we deduce that the inverse map $\iota\colon G \to G$ is absolutely continuous with respect to the right-Haar measure. Since ι^{-1} is $= \iota$, we conclude that ι is absolutely continuous also with respect the left-Haar measure, and that $\mu(E^{-1}) = 0$ if and only if $\mu(E) = 0$.

We can further study the relation between the functional I_ℓ and the left-invariant measure $\mu(\iota(\cdot))$; the conclusion will be a formula showing how the integral changes when the change of variable ι is applied (Formula 1.2 below). The functional on $\mathcal{C}_c(G)$ generated by $\mu(\iota(\cdot))$ is

$$f \mapsto \int_G f(g^{-1})\,\mathrm{d}\mu(g),$$

and we prove that actually this functional coincides with I_ℓ. In fact, both are left invariant, hence there exists a constant c such that

$$\int_G f(g^{-1})\,\mathrm{d}\mu(g) = c\int_G \frac{f(g)}{\Delta(g)}\,\mathrm{d}\mu(g), \qquad \forall f \in \mathcal{C}_c(G).$$

Applying this formula to $f(g) := (h(g) + h(g^{-1}))\sqrt{\Delta(g)}$, where $h \in \mathcal{C}_c^+(G)$, we get

$$\int_G (h(g^{-1}) + h(g))\sqrt{\Delta(g^{-1})}\,\mathrm{d}\mu(g) = c\int_G \frac{(h(g) + h(g^{-1}))\sqrt{\Delta(g)}}{\Delta(g)}\,\mathrm{d}\mu(g),$$

i.e.,

$$\int_G \frac{h(g) + h(g^{-1})}{\sqrt{\Delta(g)}}\,\mathrm{d}\mu(g) = c\int_G \frac{h(g) + h(g^{-1})}{\sqrt{\Delta(g)}}\,\mathrm{d}\mu(g),$$

which proves that $c = 1$. In other words, we have proved the formula:

$$\int_G f(g)\,\mathrm{d}\mu(g) = \int_G \frac{f(g^{-1})}{\Delta(g)}\,\mathrm{d}\mu(g), \qquad \forall f \in \mathcal{C}_c(G). \tag{1.2}$$

Exercise. 1.2 Let $G := (\mathrm{GL}^+(n, \mathbb{R}), \cdot)$, the matrices with positive determinant and the 'rows times columns' operation as product.

1. Prove that $G' = \mathrm{SL}(n, \mathbb{R})$. Note that in this case $G' = \overline{G'}$.
2. Prove that the determinant gives an isomorphism of $G/\overline{G'}$ with $(\mathbb{R}^+, \cdot)$, so that Δ' becomes a continuous morphism $(\mathbb{R}^+, \cdot) \to (\mathbb{R}^+, \cdot)$.
3. Let f be a continuous morphism $(\mathbb{R}^+, \cdot) \to (\mathbb{R}^+, \cdot)$ (i.e., $f(xy) = f(x)f(y)$ and it is continuous). Suppose that f is a C^1 map. Prove that $f(x) = x^c$ where $c := f'(1)$.
4. The C^1 regularity assumed in previous step allows an interesting proof of the claim, but actually the conclusion holds true under the weaker hypothesis of the continuity for f. Prove it. (Hint: consider the map $F : \mathbb{R} \to \mathbb{R}$, $F(w) := \log(f(e^w))$).
5. Conclude that there exists $c \in \mathbb{R}$ such that $\Delta'(\cdot) = \det(\cdot)^c$.

Exercise. 1.3 Let $G := (\mathrm{GL}(n, \mathbb{R}), \cdot)$.

1. Prove that $G' = \mathrm{SL}(n, \mathbb{R})$.
2. Prove that the determinant gives an isomorphism of $G/\overline{G'}$ with $(\mathbb{R}^\times, \cdot)$, so that Δ' becomes a continuous morphism $(\mathbb{R}^\times, \cdot) \to (\mathbb{R}^+, \cdot)$.
3. Let f be a continuous morphism $(\mathbb{R}^\times, \cdot) \to (\mathbb{R}^+, \cdot)$. Prove that there is $\alpha \in \{0, 1\}$ and $c \in \mathbb{R}$ such that $f(x) = \mathrm{sgn}(x)^\alpha |x|^c$.
5. Conclude that there exists $c \in \mathbb{R}$ such that $\Delta'(\cdot) = |\det(\cdot)|^c$.

Note, that for $\mathrm{GL}(n, \mathbb{R})$ and for $\mathrm{GL}^+(n, \mathbb{R})$ the left and the right Haar measures coincide (see Example 1.3), so that for this group Δ is trivial ($c = 0$) and the conclusions of the previous exercise are trivially true.

Exercise. 1.4 Let $G := \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, the group of affine maps in $\mathbb{R}^n$, i.e. the group in Example 1.5.

1. Prove that $(\mathrm{GL}(n, \mathbb{R}), 0)$ is a (not normal) subgroup in G, and that $(1, \mathbb{R}^n)$ is a normal subgroup of G.
2. Prove that G' contains $(\mathrm{SL}(n, \mathbb{R}),0)$, but note that it is strictly larger, since $(\mathrm{SL}(n, \mathbb{R}),0)$ is not normal.
3. Prove that G' contains also $(1, \mathbb{R}^n)$ (Hint: pick $h \in \mathrm{GL}(n, \mathbb{R}^n)$ such that also $1 - h$ is invertible. Then compute $[(h, (1 - h)^{-1}w), (1, 0)]$).
4. Since G' contains both $(\mathrm{SL}(n, \mathbb{R}), 0)$ and $(1, \mathbb{R}^n)$, it contains also $(\mathrm{SL}(n, \mathbb{R}), \mathbb{R}^n)$ (the group of affine maps preserving the measure of sets). Conclude that G'=$(\mathrm{SL}(n, \mathbb{R}), \mathbb{R}^n)$ and that therefore $G/G' \sim \mathbb{R}^\times$, with the isomorphism given by the determinant.
5. Note that also in this case the modular function must be a power of the (absolute value of the) determinant: in fact left and right Haar measures in this case differ exactly by a factor which is equal to the determinant (see Example 1.5).

CHAPTER 2

Banach algebras

2.1. Preliminary facts and basic properties for the spectrum

A Banach algebra A on the field $\mathbb{C}$ is a $\mathbb{C}$ algebra which is also a Banach space (hence it is also complete) with respect to a norm $\|\cdot\|$, where

$$\|ab\| \leq \|a\| \cdot \|b\|, \qquad \forall a, b \in A.$$

This axiom shows that the moltiplicative structure of A and the norm are well related. We will always assume that A is associative. Moreover, without loss of generality we can further assume that A is *unital*, i.e. that A contains a multiplicative identity 1_A, and that $\|1_A\| = 1$ (see Exercises 2.1– 2.3).

Exercise. 2.1 Let A be a $\mathbb{C}$-algebra. Let $A' := A \times \mathbb{C}$, with the pointwise sum and $\mathbb{C}$ product, and further set

$$(a, \lambda) \cdot (b, \mu) := (ab + \lambda b + \mu a, \lambda\mu).$$

1. Prove that A' is $\mathbb{C}$-algebra with respect to these operations, and that $(0, 1)$ is a multiplicative identity in A'.
2. Note that $\{(a, 0) \colon a \in A\}$ is a subalgebra in A' which can be identified with A.
3. Suppose that A is also a Banach algebra with respect to $\|\cdot\|_A$. Set $\|(a, \alpha)\|_{A'} := \|a\|_A + |\lambda|$. Prove that A' is a Banach algebra with respect to this norm, and that the inclusion of A into A' is a continuous map which preserves the norm.
4. Note that $\|(0, 1)\|_{A'} = 1$.

Exercise. 2.2 Let A be a complex Banach algebra with a multiplicative identity 1_A. For every $a \in A$, let $m_a \colon A \to A$ the (left)-multiplicative map: $m_a(b) := ab$, considered as linear map on the $\mathbb{C}$-vector space A.

1. Shows that
$$\|a\|_\infty := \sup_{x \in A, x \neq 0} \frac{\|m_a(x)\|}{\|x\|} = \sup_{\substack{x \in A \\ \|x\| \leq 1}} \frac{\|m_a(x)\|}{\|x\|} = \sup_{\substack{x \in A \\ \|x\| = 1}} \|m_a(x)\|.$$
2. Prove that
$$\frac{\|a\|}{\|1_A\|} \leq \|a\|_\infty \leq \|a\|$$
so that the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent and the converging sequences are the same.
3. Prove that A is a Banach algebra also with respect to $\|\cdot\|_\infty$.
4. Note that $\|1_A\|_\infty = 1$.

Exercise. 2.3 Let A be a complex Banach algebra with a multiplicative identity 1_A and such that $\|1_A\| = 1$. Prove that the operatorial norm of the multiplication by a coincides with $\|a\|$ (i.e. $\|a\|_\infty = \|a\|$, where $\|\cdot\|_\infty$ is defined in Exercise 2.2).

Let $a \in A$, and recall the polynomial identity

$$(1-a)\Big(\sum_{n=0}^{N} a^n\Big) = 1 - a^{N+1} = \Big(\sum_{n=0}^{N} a^n\Big)(1-a).$$

Assume that $\|a\| < 1$. Then $\sum_{n=0}^{+\infty} a^n$ converges by Weierstrass' test, because $\|a^n\| \leq \|a\|^n$ and $\sum_{n=0}^{+\infty} \|a\|^n < \infty$, and the previous identity shows both that $1-a$ is invertible, and that

$$(1-a)^{-1} = \sum_{n=0}^{+\infty} a^n.$$

This identity also shows that

$$\|(1-a)^{-1}\| \leq \sum_{n=0}^{+\infty} \|a^n\| \leq \sum_{n=0}^{+\infty} \|a\|^n = \frac{1}{1-\|a\|}.$$

This easy computation is the key ingredient for the proof of the following facts.

Proposition 2.1 *Let $A^\times$ be the set of invertible elements in A. Then $A^\times$ is an open set. The inverse map $\iota\colon A^\times \to A^\times$, $\iota(a) := a^{-1}$ is a continuous map (actually it is locally Lipschitz).*

Proof. Let $a \in A^\times$, and take $b \in A$, with $\|b - a\| < 1/\|a^{-1}\|$. Then

$$\|1 - a^{-1}b\| = \|a^{-1}(a-b)\| \leq \|a^{-1}\| \cdot \|a - b\| < 1.$$

Thus, $a^{-1}b = 1 - (1 - a^{-1}b)$ is invertible, and hence $b = a \cdot (a^{-1}b)$ is invertible as well. This shows that the open ball $B_{1/\|a^{-1}\|}(a)$ is in $A^\times$, which is therefore an open set.
The computation also shows that

$$\|b^{-1}a\| = \|(a^{-1}b)^{-1}\| = \|(1-(1-a^{-1}b))^{-1}\| \leq \frac{1}{1 - \|a^{-1}\| \cdot \|a-b\|}.$$

Now, suppose that b satisfies the more restrictive assumption $\|b - a\| \leq \frac{1}{2\|a^{-1}\|}$. Then

$$\|b^{-1}a\| \leq \frac{1}{1 - \|a^{-1}\| \cdot \|a-b\|} \leq \frac{1}{1 - \|a^{-1}\| \cdot \frac{1}{2\|a^{-1}\|}} = 2,$$

so that $\|b^{-1}\| = \|b^{-1}a \cdot a^{-1}\| \leq 2\|a^{-1}\|$. This shows that

$$\|a^{-1} - b^{-1}\| = \|a^{-1}(a-b)b^{-1}\| \leq \|a^{-1}\| \cdot \|a-b\| \cdot \|b^{-1}\| \leq 2\|a^{-1}\|^2 \cdot \|a-b\|$$

proving that ι is a Lipschitz map in a neighborhood of a. ■

To each $a \in A$ we associate its *spectrum* in A, i.e. the set

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C}\colon \lambda 1_A - a \notin A^\times\}.$$

Its complementary set is called *resolvent set* of a:

$$\mathrm{Res}(a) := \mathbb{C} \setminus \mathrm{sp}(a) = \{\lambda \in \mathbb{C}\colon \lambda 1_A - a \in A^\times\}.$$

We also introduce the *radius of the spectrum*, which is simply

$$r(a) := \sup\{|\lambda|\colon \lambda \in \mathrm{sp}(a)\}$$

(it is well defined because $\mathrm{sp}(a)$ is never empty, as we will see in a moment).

Let $p \in \mathbb{C}[x]$, $p(x) = \sum_{n=0}^{N} c_n x^n$, say. Then

$$p(\lambda)1_A - p(a) = \sum_{n=0}^{N} c_n(\lambda^n 1_A - a^n) = \sum_{n=1}^{N} c_n(\lambda^n 1_A - a^n)$$

$$= \sum_{n=1}^{N} c_n \sum_{j=0}^{n-1} (\lambda^j a^{n-1-j})(\lambda 1_A - a)$$
$$= b(\lambda 1_A - a) = (\lambda 1_A - a)b$$

with $b := \sum_{n=1}^{N} c_n \sum_{j=0}^{n-1}(\lambda^j a^{n-1-j})$. Note that b commutes with $\lambda 1_A - a$.
Suppose that $p(\lambda)$ belongs to the resolvent set for $p(a)$. Then there exists $c \in A$ such that $c(p(\lambda)1_A - p(a)) = 1_A = (p(\lambda)1_A - p(a))c$. The previous computation shows that cb is a left inverse for $(\lambda 1_A - a)$ and that bc is a right inverse for $(\lambda 1_A - a)$. In associative rings this is sufficient to conclude that $(\lambda 1_A - a)$ is invertible[1], so that λ belongs to the resolvent for a. In other words, this shows that $p(\mathrm{sp}(a)) \subseteq \mathrm{sp}(p(a))$. (See also Exercise 2.4 below; in Theorem 2.2 we will see that this is true as equality, also for much more general functions).

The following proposition contains some fundamental properties of the spectrum of an element.

Proposition 2.2 *Let A be a complex unital Banach algebra. Let $a \in A$. Then*

1. $r(a) \leq \|a\|$, *so that* $\mathrm{sp}(a) \subseteq \overline{B}_{\|a\|}(0)$,
2. $\mathrm{sp}(a)$ *is a compact set,*
3. $\mathrm{sp}(a) \neq \emptyset$,
4. *the limit* $\lim_{n\to\infty} \|a^n\|^{1/n}$ *exists and equals* $r(a)$, *i.e.*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Proof. Assume that $|\lambda| > \|a\|$. Then $\lambda 1_A - a = \lambda(1_A - \lambda^{-1}a)$ and $\|\lambda^{-1}a\| = \frac{\|a\|}{|\lambda|} < 1$, proving that $1_A - \lambda^{-1}a$ (and hence $\lambda 1_A - a$) is invertible. This shows that $r(a) \leq \|a\|$. Applying the same argument to a^n, we get $r(a^n) \leq \|a^n\|$. Specializing the previous computation to $p(x) = x^n$ we get also that $r(a)^n \leq r(a^n)$, so that $r(a)^n \leq \|a^n\|$ and the arbitrariness of n produces the bound

$$r(a) \leq \liminf_{n\to\infty} \|a^n\|^{1/n}. \tag{2.1}$$

Moreover, the resolvent of a is the preimage of the map $\mathbb{C} \to A$ mapping λ to $\lambda 1_A - a$, of the open set $A^\times$. Since this map is continuous, the preimage is open. Therefore the spectrum (being the complementary set of the resolvent set) is closed. Since we already know that it is bounded (by $\|a\|$), we conclude that $\mathrm{sp}(a)$ is compact.
The proof of the remaining statements is more difficult, and relates different fields of the analysis. Let ϕ be any topological functional in A^*. Thus it is a map $A \to \mathbb{C}$ which is $\mathbb{C}$ linear and continuous. Let $f_\phi\colon \mathrm{Res}(a) \subseteq \mathbb{C} \to \mathbb{C}$:

$$f_\phi(\lambda) := \phi((\lambda 1_A - a)^{-1}).$$

Note that $\mathrm{Res}(a)$ is an open set, which is not empty (for example because it contains every λ with $|\lambda| > \|a\|$). We prove that f_ϕ is a holomorphic function. In fact, let $\lambda \in \mathrm{Res}(a)$ and take μ small enough. Then

$$\begin{aligned} f_\phi(\lambda - \mu) &= \phi\big(((\lambda - \mu)1_A - a)^{-1}\big) \\ &= \phi\big(\big[(\lambda 1_A - a)(1 - \mu(\lambda 1_A - a)^{-1})\big]^{-1}\big) \\ &= \phi\big((1 - \mu(\lambda 1_A - a)^{-1})^{-1}(\lambda 1_A - a)^{-1}\big) \end{aligned}$$

[1] In fact, suppose that in an associative ring R one has $\alpha\beta = 1 = \gamma\alpha$. Then $\gamma = \gamma \cdot 1 = \gamma(\alpha\beta) = (\gamma\alpha)\beta = 1 \cdot \beta = \beta$.

$$= \phi\Big(\sum_{m=0}^{+\infty} \mu^m (\lambda 1_A - a)^{-m} (\lambda 1_A - a)^{-1}\Big) \quad (\mu \text{ is small, } \Sigma \text{ converges})$$

$$= \phi\Big(\sum_{m=0}^{+\infty} \mu^m (\lambda 1_A - a)^{-m-1}\Big) = \sum_{m=0}^{+\infty} \mu^m \phi\big((\lambda 1_A - a)^{-m-1}\big). \quad (\phi \text{ is continuous})$$

This shows that f_ϕ is locally represented by a power series, hence it is holomorphic. Suppose $|\lambda| > \|a\|$. Then

$$f_\phi(\lambda) = \frac{1}{\lambda}\phi((1_A - \lambda^{-1}a)^{-1}) = \frac{1}{\lambda}\phi\Big(\sum_{m=0}^{+\infty} \frac{a^m}{\lambda^m}\Big) = \frac{1}{\lambda}\sum_{m=0}^{+\infty} \frac{\phi(a^m)}{\lambda^m}. \tag{2.2}$$

In particular, recalling that ϕ is continuous,

$$|f_\phi(\lambda)| \le \frac{1}{|\lambda|}\sum_{m=0}^{+\infty} \frac{|\phi(a^m)|}{|\lambda|^m} \le \frac{1}{|\lambda|}\sum_{m=0}^{+\infty} \frac{\|\phi\|\|a^m\|}{|\lambda|^m} \le \frac{1}{|\lambda|}\sum_{m=0}^{+\infty} \frac{\|\phi\|\|a\|^m}{|\lambda|^m} = \frac{\|\phi\|}{|\lambda| - \|a\|}.$$

This computations shows that $f_\phi(\lambda) \to 0$ as $\lambda \to \infty$. Suppose that $\mathrm{sp}(a) = \emptyset$. Then $\mathrm{Res}(a) = \mathbb{C}$, so that f_ϕ is an entire function (i.e. holomorphic on $\mathbb{C}$) which is bounded (because it goes to 0 for $\lambda \to \infty$). By Liouville's theorem f_ϕ is constant, and hence it is identically 0. This happens for every choice of the functional ϕ. However, the Hahn–Banach theorem states that there is always a continuous functional ϕ' whose value at $(1_A - a)^{-1}$ is not 0 (because $(1_A - a)^{-1}$ is evidently not 0), and this is a contradiction with the fact that $\phi'((1_A - a)^{-1}) = f_{\phi'}(1) = 0$. This proves that $\mathrm{sp}(a) \neq \emptyset$.
The formula (2.2) has been proved under the hypothesis $|\lambda| > \|a\|$, but we also know that f_ϕ is holomorphic in $\mathrm{Res}(a)$, which is an open set containing the set $|\lambda| > \|a\|$. As a consequence, for every $r > r(a)$, the formula holds also for $|\lambda| \ge r$, and actually the series converges uniformly here (this is a typical application of the expansion of holomorphic functions as Laurent series, it comes out from the fact that the set $|\lambda| \ge r$ is connected, the function is holomorphic in an open set containing this set, and the power series is a Laurent series representing f_ϕ outside the disk $|\lambda| \le \|a\|$). Thus, we can recover the coefficient of the series integrating the formula along the circle with radius r. In this way we get:

$$\int_0^1 e^{2\pi(n+1)i\theta} f_\phi(re^{2\pi i\theta})\,\mathrm{d}\theta = \int_0^1 \frac{e^{2\pi(n+1)i\theta}}{re^{2\pi i\theta}} \sum_{m=0}^{+\infty} \phi(a^m) r^{-m} e^{-2\pi i m\theta}\,\mathrm{d}\theta$$
$$= \sum_{m=0}^{+\infty} \phi(a^m) r^{-m-1} \int_0^1 e^{2\pi(n-m)i\theta}\,\mathrm{d}\theta = \phi(a^n) r^{-n-1}.$$

Hence,

$$|\phi(a^n)| r^{-n-1} \le \Big|\int_0^1 e^{2\pi(n+1)i\theta} f_\phi(re^{2\pi i\theta})\,\mathrm{d}\theta\Big| \le \int_0^1 \Big|f_\phi(re^{2\pi i\theta})\Big|\,\mathrm{d}\theta$$
$$= \int_0^1 \big|\phi((re^{2\pi i\theta}1_A - a)^{-1})\big|\,\mathrm{d}\theta \le \int_0^1 \|\phi\| \cdot \|(re^{2\pi i\theta}1_A - a)^{-1}\|\,\mathrm{d}\theta$$
$$\le \|\phi\| \cdot M(r)$$

where $M(r) := \sup_{\theta \in [0,1]} \|(re^{2\pi i\theta}1_A - a)^{-1}\|$. Hence

$$|\phi(a^n)| \le \|\phi\| r^n \cdot rM(r).$$

This bound holds for every choice of ϕ, every integer n and for every $r > r(a)$. Note that $M(r)$ does not depend on n and ϕ. Fix the integer n. Consider the linear subspace $\{\alpha \cdot a^n \colon \alpha \in \mathbb{C}\}$. The map $\phi'(\alpha a^n) := \alpha\|a^n\|$ is a continuous functional on the subspace, and its norm is evidently 1. By Hahn–Banach theorem this functional admits an extension to a continuous functional on A whose norm is still 1. When we apply the previous formula to this functional we get that

$$\|a^n\| \le r^n \cdot rM(r), \qquad \text{i.e.,} \qquad \|a^n\|^{1/n} \le r \cdot (rM(r))^{1/n}.$$

Since $M(r)$ is independent of n this formula shows that

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le r.$$

This is true for every r which is by assumption any number larger than $r(a)$, thus we conclude that

$$\limsup_{n\to\infty} \|a^n\|^{1/n} \le r(a).$$

With (2.1), this shows that $\lim_{n\to\infty} \|a^n\|^{1/n}$ exists and equals $r(a)$. ■

Exercise. 2.4 Let A be a (unital, associative) complex algebra. Let $f(x) := \sum_{n=0}^{+\infty} c_n x^n$ be a complex power series with a positive convergence radius ρ. Let $a \in A$, with $\|a\| < \rho$.

1. Prove that $f(a) := \sum_{n=0}^{+\infty} c_n a^n$ converges in A.
2. Prove that $f(\mathrm{sp}(a)) \subseteq \mathrm{sp}(f(a))$.

Hint: for the first part use Weierstrass test, for the second part repeat what we have done for polynomials.

Corollary 2.1 (Gelfand–Mazur) *Let A be a complex Banach algebra (recall that we are also assuming that it is unital and associative). If it is a division algebra (i.e., every nonzero element is invertible) then it is isomorphic (as algebra) to $\mathbb{C}$.*

Proof. Let $a \in A$, $a \neq 0$. Proposition 2.2 proves that $\mathrm{sp}(a) \neq \emptyset$. Let $\lambda \in \mathrm{sp}(a)$. Then $\lambda 1_A - a$ is not invertible, therefore it is zero, because we are assuming that A is a division algebra; in other words $a = \lambda 1_A$. Evidently λ is uniquely determined by this property, and the map $a \mapsto \lambda$ gives the isomorphism with $\mathbb{C}$. ■

Mazur proved also a similar result for *real* algebras: the algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (the quaternion) are the unique real Banach (unital and associative) algebras which are division algebras.

Remark. 2.1 Let $a \in A$. Then a is invertible if and only if $0 \notin \mathrm{sp}(a)$. Moreover, when $a \in A^\times$ and $\lambda \neq 0$ on has the identity $\lambda 1_A - a = -\lambda a(\lambda^{-1}1_A - a^{-1})$ proving that $\lambda \in \mathrm{sp}(a)$ if and only if $\lambda^{-1} \in \mathrm{sp}(a^{-1})$. i.e. that $\mathrm{sp}(a^{-1}) = (\mathrm{sp}(a))^{-1}$. □

Let A be a complex Banach algebra, and let J be a two-sided ideal in A (where only the structure as a ring for A matters here). Then A/J is ring. Actually it is an algebra, since $\lambda \cdot J = (\lambda 1_A) \cdot J \subseteq J$ for every $\lambda \in \mathbb{C}$ so that J is automatically also a complex vector space (where the distributive properties are inherited from A). In A/J we define the map

$$\|a + J\|_{A/J} := \inf_{w\in J} \|a + w\|_A.$$

It is easy to check that it is a semi-norm, indeed. It is also submultiplicative, since

$$\begin{aligned} \|ab + J\|_{A/J} &\le \inf_{w,w'\in J} \|ab + (wb + aw' + ww')\|_A = \inf_{w,w'\in J} \|(a+w)(b+w')\|_A \\ &\le \inf_{w,w'\in J} \|a+w\|_A \cdot \|b+w'\|_A = \inf_{w\in J} \|a+w\|_A \cdot \inf_{w'\in J} \|b+w'\|_A \\ &= \|a+J\|_{A/J} \cdot \|b+J\|_{A/J} \end{aligned}$$

(for the first step, recall that J is a two-sided ideal, so that $wb + aw' + ww'$ ranges in J when w, w' are in J).
However, it is a norm if and only if J is closed. In that case A/J is complete, so that it becomes a complex Banach algebra in itself.

Remark. 2.2 Let J be a two-sided ideal in A. Then $\overline{J}$ is again a two-sided ideal. In fact it is a $\mathbb{C}$ vector space (because J is a subgroup of the additive group A, and we apply Proposition 1.4[iv.]). Moreover, if x_n is a sequence in J converging to x, then for every $a \in A$, ax_n is a new sequence in J (because J is an ideal), converging to ax since $\|ax_n - ax\| = \|a(x_n - x)\| \leq \|a\| \cdot \|x_n - x\| \to 0$. □

2.2. Gelfand transform

Let A be a complex (associative, unital) Banach algebra. Then, A is in particular a topological $\mathbb{C}$ vector space, so that we can introduce the set A^* which by definition is the set $\mathrm{Hom}_{\mathrm{top}}(A, \mathbb{C})$, i.e. the set of its linear functionals which are continuous[2]. We introduce also a second family of maps, better related to the structure of A as algebra. By definition a *character* χ for A is a nontrivial morphism $A \to \mathbb{C}$ as complex algebras, so that

$$\chi(a+b) = \chi(a) + \chi(b), \quad \chi(\lambda a) = \lambda\chi(a), \quad \chi(ab) = \chi(a)\chi(b)$$

for every $a, b \in A$, $\lambda \in \mathbb{C}$, and χ is not identically 0. Since $\chi(a) \neq 0$ for some a, then $\chi(a) = \chi(1_A a) = \chi(1_A)\chi(a)$ forces $\chi(1_A) = 1$, and $\chi(\lambda 1_A) = \lambda$ for every $\mathbb{C}$. Therefore any character is automatically surjective.
The set of characters of A is denoted $\hat{A}$. Note that the continuity of χ is *not* assumed. In fact, we will see in a moment that a character is necessarily continuous[3].

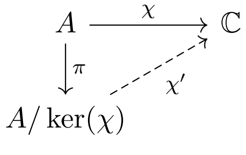
Proposition 2.3 *Let A be an* abelian *complex Banach algebra. Then*

i. *Every maximal ideal M is closed;*
ii. *There is a bijection $\hat{A} \to \{$maximal ideals$\}$;*
iii. *Every character is continuous;*
iv. *for every $a \in A$, $\mathrm{sp}(a) = \{\chi(a) \colon \chi \in \hat{A}\}$.*

Note that the proposition deals only *abelian* algebras: this is a strong limitation of the proposition, but it cannot be removed.

Proof.

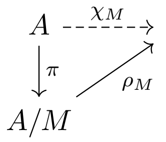
i. Let M be a maximal ideal. Then $\overline{M}$ is again an ideal, which contains M. Maximality implies that if $\overline{M} \neq M$, then $\overline{M} = A$. In this case $1_A \in \overline{M}$. Let x_n be a sequence in M converging to 1_A. We know that $A^\times$ is open, therefore the sequence is eventually in $A^\times$. As a consequence M contains an invertible element, but this is impossible since M is a maximal ideal.
ii. To each $\chi \in \hat{A}$ we associate its kernel $\ker(\chi)$. It is a maximal ideal since χ is surjective in $\mathbb{C}$, so that $A/\ker(\chi)$ is isomorphic to $\mathbb{C}$ and hence is a field. This shows that the map $\chi \mapsto \ker(\chi)$ is actually a map $\hat{A} \to \{$maximal ideal$\}$.



On the other hand, let M be a maximal ideal. Then M is closed (by the first part of this proposition), and A/M is a complex Banach algebra which is also a field, *because we are assuming that A is commutative.* By Corollary 2.1 we know that it is isomorphic

[2]This requirement is necessary because $\mathrm{Hom}(A, \mathbb{C})$, the set of all linear functionals, is a much larger set which is not well related with the topological structure of A.
[3]This is a big difference with functionals, which are not necessarily continuous.

to $\mathbb{C}$. Let ρ_M denote this isomorphism. Then $\chi_M := \rho_M \circ \pi$ is a character of A.



It is easy to check that these constructions realize the bijection, i.e. that $\chi_{\ker(\chi)} = \chi$ and that $\ker(\chi_M) = M$.

iii. Let χ be a character, which we describe as $\rho_M \circ \pi$, with $M := \ker(\chi)$. Corollary 2.1 shows that A/M is isomorphic to $\mathbb{C}$, and the proof of this result shows that the isomorphism is produced by the map associating to every element $a + M$ the unique element of its spectrum. Let U be an open set in $\mathbb{C}$, then $\rho_M^{-1}(U) = \{u1_A + M : u \in U\}$ and $\pi^{-1}(\{u1_A + M : u \in U\}) = \{u1_A + m : u \in U,\ m \in M\} = \bigcup_{m\in M}(U1_A + m)$, which is open (each $U1_A + m$ is open in A). This proves that $\chi^{-1}(U) = (\rho_M \circ \pi)^{-1}(U)$ is an open set, i.e. that χ is continuous.

iv. Let $\lambda \in \mathbb{C}$. Then $\lambda \in \mathrm{sp}(a)$ if and only if $\lambda 1_A - a$ is not invertible, and this happens if and only if it belongs to some maximal ideal (by Zorn Lemma), and hence this happens if and only if it belongs to the kernel of some character χ. But $0 = \chi(\lambda 1_A - a) = \lambda\chi(1_A) - \chi(a) = \lambda - \chi(a)$, i.e. $\lambda = \chi(a)$.

■

Each element a in A generates a functional ev_a on A^*, via the map $\mathrm{ev}_a(\phi) := \phi(a)$. The weak-$*$ topology in A^* is the weakest topology which makes all these maps continuous. In other words, a sequence of functionals $\phi_n \in A^*$ converges to an element ϕ if and only if $\mathrm{ev}_a(\phi_n)$ tends to $\mathrm{ev}_a(\phi)$ for every a. Given its definition, this means that the sequence converges if and only if $\phi_n(a) \to \phi(a)$ for all $a \in A$. The weak* topology, hence, coincides with the topology of the pointwise convergence. It is known that it is a Hausdorff topology. According to the previous proposition $\hat{A} \subseteq A^*$; the topology induced on $\hat{A}$ by the weak* topology of A^* is called *Gelfand's topology.*

Proposition 2.4 *Let A be an* abelian *complex Banach algebra. Then $\hat{A}$ is a compact set in Gelfand's topology.*

Proof. Let χ be a character. Then $\chi(a)$ is contained in the spectrum of a (by Prop. 2.3[iv]), and hence is bounded by $\|a\|$ (by Prop. 2.2[1]). This proves that $\hat{A}$ is a subset of the 1 ball of A^*, which is the set $\{\phi \in A^* : |\phi(a)| \leq \|a\| \forall a \in A\}$. Banach–Alaoglu theorem states that the closed unit ball is weak* compact, hence it is sufficient to prove that $\hat{A}$ is a closed subset in Gelfand's topology, i.e. in weak* topology, i.e. in pointwise topology. This is immediate, because if a sequence of characters χ_n converges (pointwise) to a functional χ, then χ itself is a character. ■

It is customary to denote $\hat{a}$ the map ev_a when it is restricted to the subspace of characters $\hat{A}$; in other words, $\hat{a}(\chi) := \mathrm{ev}_a(\chi) = \chi(a)$. By construction, the map $\hat{a}$ is a continuous map $\hat{A} \to \mathbb{C}$, when in $\hat{A}$ Gelfand's topology is considered.
Let $\mathcal{C}(\hat{A})$ be the set of all continuous maps $\hat{A} \to \mathbb{C}$. With the sup norm (recall that $\hat{A}$ is weak* compact, the sup norm is well defined) it is a (unital, associative) complex Banach algebra, and in some sense it is the prototypical Banach algebra, as we will see in a moment. The map

$$\Gamma\colon A \longrightarrow \mathcal{C}(\hat{A}),$$
$$a \mapsto \Gamma(a) := \hat{a}.$$

is called *Gelfand's transform.*

Theorem 2.1 *Let A be a (unital, associative) commutative complex Banach algebra. Then:*

i. *Gelfand's transform $\Gamma\colon A \to \mathcal{C}(\hat{A})$ is a norm-decreasing homomorphism of unital complex algebras;*
ii. *The image of Γ separates the points in $\hat{A}$;*
iii. *for every $a \in A$, $\hat{a}(\hat{A}) = \mathrm{sp}(a)$ and $\|\hat{a}\|_\infty = r(a)$, the spectral radius of a;*
iv. *the kernel of Γ is the Jacobson radical of A; i.e., it is the intersection of all maximal ideals of A. In other words, the kernel of Γ is the set of all elements in A whose spectral radius is 0;*
v. *Γ is injective if and only if the Jacobson radical is trivial, i.e. if and only if A is semisimple.*

Proof.

i. Identities $\Gamma(a+b) = \Gamma(a) + \Gamma(b)$, $\Gamma(\lambda a) = \lambda\Gamma(a)$ and $\Gamma(ab) = \Gamma(a)\Gamma(b)$ for every $a, b \in A$, $\lambda \in \mathbb{C}$ follow immediately from the definitions. Moreover, for every $a \in A$ and $\chi \in \hat{A}$,
$$|\Gamma(a)(\chi)| = |\hat{a}(\chi)| = |\chi(a)| \le r(a) \le \|a\|$$
(because $\chi(a)$ belongs to the spectrum of a). Hence $\|\Gamma(a)\|_\infty \le \|a\|$;
ii. This is immediate, since if $\chi, \psi \in \hat{A}$ and $\chi \ne \psi$, then there is $a \in A$ with $\chi(a) \ne \psi(a)$, i.e. with $\Gamma(a)(\chi) \ne \Gamma(a)(\psi)$;
iii. By Proposition 2.3[iv.] we know that $\mathrm{sp}(a) = \{\chi(a)\colon \chi \in \hat{A}\}$, and this is $= \hat{a}(\hat{A})$. The equality $\|\hat{a}\|_\infty = r(a)$ is immediate;
iv. the element $a \in A$ belongs to the kernel of Γ if and only if $\hat{a}(\chi) = 0$ for every $\chi \in \hat{A}$, i.e. if and only if $\chi(a) = 0$ for every χ, if and only if $a \in \bigcap_\chi \ker(\chi)$. By Proposition 2.3[ii.] this happens if and only if a belongs to Jacobson's radical of A.
v. Trivial (an algebra is semisimple if and only if its radical is trivial, by definition).

■

Note that the previous proposition does not say anything about the range of Γ, in particular we do not know that it is surjective. In fact, for a general algebra this is *not* the case. Also about the possible injectivity of Γ we have only very poor control: usually it is very difficult to compute the radical of an algebra. These difficulties force us to further select as algebra a special family of algebras: actually commutative subalgebras of the algebra of endomorphisms of a Hilbert space.

2.3. Stone-Weierstrass theorem

Theorem 2.2 (Stone–Weierstrass) *Let X be a compact Hausdorff space. If $\mathfrak{A}$ is a closed (with respect to the uniform topology) self-adjoint subalgebra of $\mathcal{C}(X)$ (i.e. such that $f \in \mathfrak{A}$ implies $\overline{f} \in \mathfrak{A}$) which separates the points of X and contains 1, then $\mathfrak{A} = \mathcal{C}(X)$.*

Proof. (Following [D], Section 2.40.) Let $\mathfrak{A}_r$ and $\mathcal{C}_r(X)$ be the subsets of real functions. Then, it is sufficient to prove that under those hypotheses $\mathfrak{A}_r = \mathcal{C}_r(X)$. Note that $\mathfrak{A}_r$ is closed in $\mathcal{C}_r(X)$, contains 1, and separates the points[4].
We begin by showing that if $f \in \mathfrak{A}_r$, then also $|f| \in \mathfrak{A}_r$. Without loss of generality we can assume that $\|f\|_\infty \le 1$. Let $\delta \in (0,1)$, and take $g_\delta(x) := \delta + (1-\delta)f^2(x)$. Note that it is in $\mathfrak{A}_r$, because 1 and f are in $\mathfrak{A}_r$, by hypothesis, and $\mathfrak{A}_r$ is an algebra. Note also that $\delta \le g_\delta(x) \le 1$, for every $x \in X$.

[4] In fact, let $x, y \in X$, $x \ne y$. Let $f \in \mathfrak{A}$ be the element separating x and y, i.e. such that $f(x) \ne f(y)$. Then at least one of the inequalities $(\mathrm{Re}\, f)(x) \ne (\mathrm{Re}\, f)(x)$, $(\mathrm{Im}\, f)(x) \ne (\mathrm{Im}\, f)(x)$ holds. But $\mathrm{Re}\, f = \frac{1}{2}(f + \overline{f})$ and $\mathrm{Im}\, f = \frac{1}{2i}(f - \overline{f})$, and these formulas show that both $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ are in $\mathfrak{A}$, and hence in $\mathfrak{A}_r$.

Consider the power series $\sum_{n=0}^{+\infty}\binom{1/2}{n}z^n$, which in $(-1,1]$ converges to $(1+z)^{1/2}$ (uniformly on compact subsets of $(-1,1]$, by Abel Lemma). Then $g_\delta(x)-1\in[-1+\delta,0]$, so that

$$(\delta+(1-\delta)f^2)^{1/2}=(1+(g_\delta-1))^{1/2}=\sum_{n=0}^{+\infty}\binom{1/2}{n}(g_\delta-1)^n,$$

and the series converges in $\mathcal{C}_r(X)$ (by uniform convergence) and is in $\mathfrak{A}_r$ (because it is a closed algebra). This equality shows that $(\delta+(1-\delta)f^2)^{1/2}\in\mathfrak{A}_r$, for every δ. At last, note that

$$(\delta+(1-\delta)f^2)^{1/2}-|f|=\frac{(1-f^2)\delta}{(\delta+(1-\delta)f^2)^{1/2}+|f|}\leq\sqrt{\delta}$$

proving that $|f|$ is the limit (in sup norm) of $(\delta+(1-\delta)f^2)^{1/2}$, and hence belongs to $\mathfrak{A}_r$. This shows that for every $f,g\in\mathfrak{A}_r$, also

$$f\wedge g:=\min(f,g)=\tfrac{1}{2}(f+g-|f-g|)$$

and

$$f\vee g:=\max(f,g)=\tfrac{1}{2}(f+g+|f-g|)$$

are in $\mathfrak{A}_r$.
Moreover, let $x\neq y\in X$. By hypothesis there is $h\in\mathfrak{A}_r$ such that $h(x)\neq h(y)$. Thus, for every $\alpha,\beta\in\mathbb{R}$, the function g defined as

$$g(z):=\beta+(\alpha-\beta)\frac{h(z)-h(y)}{h(x)-h(y)}$$

belongs to $\mathfrak{A}_r$ and is such that $g(x)=\alpha$, $g(y)=\beta$.
Now the construction of a uniform approximation in $\mathfrak{A}_r$ of any $f\in\mathcal{C}_r(X)$. Let $f\in\mathcal{C}_r(X)$ and $\epsilon>0$ be fixed. We further set a point $x'\in X$. For every $x\in X$ there is a function $g_x\in\mathfrak{A}_r$ with $g_x(x')=f(x')$ and $g_x(x)=f(x)$: for $x\neq x'$ this is what we have proved in previous step, in case $x=x'$ we set $g_x(z):=f(x')$ for every z i.e., it is the constant function.
By continuity, for every ϵ there is an open neighborhood $U^+(x)$ where $g_x(z)<f(z)+\epsilon$ for every $z\in U^+(x)$. The family of $U^+(x)$'s when x ranges in X is an open covering of X. The compactness of X allows to extract a finite covering, i.e. a finite set of points $\{x_j\}_{j=1}^N$ such that functions $g_{x_j}(x)<f(x)+\epsilon$ for every $x\in U^+(x_j)$, and $\bigcup_{j=1}^N U^+(x_j)=X$. As a consequence, the function $f_{x'}:=g_{x_1}\wedge\cdots\wedge g_{x_N}=\min(g_{x_1},\ldots,g_{x_N})$ is such that $f_{x'}(z)<f(z)+\epsilon$ for every $z\in X$. Note that $f_{x'}\in\mathfrak{A}_r$ and that $f_{x'}(x')=f(x')$.
We repeat the construction for every $x'\in X$. This produces for every $x'\in X$ a function $f_{x'}\in\mathfrak{A}_r$ such that $f_{x'}(x')=f(x')$ and still satisfying $f_{x'}(z)<f(z)+\epsilon$ for every $z\in X$. By continuity, to every x' we associate an open set $U^-(x')$ where $f_{x'}(z)>f(z)-\epsilon$ for every $z\in U^-(x')$. The collection of $U^-(x')$ for $x'\in X$ is an open covering of X from which we can therefore extract a finite covering: $\bigcup_{j=1}^{N'}U^-(x'_j)$. Then the function $F:=f_{x'_1}\vee\cdots\vee f_{x'_{N'}}=\max(f_{x_1},\ldots,f_{x_{N'}})$ is such that $F(z)>f(z)-\epsilon$ for every $z\in X$. Note that $F\in\mathfrak{A}_r$, and that $F(z)<f(z)+\epsilon$ (because each $f_{x'_j}$ has this property). Hence we have found a function $F\in\mathfrak{A}_r$ such that $\|F-f\|_\infty\leq\epsilon$. ■

We need to extend a little bit the previous result, to cover the case of a non-compact set X. So, let X be any locally compact Hausdorff space. Let ∞ denote any (new) object, and let $X':=X\cup\{\infty\}$. We extend the topology of X to X' deciding that the complementary sets in X' of compact sets in X are a family of open neighborhood of ∞. With this choice the resulting space X' becomes a compact and Hausdorff topological space: the so called

Aleksandrov's one-point compactification of X (see [K], Theorem 21 p. 150). We identify the set of functions in $\mathcal{C}(X')$ and assuming the value 0 in ∞ with the subset

$$\mathcal{C}_0(X) := \{f \in \mathcal{C}(X) \colon \forall \epsilon > 0 \; \exists \text{ compact } K \text{ s.t. } |f(x)| \le \epsilon \text{ for } x \in K^c\}$$

(i.e. the subset of $\mathcal{C}(X)$ of functions which 'tend to 0' as the variable tends to ∞). It is a Banach space with respect the sup norm.

Corollary 2.2 *Let X be a locally compact Hausdorff space. If $\mathfrak{A}$ is a closed (with respect to the uniform topology) self-adjoint subalgebra of $\mathcal{C}_0(X)$ which separates the points of X and such that for every $x_0 \in X$ there is a function $f \in \mathfrak{A}$ such that $f(x_0) \neq 0$, then $\mathfrak{A} = \mathcal{C}_0(X)$.*

Proof. Let $\mathfrak{A}'$ be the algebra generated by $\mathfrak{A}$ and the constants. We identify the elements in $\mathfrak{A}$ with the elements in $\mathfrak{A}'$ which are null in ∞. The algebra $\mathfrak{A}'$ separates the points of X (because $\mathfrak{A}$ already does) and separate also ∞ (because by hypothesis for every $x_0 \in X$ there is an element in $\mathfrak{A}$ which is not zero in x_0). It is also evidently self-adjoint and unital, hence it coincides with $\mathcal{C}(X')$. Let $f \in \mathcal{C}_0(X)$; then $f \in \mathcal{C}(X')$ (with $f(\infty) := 0$), and hence $f \in \mathfrak{A}'$. Hence there is $g \in \mathfrak{A}$ and a constant $\lambda \in \mathbb{C}$ such that $f = g + \lambda$. Evaluating this identity in ∞ we see that $\lambda = 0$, hence $f = g$, i.e. $\mathcal{C}_0(X) \subseteq \mathfrak{A}$. ■

Remark. 2.3 Let X be a compact Hausdorff space, and let $X' = X \cup \{\infty\}$ be the one-point compactification of X. Then ∞ is an isolated point of X', and $\mathcal{C}_0(X)$ coincides with the set of functions f in $\mathcal{C}(X)$ each one extended to ∞ setting $f(\infty) := 0$. Let $\mathfrak{A}$ be a closed, separating and self-adjoint subalgebra of $\mathcal{C}(X)$. Moreover, assume that for every x_0 there exists $f \in \mathfrak{A}$ with $f(x_0) \neq 0$. (Notice that we do not assume that $1 \in \mathfrak{A}$). Let $\mathfrak{A}'$ be the subalgebra of $\mathcal{C}(X')$ generated by all the elements in $\mathfrak{A}$ (extended as 0 at ∞) and by 1. Then $\mathfrak{A}'$ is self-adjoint (evident), separating[5], and closed in $\mathcal{C}(X')$[6]. Thus, by Theorem 2.2, $\mathfrak{A}' = \mathcal{C}(X')$. Now, projecting on the subspace of functions which are zero at ∞, we recover the equality $\mathfrak{A} = \mathcal{C}(X)$. In other words, the equality $\mathfrak{A} = \mathcal{C}(X)$ holds also without the assumption $1 \in \mathfrak{A}$, proviso that for every $x_0 \in X$ we are able to find a function $f \in \mathfrak{A}$ with $f(x_0) \neq 0$. □

2.4. Hilbert spaces

Let $\mathcal{H}$ be a complex Hilbert space, with respect to the hermitian scalar product $\langle \cdot, \cdot \rangle$. Hence for every $x, y, z \in \mathcal{H}$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} &\langle x, x \rangle \ge 0 \; \forall x, \text{ and } \langle x, x \rangle = 0 \iff x = 0, \\ &\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle, \\ &\overline{\langle x, y \rangle} = \langle y, x \rangle, \end{aligned}$$

and $\mathcal{H}$ is complete with respect to the norm generated by the scalar product, saying with respect to the map $\|x\| := \sqrt{\langle x, x \rangle}$.
As usual, $\mathrm{End}(\mathcal{H})$ is the set of continuous linear maps $\mathcal{H} \to \mathcal{H}$. In $\mathrm{End}(\mathcal{H})$ we set the usual norm:

$$\|T\| := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|=1} \sqrt{\langle Tx, Tx \rangle}.$$

Recall that $\mathrm{End}(\mathcal{H})$ with this norm and the usual operations is a complex (unital, associative) Banach algebra.

[5] Let $x_1 \neq x_2 \in X'$ and $\alpha, \beta \in \mathbb{C}$. If x_1, x_2 both are in X then the separating property of $\mathfrak{A}$ shows that there is an $f \in \mathfrak{A}$ with $f(x_1) = \alpha$ and $f(x_2) = \beta$: this function f extended to ∞ as 0 becomes a separating element in $\mathfrak{A}'$. Suppose that x_1 or x_2 are not in X, say x_2. Then $x_2 = \infty$ and $x_1 \in X$ (because ∞ is the unique point of X' which is not in X, and $x_1 \neq x_2$). Let f_1 be the element in $\mathcal{C}(X)$ such that $f_1(x_1) \neq 0$ (it exists, by hypothesis). Set $g(x) := (\alpha - \beta) f_1(x)/f_1(x_1) + \beta$. Then $g(x_1) = \alpha$ and $g(\infty) = \beta$.

[6] Because the convergence in sup norm in X' means a convergence in sup norm at X plus a pointwise convergence in ∞, since ∞ is isolated in X'.

This algebra has a further property, which is very important for our applications: via the Riesz theorem, to every $T \in \mathrm{End}(\mathcal{H})$ we can associate its adjoint T^*, which is the unique linear map such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

holds for every $x, y \in \mathcal{H}$[7]. This construction does not make evident that T^* is continuous. However, the Cauchy–Schwarz inequality and the continuity of T give

$$|\langle Tx, y\rangle| \leq \|Tx\| \cdot \|y\| \leq \|T\| \cdot \|x\| \cdot \|y\|, \qquad \forall x, y \in \mathcal{H}.$$

Setting $x = T^*y$, this inequality gives

$$\|T^*y\|^2 = |\langle T^*y, T^*y\rangle| = |\langle TT^*y, y\rangle| \leq \|T\| \cdot \|T^*y\| \cdot \|y\| \qquad \forall y \in \mathcal{H},$$

proving that

$$\|T^*y\| \leq \|T\| \cdot \|y\| \qquad \forall y \in \mathcal{H}.$$

This shows both that T^* is bounded (and hence is in $\mathrm{End}(\mathcal{H})$), and that $\|T^*\| \leq \|T\|$. Since $(T^*)^* = T$, this inequality immediately proves that

$$\|T^*\| = \|T\|.$$

Moreover, a similar computation shows that

$$\|Tx\|^2 = |\langle Tx, Tx\rangle| = |\langle T^*Tx, x\rangle| \leq \|T^*Tx\| \cdot \|x\| \leq \|T^*T\| \cdot \|x\|^2 \qquad \forall x \in \mathcal{H},$$

proving that $\|T\|^2 \leq \|T^*T\|$. The submultiplicity of the norm gives also that $\|T^*T\| \leq \|T^*\| \cdot \|T\| = \|T\|^2$, yielding that

$$\|T^*T\| = \|T\|^2. \tag{2.3}$$

We recall that an operator T is *normal* when commutes with T^*, *unitary* when $T^* = T^{-1}$, and *self-adjoint* when $T = T^*$.

Let T be a normal operator. Then we have:

$$\begin{aligned}
\|T^2\|^2 &= \|(T^2)^*T^2\| && \text{(by (2.3) applied to } T^2\text{)}\\
&= \|T^*T^*TT\| = \|(T^*T)(T^*T)\| && \text{(because } T \text{ is normal)}\\
&= \|(T^*T)^*(T^*T)\| && \text{(because } T^*T \text{ is self-adjoint)}\\
&= \|T^*T\|^2 && \text{(by (2.3) applied to } T^*T\text{)}\\
&= \|T\|^4 && \text{(by (2.3) applied to } T\text{)}.
\end{aligned}$$

Thus, for a normal operator one has

$$\|T^2\| = \|T\|^2. \tag{2.4}$$

This formula allows to prove the following interesting result for normal operators.

Proposition 2.5 *Let T be a normal operator in $\mathrm{End}(\mathcal{H})$. Then $r(T) = \|T\|$.*

Proof. In fact, iterating (2.4) one gets

$$\|T^{2^n}\| = \|T\|^{2^n} \quad \forall n,$$

i.e.,

$$\|T^{2^n}\|^{1/2^n} = \|T\| \quad \forall n.$$

The formula for the spectral radius (Proposition 2.2[4]) immediately yields the conclusion. ∎

Proposition 2.6 *Let $T \in \mathrm{End}(\mathcal{H})$. If T is unitary then $\mathrm{sp}(T) \subseteq S^1 := \{\lambda \in \mathbb{C} \colon |\lambda| = 1\}$. If T is self-adjoint, then $\mathrm{sp}(T) \in \mathbb{R}$.*

[7] A complex algebra supporting a map $*$ having all formal properties for the adjoint operation (i.e., $(T^*)^* = T$ (involution), $(\lambda T + S)^* = \overline{\lambda} T^* + S^*$(hermitian linearity), and $(TS)^* = S^*T^*$(reverse ordering)) is called *C^*-algebra*. Everything we are proving now comes from these formal properties and hence can be extended to C^*-algebras.

Proof. Suppose T be unitary. Then $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle x, x\rangle = \|x\|^2$, proving that $\|T\| = 1$. Thus $r(T) \leq \|T\| = 1$, proving that each $\lambda \in \mathrm{sp}(T)$ has $|\lambda| \leq 1$. On the other hand, an operator S is invertible if and only if S^* is invertible, thus $\mathrm{sp}(T^*) = \overline{\mathrm{sp}(T)}$. Moreover, for invertible elements S we know that $\mathrm{sp}(S^{-1}) = (\mathrm{sp}(S))^{-1}$. It follows that for the unitary operator T (so that $T^* = T^{-1}$), $\lambda \in \mathrm{sp}(T)$ implies that and $\lambda^{-1} \in \overline{\mathrm{sp}(T)}$. In particular $|\lambda^{-1}| \leq 1$, proving that actually $\mathrm{sp}(T) \subseteq S^1$.
For every $S \in \mathrm{End}(\mathcal{H})$, the element

$$\exp(S) := \sum_{n=0}^{+\infty} \frac{S^n}{n!}$$

is well defined (use Weierstrass test to prove the convergence), and it is invertible with inverse given by $\exp(-S)$ (because S and $-S$ commute, so that the formal computation $\exp(S)\exp(-S) = \exp(S - S) = \exp(0) = 1$ holds true).
Suppose T be self-adjoint. Then $\exp(iT)$ is unitary, since

$$\begin{aligned}(\exp(iT))^* &= \Big(\sum_{n=0}^{+\infty} \frac{i^n T^n}{n!}\Big)^* = \sum_{n=0}^{+\infty} \frac{(-i)^n (T^*)^n}{n!} = \sum_{n=0}^{+\infty} \frac{(-i)^n T^n}{n!} \\ &= \exp(-iT) = \big(\exp(iT)\big)^{-1}.\end{aligned}$$

Let $\lambda \in \mathrm{sp}(T)$. Then $\exp(i\lambda) \in \mathrm{sp}(\exp(iT))$ (consider Exercise 2.4). Therefore $|\exp(i\lambda)| = 1$, because $\exp(iT)$ is unitary. This forces λ to be real. ■

Finally we can prove the main result in this section.

Proposition 2.7 *Let A be a unital, commutative $*$-closed and closed complex subalgebra of* $\mathrm{End}(\mathcal{H})$*. Then the Gelfand transform $\Gamma\colon A \to \mathcal{C}(\hat{A})$ is an isometric $*$-isomorphism of unital complex algebras. ($*$-closed means that if $T \in A$ then $T^* \in A$; $*$-isomorphism means that Γ is an isomorphism such that $\Gamma(T^*) = \overline{\Gamma(T)}$ for all $T \in A$.)*

Proof. By definition, $\Gamma(T) = \hat{T}$, and $\|\hat{T}\|_\infty = r(T)$. But T is normal, since T^* is in A (because A is $*$-closed) and all elements in A commutes. Therefore $r(T) = \|T\|$, proving that

$$\|\Gamma(T)\|_\infty = \|\hat{T}\|_\infty = r(T) = \|T\|$$

proving that Γ preserves the norm. In particular Γ is injective.
Given $T \in A$, let

$$T_+ := \tfrac{1}{2}(T + T^*) \quad , \quad T_- := \tfrac{1}{2i}(T - T^*).$$

Elements $T_\pm$ are self-adjoint, hence their spectrum is real. Since the spectrum of $T_\pm$ coincides with the range of $\hat{T}_\pm$, this proves that

$$\Gamma(T_\pm)(\chi) = \chi(T_\pm) = \overline{\chi(T_\pm)} = \overline{\Gamma(T_\pm)}(\chi).$$

Thus

$$\begin{aligned}\Gamma(T^*)(\chi) &= \Gamma(T_+ - iT_-)(\chi) = \Gamma(T_+)(\chi) - i\Gamma(T_-)(\chi) \\ &= \overline{\Gamma(T_+)(\chi) + i\Gamma(T_-)(\chi)} = \overline{\Gamma(T_+ + iT_-)(\chi)} \\ &= \overline{\Gamma(T)(\chi)} = \overline{\Gamma(T)}(\chi)\end{aligned}$$

proving that $\Gamma(A)$ is a self-adjoint algebra in $\mathcal{C}(\hat{A})$. It contains 1 since $\Gamma(1_A) = 1$, separates the points of $\hat{A}$ (by Theorem 2.1[ii.]) and is closed, since A is closed and Γ is unitary. Thus, by Stone–Weierstrass we conclude that $\Gamma(A) = \mathcal{C}(\hat{A})$, i.e. that Γ is surjective. ■

2.5. The spectral theorem (functional calculus version)

Let $T \in \mathrm{End}(\mathcal{H})$ be a normal operator. Let A_T be the least (i.e. intersection of all) closed complex subalgebra of $\mathrm{End}(\mathcal{H})$ containing T and T^*. It is clearly the closure of the algebra generated by $1_{\mathcal{H}}$, T and T^*. Note that A_T is commutative, since T and T^* commute. Each element which is invertible in A_T is evidently invertible also in $\mathrm{End}(\mathcal{H})$, thus $\mathrm{Res}_{A_T}(T) \subseteq \mathrm{Res}(T)$, or, which is the same, $\mathrm{sp}(T) \subseteq \mathrm{sp}_{A_T}(T)$. The opposite inclusion is also true, but its proof needs a sofisticate argument. It is a part of the following theorem.

Theorem 2.3 *Let $T \in \mathrm{End}(\mathcal{H})$ be a normal operator. Let A_T as above. Then there exists a map $\Phi\colon \mathcal{C}(\mathrm{sp}(T)) \to A_T$ which is an isometry and a $*$-isomorphism of unital algebras. Moreover, $\Phi(i_{\mathrm{sp}(T)}) = T$, and $\mathrm{sp}(\Phi(f)) = f(\mathrm{sp}(T))$ for every $f \in \mathcal{C}(\mathrm{sp}(T))$. (The map $i_{\mathrm{sp}(T)}$ is the inclusion of $\mathrm{sp}(T) \hookrightarrow \mathbb{C}$).*

Essentially, this theorem states that to every continuous function f which can be defined on the spectrum of a normal operator T corresponds a unique element $\Phi(f)$ in the algebra A_T (so that in particular it commutes with T), and that the correspondence is via an isomorphism of algebras. The claim that the spectrum of $\Phi(f)$ is exactly $f(\mathrm{sp}(T))$ settles in the larger setting of continuous functions what we have stated (without proof) firstly for polynomials and then for power series.

Proof. The map $\hat{T}\colon \hat{A_T} \to \mathbb{C}$ is continuous (because in $\hat{A_T}$ we are considering the weak* topology). Moreover, it is injective. In fact, let χ_1 and χ_2 in $\hat{A_T}$, such that $\hat{T}(\chi_1) = \hat{T}(\chi_2)$. This means that $\chi_1(T) = \chi_2(T)$. We have proved (Proposition 2.4) that Gelfand's map is a $*$-isomorphism, hence for every character $\chi(T^*) = \overline{\chi(T)}$. As a consequence we also know that $\chi_1(T^*) = \chi_2(T^*)$, i.e. $\hat{T^*}(\chi_1) = \hat{T^*}(\chi_2)$. Since A_T is generated by $1_{\mathcal{H}}$, T and T^*, we conclude that

$$\hat{T}(\chi_1) = \hat{T}(\chi_2) \quad \Longrightarrow \quad \hat{S}(\chi_1) = \hat{S}(\chi_2) \quad \forall S \in A_T,$$

and this means that $\chi_1(S) = \chi_2(S)$ for every $S \in A_T$, i.e. that $\chi_1 = \chi_2$ as elements in $\hat{A_T}$. Since $\hat{T}$ is a closed map (because $\hat{A_T}$ is a compact), it is a homeomorphism on its image, which is $\mathrm{sp}_{A_T}(T)$ (by Prop. 2.3[iv.]), so

$$\hat{T}\colon \hat{A_T} \xrightarrow{\sim} \mathrm{sp}_{A_T}(T).$$

Let Ψ be defined in the following way:

$$\begin{aligned} \Psi\colon \mathcal{C}(\mathrm{sp}_{A_T}(T)) &\longrightarrow \mathcal{C}(\hat{A_T}) \\ f &\mapsto \Psi(f) := f \circ \hat{T} \end{aligned} \qquad \text{i.e.} \qquad \begin{array}{ccc} \hat{A_T} & \xrightarrow{\hat{T}} & \mathrm{sp}_{A_T}(T) \\ & \overset{\Psi(f)}{\searrow} & \downarrow f \\ & & \mathbb{C} \end{array}$$

Then Ψ is an isometry and a $*$-isomorphism of complex algebras. In fact, it is evident that it is a morphism of complex algebras. Moreover,

$$\begin{aligned} \|\Psi(f)\|_\infty &= \sup_{\chi \in \hat{A_T}} |\Psi(f)(\chi)| = \sup_{\chi \in \hat{A_T}} |f \circ \hat{T}(\chi)| = \sup_{\chi \in \hat{A_T}} |f(\chi(T))| \\ &= \sup_{\lambda \in \mathrm{sp}_{A_T}(T)} |f(\lambda)| = \|f\|_{\infty, \mathrm{sp}_{A_T}(T)} \end{aligned}$$

(because $\chi(T)$ with $\chi \in \hat{A_T}$ ranges exactly on $\mathrm{sp}_{A_T}(T)$). This proves that Ψ is an isometry and hence that it is injective. It is also surjective, since $\hat{T}$ is a homeomorphism[8].

[8] so that for every $g \in \mathcal{C}(\hat{A_T})$ the map $g \circ \hat{T}^{-1}$ is in $\mathcal{C}(\mathrm{sp}_{A_T}(T))$ and $\Psi(g \circ \hat{T}^{-1}) = g \circ \hat{T}^{-1} \circ \hat{T} = g$; this proves that Ψ is surjective.

Both Ψ and Gelfand's map Γ are isometries and $*$-isomorphism. Thus the map Φ:

$$\begin{aligned}\Phi\colon \mathcal{C}(\mathrm{sp}_{A_T}(T)) &\longrightarrow A \\ \Phi &:= \Gamma^{-1}\circ\Psi\end{aligned} \qquad \text{i.e.} \qquad \begin{array}{ccc} \mathcal{C}(\mathrm{sp}_{A_T}(T)) & \xrightarrow{\Psi} & \mathcal{C}(\hat{A_T}) \\ & \overset{\Phi}{\searrow} & \uparrow \Gamma\, \sim \\ & & A_T \end{array}$$

is well defined, and is itself an isometry and a $*$-isomorphism.
Now we prove that $\Phi(i_{\mathrm{sp}_{A_T}(T)}) = T$. This happens if and only if $\Gamma(\Phi(i_{\mathrm{sp}_{A_T}(T)})) = \Gamma(T)$, i.e., if and only if $\Psi(i_{\mathrm{sp}_{A_T}(T)}) = \Gamma(T)$. This is true, since directly from its definition

$$\Psi(i_{\mathrm{sp}_{A_T}(T)}) = i_{\mathrm{sp}_{A_T}(T)} \circ \hat{T} = \hat{T}$$

since the image of $\hat{T}$ is in $\mathrm{sp}_{A_T}(T)$. A similar computation shows that $\Phi(1) = 1_{\mathcal{H}}$.
Thus everything has been proved, but the equalities

$$\mathrm{sp}_{A_T}(T) = \mathrm{sp}(T) \qquad \text{and} \qquad \mathrm{sp}(\Phi(f)) = f(\mathrm{sp}(T)).$$

We prove both of them with a unique argument.
Let $f \in \mathcal{C}(\mathrm{sp}_{A_T}(T))$. Then

$$\begin{aligned} f(\mathrm{sp}_{A_T}(T)) &= f(\hat{T}(\hat{A_T})) && \text{(formula for the spectrum)} \\ &= (\Psi(f))(\hat{A_T}) && \text{(definition of } \Psi) \\ &= ((\Gamma\circ\Phi)(f))(\hat{A_T}) && \text{(definition of } \Phi) \\ &= (\widehat{\Phi(f)})(\hat{A_T}) && \text{(definition of } \Gamma) \\ &= \mathrm{sp}_{A_T}(\Phi(f)) && \text{(formula for the spectrum),} \end{aligned} \tag{2.5}$$

which is similar to what we would like to prove, but it involves the spectrum on A_T, not the one in $\mathcal{H}$.

Fix a $\lambda \in \mathrm{sp}_{A_T}(T)$. Let $\epsilon > 0$, and let $\delta > 0$ be such that

$$\begin{cases} |\lambda-\mu| \le \delta \\ \mu \in \mathrm{sp}_{A_T}(T) \end{cases} \qquad \Longrightarrow \qquad |f(\lambda) - f(\mu)| \le \epsilon$$

(δ exists, because f is continuous). Let $g \in \mathcal{C}(\mathrm{sp}_{A_T}(T))$ be a function such that

$$\|g\|_{\infty,\mathrm{sp}_{A_T}(T)} = 1, \qquad \begin{cases} |\mu-\lambda| \ge \delta \\ \mu \in \mathrm{sp}_{A_T}(T) \end{cases} \Longrightarrow g(\mu) = 0.$$

Let $P := \Phi(g)$. Then, since Φ is an isometry and an isomorphism of algebras, from the choice of g (and the fact that Φ is a morphism of algebras, with $\Phi(1) = 1$), we get

$$\begin{aligned} \|(f(\lambda)1_{\mathcal{H}} - \Phi(f))P\| &= \|\Phi^{-1}((f(\lambda)1_{\mathcal{H}} - \Phi(f))P)\|_{\infty,\mathrm{sp}_{A_T}(T)} \\ &= \|\Phi^{-1}(f(\lambda)1_{\mathcal{H}} - \Phi(f))\Phi^{-1}(P)\|_{\infty,\mathrm{sp}_{A_T}(T)} \\ &= \|(f(\lambda) - f)g\|_{\infty,\mathrm{sp}_{A_T}(T)} \le \epsilon \end{aligned}$$

(for the last bound recall that $|((f(\lambda) - f)g)(\mu)| = |(f(\lambda) - f(\mu))g(\mu)|$ is bounded by ϵ on the support of g).
Assume now by absurd that $f(\lambda)1_{\mathcal{H}} - \Phi(f)$ is invertible in $\mathrm{End}(\mathcal{H})$. Then

$$\begin{aligned} 1 = \|g\|_{\infty,\mathrm{sp}_{A_T}(T)} &= \|\Phi(g)\| = \|P\| = \|(f(\lambda)1_{\mathcal{H}} - \Phi(f))^{-1}(f(\lambda)1_{\mathcal{H}} - \Phi(f))P\| \\ &\le \|(f(\lambda)1_{\mathcal{H}} - \Phi(f))^{-1}\| \cdot \|(f(\lambda)1_{\mathcal{H}} - \Phi(f))P\| \le \|(f(\lambda)1_{\mathcal{H}} - \Phi(f))^{-1}\|\epsilon. \end{aligned}$$

But ϵ is arbitrarily small, hence this is impossible. This proves that $f(\lambda) \in \mathrm{sp}(\Phi(f))$, i.e. that

$$f(\mathrm{sp}_{A_T}(T)) \subseteq \mathrm{sp}(\Phi(f)). \tag{2.6}$$

Specializing the identity to $f = i_{\mathrm{sp}_{A_T}(T)}$, we get

$$\mathrm{sp}_{A_T}(T) \subseteq \mathrm{sp}(T).$$

Since we already know that $\mathrm{sp}_{A_T}(T) \supseteq \mathrm{sp}(T)$, we conclude that

$$\mathrm{sp}_{A_T}(T) = \mathrm{sp}(T), \tag{2.7}$$

which is the first of the equalities. Moreover, if an operator is invertible in A_T, then it is also invertible in $\mathrm{End}(\mathcal{H})$, so

$$\mathrm{sp}(\Phi(f)) \subseteq \mathrm{sp}_{A_T}(\Phi(f)). \tag{2.8}$$

Thus,

$$\mathrm{sp}(\Phi(f)) \overset{(2.8)}{\subseteq} \mathrm{sp}_{A_T}(\Phi(f)) \overset{(2.5)}{=} f(\mathrm{sp}_{A_T}(T)) \overset{(2.6)}{\subseteq} \mathrm{sp}(\Phi(f)),$$

therefore the inclusions are actually equalities, giving

$$\mathrm{sp}(\Phi(f)) = f(\mathrm{sp}_{A_T}(T)) = f(\mathrm{sp}(T)),$$

where for the last equality we have used (2.7). ■

Proposition 2.8 *Let $T \in \mathrm{End}(\mathcal{H})$ be a normal operator. Suppose that $\mathrm{sp}(T)$ contains a unique value, λ_0. Then $T = \lambda_0 1_{\mathcal{H}}$.*

Proof. The space $\mathcal{C}(\mathrm{sp}(T))$ of all continuous functions on $\mathrm{sp}(T)$ coincides with $\mathbb{C}$, as an algebra. Hence the algebra A_T generated by T and T^* is isomorphic to $\mathbb{C}$, By Theorem 2.3. Thus $\lambda_0 1_{\mathcal{H}} - T$ is 0, because it is not invertible and 0 is the unique element which is not invertible in A_T. ■

The following proposition shows how injectivity/surjectivity of T and T^* are related.

Proposition 2.9 *Let $T \in \mathrm{End}\,\mathcal{H}$. Then the following claims are equivalent:*

i. T is invertible in $\mathrm{End}(\mathcal{H})$;

ii. T^* is invertible in $\mathrm{End}(\mathcal{H})$;

iii. T and T^* are both injective and $T(\mathcal{H})$ is closed;

iv. T and T^* are both bounded away from 0.

Proof.

i. $\iff$ ii. evident, since $(T^*)^{-1} = (T^{-1})^*$;

i.+ii. $\implies$ iii. evident.

iii. $\implies$ i. The definition of the adjoint operator shows that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, thus $\ker T^*$ coincides with $T(\mathcal{H})^\perp$. Thus, $T(\mathcal{H})^\perp = \{0\}$ when T^* is injective, and hence $T(\mathcal{H}) = \overline{T(\mathcal{H})} = ((T(\mathcal{H}))^\perp)^\perp = \mathcal{H}$. Thus T is surjective. By hypothesis it is also injective, so that it is a bijective operator. The open mapping theorem ensures that such operator is invertible.

i.+ii. $\implies$ iv. Let T be invertible. Then $x = T^{-1}Tx$ shows that $\|x\| \leq \|T^{-1}\| \cdot \|Tx\|$, i.e. that $\|Tx\| \geq (1/\|T^{-1}\|)\|x\|$, proving that T is bounded away from 0. The same happens to T^*, by [ii.].

iv. $\implies$ iii. Suppose that both T and T^* are bounded away from 0. Then T and T^* are injective. Moreover, by assumption there exists a constant ℓ such that $\|Tx - Ty\| = \|T(x-y)\| \geq \ell\|x-y\|$ for every $x, y \in \mathcal{H}$. Thus, every Cauchy sequence in $T(\mathcal{H})$ comes from a Cauchy sequence in $\mathcal{H}$, and hence converges. This proves that $T(\mathcal{H})$ is closed.

■

Let $\lambda \in \mathrm{sp}(T)$. Then there is a sequence of unitary vectors v_n such that one of the limits $\|(T-\lambda 1_{\mathcal{H}})v_n\| \to 0$, $\|(T^*-\overline{\lambda}1_{\mathcal{H}})v_n\| \to 0$ holds. In fact, if both are false then both $T-\lambda 1_{\mathcal{H}}$ and $(T-\lambda 1_{\mathcal{H}})^*$ are bounded away from 0, but then $T-\lambda 1_{\mathcal{H}}$ would be invertible, by the part [iv] of the previous proposition.
Suppose now that T is normal. Then $T-\lambda 1_{\mathcal{H}}$ is normal as well. But then $\|(T-\lambda 1_{\mathcal{H}})v_n\| = \|(T^*-\overline{\lambda}1_{\mathcal{H}})v_n\|$ (because if S is normal then $\|Sv\|^2 = \langle Sv, Sv\rangle = \langle S^*Sv, v\rangle = \langle SS^*v, v\rangle = \langle S^*v, S^*v\rangle = \|S^*v\|^2$), thus we deduce the following proposition showing that each number in the spectrum of T is 'almost' an eigenvalue.

Proposition 2.10 *Let T be a normal operator in* $\mathrm{End}(\mathcal{H})$. *Let $\lambda \in \mathrm{sp}(T)$. Then there is a sequence of unitary vectors such that $\|Tv_n - \lambda v_n\| \to 0$. If λ is isolated, then λ is an eigenvalue.*

Proof. We have already proved the first claim. Let λ be isolated in $\mathrm{sp}(T)$. Then the function $f\colon \mathrm{sp}(T) \to \mathbb{C}$ defined as

$$f(x) = \begin{cases} 1 & \text{if } x = \lambda \\ 0 & \text{if } x \neq \lambda \end{cases}$$

is a continuous map on $\mathrm{sp}(T)$. By Theorem 2.3, let $P := \Phi(f)$. Then

$$\|(\lambda 1_{\mathcal{H}} - T)P\| = \|\Phi^{-1}((\lambda 1_{\mathcal{H}} - T)P)\|_{\infty,\mathrm{sp}(T)} = \|(\lambda - i_{\mathrm{sp}(T)})f\|_{\infty,\mathrm{sp}(T)} = 0.$$

Hence $(\lambda 1_{\mathcal{H}} - T)P$ is the trivial operator. The operator P is not trivial, since f is not trivial, therefore there is a vector v such that $(\lambda 1_{\mathcal{H}} - T)w = 0$, where $w := Pv \neq 0$. ■

By definition, an operator $T \in \mathrm{End}(\mathcal{H})$ is *positive* when it is self-adjoint and $\langle Tv, v\rangle \geq 0$ for every v. The following proposition characterizes positive operators.

Proposition 2.11 *Let T be a normal operator in* $\mathrm{End}(\mathcal{H})$. *It is self-adjoint if and only if* $\mathrm{sp}(T) \subset \mathbb{R}$, *and it is positive if and only if* $\mathrm{sp}(T) \subset \mathbb{R}_{\geq 0}$. *Moreover, T is positive if and only if there exists a self-adjoint positive operator S such that $T = S^2$. The operator S is uniquely determined by T and is called the* square root *of T. It commutes with all operators commuting with T.*

Proof. We already know that $\mathrm{sp}(T) \subset \mathbb{R}$ for a self-adjoint operator. Let T be positive and let $\lambda \in \mathrm{sp}(T)$. By Proposition 2.10 there is a sequence of unitary vectors v_n such that $\|Tv_n - \lambda v_n\| \to 0$, so that

$$\begin{aligned} 0 \leq \langle Tv_n, v_n\rangle &= \langle \lambda v_n + (T-\lambda 1_{\mathcal{H}})v_n, v_n\rangle = \lambda + O(|\langle (T-\lambda 1_{\mathcal{H}})v_n, v_n\rangle|) \\ &= \lambda + O(\|(T-\lambda 1_{\mathcal{H}})v_n\| \cdot \|v_n\|) = \lambda + O(\|(T-\lambda 1_{\mathcal{H}})v_n\|) \\ &= \lambda + o(1) \end{aligned}$$

which proves that $\lambda \geq 0$, i.e. that $\mathrm{sp}(T) \subset \mathbb{R}_{\geq 0}$.
On the contrary, suppose that $\mathrm{sp}(T) \subset \mathbb{R}$. Then $i_{\mathrm{sp}(T)}$ (the identity map on $\mathrm{sp}(T)$) is a real map, i.e. it is fixed by the conjugation. This holds true also for $\Phi(i_{\mathrm{sp}(T)})$ (because Φ is a $*$-isomorphism), and hence T is self-adjoint (because $T = \Phi(i_{\mathrm{sp}(T)})$). At last, suppose that $\mathrm{sp}(T) \subset \mathbb{R}_{\geq 0}$. Then the map $f(\lambda) := \sqrt{\lambda}$ is well defined on $\mathrm{sp}(T)$. Then $S := \Phi(f)$ is such that $\Phi(f)^2 = \Phi(f^2) = \Phi(i_{\mathrm{sp}(T)}) = T$, and is self-adjoint (because $\mathrm{sp}(S) = \mathrm{sp}(\Phi(f)) = f(\mathrm{sp}(T)) = (\mathrm{sp}(T))^{1/2} \subset \mathbb{R}_{\geq 0}$). Thus

$$\langle Tv, v\rangle = \langle S^2 v, v\rangle = \langle Sv, Sv\rangle \geq 0,$$

proving that T is positive.
At last, we notice that $S = \Phi(\sqrt{\cdot})$ belongs to A_T, i.e. to the closed algebra generated by T and T^*, but this algebra coincides with the polynomial algebra generated by T alone

because T is self-adjoint (by hypothesis). Therefore, if W is an operator which commutes with T, then it commutes also with S. ■

CHAPTER 3

Representations

Let G be locally compact group, and V be a vector space on a field k. From a purely algebraic point of view a *representation* of G in V is simply a morphism $\rho\colon G \to \mathrm{Aut}(V)$, i.e. a map such that

$$\rho(gg') = \rho(g)\rho(g') \qquad \forall g, g' \in G,$$

where $\rho(g)$ and $\rho(g')$ are in $\mathrm{Aut}(V)$.

Example. 3.1 Let $G := \{z \in \mathbb{C}\colon |z| = 1\}$ (the circle), and let $V = \mathbb{C}^2$, as a complex vector space. For every $n \in \mathbb{Z}$, the map

$$\rho_n\colon G \to \mathrm{GL}(2,\mathbb{C}), \qquad \rho_n(z) := \left(\begin{smallmatrix} z^n & 0 \\ 0 & z^{-n} \end{smallmatrix}\right)$$

is a representation of G (in $\mathrm{SL}(2,\mathbb{C})$). □

Exercise. 3.1 Let $G := U(2,\mathbb{C})$, the set of unitary automorphisms of $\mathbb{C}^2$.

1. Prove that
$$G = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : \ a, b \in \mathbb{C}, \ \ |a|^2 + |b|^2 = 1 \right\}.$$
Note that G is a non-abelian compact group.
2. Prove that each element in G is conjugated to some element $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$ for a $\theta \in [-\pi, \pi]$, and that these elements are pairwise conjugated, with the element associated with θ conjugated with the one associated to $-\theta$. Deduce that each conjugation class contains a unique element of that type with $\theta \in [0, \pi]$.
3. For every $n \in \mathbb{N}$, let
$$\mathcal{H}_n := \{p \in \mathbb{C}[x, y]\colon \ p(\lambda x, \lambda y) = \lambda^n p(x, y)\},$$
the set of complex polynomials in two variables which are homogeneous with degree n. Prove that $\mathcal{H}_n = \mathrm{Span}_{\mathbb{C}}(\{x^j y^{n-j}\}_{j=0}^n)$. In particular, $\dim(\mathcal{H}_n) = n + 1$.
4. Let $\rho_n\colon G \to \mathrm{Aut}(\mathcal{H}_n)$, be defined as
$$(\rho_n(g)p)(x, y) := p(g^{-1}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)).$$
Prove that ρ_n is a representation of G.

Remark: This exercise continues with Exercises 3.2 and 3.5.

When the group is not finite, in order to have a better control on its representation it is a good idea to select the representations having also good properties with respect to other (usually non algebraic) structures; typically with respect to the topology of G. Moreover, for our application we can restrict our definition to the case where V is actually a complex Hilbert space $\mathcal{H}$. In this setting, a *topological representation* of G in $\mathcal{H}$ is a map

$$\rho\colon G \to \mathrm{Aut}(\mathcal{H})$$

such that the map

$$\mathcal{W}\colon G \times \mathcal{H} \longrightarrow \mathcal{H}$$

$$(g, x) \mapsto \mathcal{W}(g, x) := \rho(g)x$$

is continuous, where in $G \times \mathcal{H}$ we take the product topology.

Proposition 3.1 *Let G be a locally compact group, and let ρ be a representation of G in a Hilbert space $\mathcal{H}$. Then ρ is a topological representation if and only if both the following conditions hold true:*

1. *For every compact $K \subseteq G$, the family of automorphisms in $\rho(K)$ is equicontinuous;*
2. *For every $x \in \mathcal{H}$, the map $g \mapsto \rho(g)x$ is continuous as a map $G \to \mathcal{H}$.*

In particular, ρ is a topological representation if and only if $\mathcal{W}$ is continuous in each argument. As a consequence, if we already know that $\rho(g) \in \mathrm{Aut_{top}}(\mathcal{H})$ for every g, then ρ is a topological representation if and only if 2) holds true, i.e. if and only if $g \mapsto \rho(g)x$ is continuous, for every x.

Proof. When ρ is a topological representation, Properties 1) and 2) are evident. for the opposite implication see [RV], Proposition 2.1 (p. 48–49).
For the second part of the claim, suppose we know that $\mathcal{W}$ is continuous in each argument. For every $x \in \mathcal{H}$, the map $g \mapsto \rho(g)x$ is $\mathcal{W}(\cdot, x)$ and by hypothesis it is continuous. Let $K \subseteq G$ be any compact set. Then for every fixed x, the set $\{\rho(g)x\colon g \in K\}$ is a compact set in $\mathcal{H}$, in particular it is bounded. Thus, each $\rho(g)$ is in $\mathrm{Aut_{top}}(\mathcal{H})$, because it is the map $\mathcal{W}(g, \cdot)$ which is continuous by hypothesis, and the family $\{\rho(g)\colon g \in K\}$ is pointwise bounded. By Banach–Steinhaus theorem this family is equibounded, i.e.,

$$M := \sup_{g \in K} \sup_{x\colon \|x\|=1} \|\rho(g)x\| < \infty.$$

as a consequence $\|\rho(g)x - \rho(g)y\| = \|\rho(g)(x - y)\| \leq M\|x - y\|$ when g ranges in K, and this proves that the family $\rho(K)$ is equicontinuous. This claim is also proved in [RV], Corollary 2.2 (p. 49). ■

Let $\rho\colon G \to \mathrm{Aut}(\mathcal{H})$ be a representation. A subspace W of $\mathcal{H}$ is *ρ-invariant* when $\rho(g)(W) \subseteq W$ for every $g \in G$. The representation is called *algebraically irreducible* when $\{0\}$ and $\mathcal{H}$ are the unique invariant subspaces; if the representation is a topological representation, it is called *topologically irreducible* when $\{0\}$ and $\mathcal{H}$ are the unique *closed* subspaces which are invariant.
Note that for a topological representation both notions of irreducibility are meaningful, and that the algebraic notion is stronger than the topological one, but they coincides when $\mathcal{H}$ is a finite dimensional vector space (however, the notions are strictly different: there are representations which are topologically irreducible but not algebraically irreducible).

Another definition: two representations ρ and ρ' of a given group G in Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ (not necessarily the same) are *interlaced* by a map $T \in \mathrm{Hom}(\mathcal{H}, \mathcal{H}')$ when $T\rho(g) = \rho'(g)T$ for every $g \in G$, i.e. when the following diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{T} & \mathcal{H}' \\ \rho(g)\downarrow & & \downarrow\rho'(g) \\ \mathcal{H} & \xrightarrow[T]{} & \mathcal{H}' \end{array}$$

commutes. Moreover, they are called *equivalent* when they are interlaced via any isomorphism of vector spaces. Analogously, two topological representations ρ and ρ' are *topologically interlaced* by a map T when T is continuous (as a map $\mathcal{H} \to \mathcal{H}'$), and are *topologically equivalent* when T is a continuous isomorphism.

At last, a representation is called *unitary* when $\rho(g)$ is unitary for all $g \in G$, i.e. when $\langle \rho(g)x, \rho(g)x \rangle_{\mathcal{H}'} = \langle x, x \rangle_{\mathcal{H}}$ for every $x \in \mathcal{H}$. The following proposition shows an interesting fact.

Proposition 3.2 *Let $(\rho, \mathcal{H})$ and $(\rho', \mathcal{H}')$ be topological unitary representations which are topologically equivalent. Then they are unitarily equivalent (i.e. the map $T\colon \mathcal{H} \to \mathcal{H}'$*

realizing the equivalence of ρ and ρ' can be modified to another map still realizing the equivalence of ρ and ρ', and which is unitary).

Proof. Let $T\colon \mathcal{H} \to \mathcal{H}'$ be the topological isomorphism such that $T\rho(g) = \rho'(g)T$ for every g. Let $T^*\colon \mathcal{H}' \to \mathcal{H}$ be the operator defined via the equation

$$\langle Tx, y\rangle_{\mathcal{H}'} = \langle x, T^*y\rangle_{\mathcal{H}} \qquad \forall x \in \mathcal{H},\ \forall y \in \mathcal{H}'.$$

(This is a generalization of the adjoint, but all properties are still true. In particular T^* is continuous). Then TT^* is positive and continuous endomorphism of $\mathcal{H}'$. By Proposition 2.11 there is a self-adjoint operator U such that $U^2 = TT^*$, and which commutes with all operators commuting with TT^*. Moreover,

$$(U^{-1}T) \cdot (U^{-1}T)^* = (U^{-1}TT^*) \cdot (U^{-1})^* = U \cdot U^{-1} = 1_{\mathcal{H}'},$$

proving that $U^{-1}T$ is a unitary and continuous isomorphism of $\mathcal{H}'$. Finally, we know that

$$T\rho(g) = \rho'(g)T \qquad \forall g \in G.$$

Passing to the adjoint operators yields $\rho(g)^*T^* = T^*\rho'(g)^*$. But $\rho(g)$ and $\rho'(g)$ are unitary, thus $\rho(g)^{-1}T^* = T^*\rho'(g)^{-1}$. The representations are morphisms, hence this equality means that $\rho(g^{-1})T^* = T^*\rho'(g^{-1})$. Since this is true for every $g \in G$, we conclude that

$$\rho(g)T^* = T^*\rho'(g) \qquad \forall g \in G.$$

Therefore,

$$TT^*\rho'(g) = T\rho(g)T^* = \rho'(g)TT^*$$

proving that TT^* commutes with $\rho'(g)$ for every g. Hence U^{-1} commutes as well, producing:

$$(U^{-1}T)\rho(g) = U^{-1}\rho'(g)T = \rho'(g)(U^{-1}T)$$

proving that ρ and ρ' are unitarily equivalent. ■

Theorem 3.1 (Schur's lemma)

1. *Let $(\rho, \mathcal{H})$ and $(\rho', \mathcal{H})$ be irreducible representations of a group G. Let $T\colon \mathcal{H} \to \mathcal{H}'$ be a morphism interlacing ρ and ρ'. Then either T is identically 0, or it is an isomorphism.*
2. *Suppose that G is locally compact and let $(\rho, \mathcal{H})$ be a topologically irreducible unitary representation of G. Let $T \in \operatorname{End}(\mathcal{H})$ such that $\rho(g)T = T\rho(g)$ for every $g \in G$. If T is normal, then it is a scalar operator (i.e. a multiple of the identity). In particular, TT^* is always a scalar operator.*

Proof.

1. By hypothesis $T\rho(g) = \rho'(g)T$ for every $g \in G$. This shows that $\ker T$ is an invariant space for ρ and $\operatorname{Imm} T$ an invariant space for ρ'. Both ρ and ρ' are irreducible, by hypothesis. Thus, if T is not trivial then $\ker T = \{0\}$ and $\operatorname{Imm} T = \mathcal{H}'$.
2. Let λ be any number in $\operatorname{sp}(T)$. If λ is the unique element in $\operatorname{sp}(T)$, then T is $\lambda 1_{\mathcal{H}}$, by Proposition 2.8. Suppose therefore that this is not the case, i.e. that $\operatorname{sp}(T)$ contains also a second element $\mu \neq \lambda$. Then there are functions f_λ, f_μ in $\mathcal{C}(\operatorname{sp}(T))$ which are not identically 0 but that are zero in suitable open neighborhoods of λ and μ which do not overlap (so that the product $f_\lambda f_\mu$ is identically zero). Let $\Phi(f_\lambda)$ be the operator associated to f_λ via the spectral theorem. It belongs to A_T (the closed complex algebra generated by T and T^*). By hypothesis every element in $\rho(G)$ commutes with T, and taking the adjoint we verify that it commutes also with T^* (because ρ is unitary). Hence every element of $\rho(G)$ commutes with A_T, and hence in particular with $\Phi(f_\lambda)$. Let W be the closure of $\Phi(f_\lambda)\mathcal{H}$ in $\mathcal{H}$. Note that W is not $\{0\}$, because in this case $\|\Phi(f_\lambda)\| = 0$, while $\|f\|_{\infty,\operatorname{sp}(T)} \neq 0$ and Φ is unitary.
   This space is invariant for ρ, because $\Phi(f_\lambda)$ and $\rho(g)$ commute for every g. By assumption ρ is topologically irreducible, hence W is $\mathcal{H}$. The same holds true for $\Phi(f_\mu)\mathcal{H}$.

However, this is impossible, since Φ is a morphism, so that $0 \cdot 1_{\mathcal{H}} = \Phi(0) = \Phi(f_\lambda f_\mu) = \Phi(f_\lambda)\Phi(f_\mu)$, which contradicts the fact that $\overline{\Phi(f_\lambda)\Phi(f_\mu)\mathcal{H}} = \overline{\Phi(f_\lambda)\mathcal{H}} = \mathcal{H}$.

■

Remark. 3.1 For the second point, we could try the following approach. Let G, $(\rho, \mathcal{H})$ and T as in Theorem 3.1. Let $\lambda \in \mathrm{sp}(T)$. Then $T - \lambda 1_{\mathcal{H}}$ is a morphism of $\mathcal{H}$, evidently commuting with $\rho(g)$ for every $g \in G$. By the first part of the proposition it is either trivial or an isomorphism, but the second case is impossible (because λ belongs to the spectrum of T), thus it is trivial, i.e. $T = \lambda 1_{\mathcal{H}}$. This argument is tantalizing, but it does not work in general: the assumed topological irreducibility of ρ is weaker than the irreducibility, thus actually we cannot apply the first part of the theorem to conclude. However, the argument runs well in case $\mathcal{H}$ is finite dimensional, because in that case the two notions of irreducibility collapse. □

For unitary representations the second claim in Schur's lemma is actually a characterization of irreducibility. In fact, the following fact is true.

Proposition 3.3 *Let G be a locally compact group, and let $(\rho, \mathcal{H})$ be a topological unitary representation of G in $\mathcal{H}$. Then ρ is topologically irreducible if and only if the constant multiples of the identity $1_{\mathcal{H}}$ are the unique elements in* $\mathrm{End}(\mathcal{H})$ *which are normal and which commute with $\rho(g)$ for all g.*

Proof. Theorem 3.1 shows that this is what happens for unitary and topologically irreducible representations. Thus, we have only to prove that in case ρ is unitary and not topologically irreducible, there exists a normal $T \in \mathrm{End}(\mathcal{H})$ commuting with $\rho(g)$ for every g and which is not a multiple of the identity. The representation is not topologically irreducible, hence it admits a closed and non-trivial invariant subspace W. Let $W^\perp$ be the orthogonal complement in $\mathcal{H}$ of W; it is not trivial either, i.e. it is not $\{0\}$ or $\mathcal{H}$, because W is closed and is not $\{0\}$ or $\mathcal{H}$ (the fact that W is closed matters here!). By hypothesis ρ is unitary, and $\rho(g)W \subseteq W$ for every $g \in G$. We show that this implies that $\rho(g)W^\perp \subseteq W^\perp$ for every $g \in G$, too. In fact, pick any $w \in W$ and $w' \in W^\perp$. Then

$$\begin{aligned}
\langle \rho(g)w', w\rangle &= \langle w', \rho(g)^* w\rangle && \text{(definition of adjoint)}\\
&= \langle w', \rho(g)^{-1} w\rangle && (\rho(g) \text{ is unitary})\\
&= \langle w', \rho(g^{-1}) w\rangle && (\rho \text{ is a morphism})\\
&= 0 && (\rho(g^{-1})w \text{ is in } W \text{ for every } g).
\end{aligned}$$

Let T be the orthogonal projector on W: note that it is not a multiple of the identity (because its image W and its kernel $W^\perp$ both are non-trivial). Write any $v \in \mathcal{H}$ as $Tv + (v - Tv)$ (where $v - Tv \in W^\perp$). Then

$$\begin{aligned}
(T\rho(g))v &= (T\rho(g))(Tv + (v - Tv)) = (T\rho(g))Tv + (T\rho(g))(v - Tv)\\
&= T(\rho(g)Tv) + T(\rho(g)(v - Tv)) = (\rho(g)T)v
\end{aligned}$$

where the last step comes from the fact that $Tv \in W$ hence $\rho(g)(Tv) \in W$ hence $T\rho(g)(Tv) = \rho(g)(Tv)$, and that $v - Tv \in W^\perp$ hence $\rho(g)(v - Tv) \in W^\perp$ hence $T(\rho(g)(v - Tv)) = 0$.
This proves that T commutes with $\rho(g)$ for every g. ■

Proposition 3.4 *Let G be an abelian locally compact group, and let $(\rho, \mathcal{H})$ be a topological unitary representation of G in $\mathcal{H}$. If ρ is topologically irreducible, then* $\dim(\mathcal{H}) = 1$.

Proof. Fix any g' in G, and consider $T_{g'} := \rho(g')$. Since G is abelian, we have $T_{g'}\rho(g) = \rho(g'g) = \rho(gg') = \rho(g)T_{g'}$, proving that $T_{g'}$ commutes with $\rho(G)$. $T_{g'}$ is also unitary, so that in particular it is normal. By Theorem 3.1[ii.], it is a multiple of the identity, so that there exists a number, $\chi(g') \in \mathbb{C}$, such that $T_{g'} = \chi(g')1_{\mathcal{H}}$. This holds true for every

$g' \in G$, hence for every g' there is $\chi(g')$ such that $\rho(g') = \chi(g')1_{\mathcal{H}}$. Let x be any nonzero vector in $\mathcal{H}$; then $\mathbb{C}x$ is a closed subspace which is invariant by $\rho(G)$, thus it coincides with $\mathcal{H}$ because ρ is topologically irreducible. ■

Exercise. 3.2 Let G be a compact group with Haar measure μ. (By Proposition 1.13 it is both left and right invariant). Let $(\rho, \mathcal{H})$ be a topological representation of G on a complex Hilbert space $\mathcal{H}$. For every $x, y \in \mathcal{H}$, set

$$(x, y) := \frac{1}{\mu(G)} \int_G \langle \rho(g)x, \rho(g)y \rangle \, \mathrm{d}\mu,$$

where $\langle \cdot, \cdot \rangle$ is the hermitian scalar product in $\mathcal{H}$.

1. Prove that it is well defined (i.e. it is finite for every x, y);
2. Prove that $(\cdot, \cdot)$ is an hermitian scalar product;
3. Prove that the norm induced by $(\cdot, \cdot)$ is equivalent to the norm coming from $\langle \cdot, \cdot \rangle$, so that $\mathcal{H}$ is an Hilbert space also with respect to $(\cdot, \cdot)$;
4. Prove that ρ becomes unitary when in $\mathcal{H}$ the new scalar product $(\cdot, \cdot)$ is considered.

This exercise shows that all representations of a compact group can be considered as unitary representations. This extends to compact groups what is well known for finite groups.
Hint: for [3], recall Proposition 3.1, and that each $\rho(g)$ is a continuous automorphism of $\mathcal{H}$, so that also its inverse is bounded.

Exercise. 3.3 This exercise shows how to operate on representations to produce new representations.
Let G be a locally compact group, and let $(\rho, \mathcal{H})$ and $(\rho', \mathcal{H}')$ be topological representations of G.

1. (*direct sum*) prove that $(\rho \oplus \rho', \mathcal{H} \oplus \mathcal{H}')$, with $(\rho \oplus \rho')(g) := \rho(g) \oplus \rho'(g)$ is a topological representation of G;
2. (*tensor product*) prove that $(\rho \otimes \rho', \mathcal{H} \hat{\otimes} \mathcal{H}')$, with $(\rho \otimes \rho')(g) := \rho(g) \otimes \rho'(g)$ is a topological representation of G;
3. (*controgradient*) prove that $(\tilde{\rho}, \mathcal{H})$, with $\tilde{\rho}(g) := (\rho(g^{-1}))^*$ is a topological representation of G.
4. (*restriction*) Let H be a closed subgroup of G. Prove that $(\operatorname{Res}_H^G \rho, \mathcal{H})$, where $(\operatorname{Res}_H^G \rho)(h) := \rho(h)$ for every $h \in \mathcal{H}$, is a topological representation of H.
5. (*induction*) Let H be a closed subgroup of G whose index in G is finite. Let $(\psi, \mathcal{H})$ be a topological representation of H. Fix a set S of representatives for G/H, and take

$$\mathcal{H}_H^G := \bigoplus_{\alpha \in S} \alpha \mathcal{H},$$

i.e. the Hilbert sum of $[G : H]$ copies of $\mathcal{H}$. For every $g \in G$ and $\alpha \in S$, there is a unique $\alpha' \in S$ and a unique $h_{g,\alpha} \in H$ satisfying the equality $g\alpha = \alpha' h_{g,\alpha}$. For every $g \in G$, we define

$$(\operatorname{Ind}_H^G \psi)(g) \in \operatorname{End}(\mathcal{H}_H^G)$$

as the map

$$(\operatorname{Ind}_H^G \psi)(g) \Big(\bigoplus_{\alpha \in S} \alpha x_\alpha \Big) := \bigoplus_{\alpha \in S} \alpha' \psi(h_{g,\alpha}) x_{\alpha'}.$$

Then $(\operatorname{Ind}_H^G \psi)(g)$ is actually in $\operatorname{Aut}(\mathcal{H}_H^G)$ and $\operatorname{Ind}_H^G \psi$ defines a topological representation of G.

Remark: there are other possible (and probably better) constructions for the induced representation; some of them can be extended to cover also the case where the index of H in G is not finite.

Exercise. 3.4 Let G be a locally compact group. Let $(\rho, \mathcal{H})$ be a topological representation of G on a complex Hilbert space $\mathcal{H}$. Let W be a closed ρ-invariant subspace of $\mathcal{H}$.

1. Note that W becomes a Hilbert space itself with respect to the scalar product coming from $\mathcal{H}$. Prove that restricting each $\rho(g)$ to W produces a new topological representation (ρ_W, W);
2. Suppose that ρ is unitary. Prove that $W^\perp := \{y \in \mathcal{H} \colon \langle y, x\rangle = 0 \ \forall x \in W\}$ is ρ invariant as well;
3. Notice that $\mathcal{H} = W \oplus W^\perp$; conclude that under the hypotheses in [2.], ρ is topologically isomorphic to $\rho_W \oplus \rho_{W^\perp}$.
4. Let ρ be a unitary representation in a finite dimensional Hilbert space $\mathcal{H}$. Prove that ρ can be decomposed as finite direct sum of topological irreducible representations.

Remark: in the case of compact group G, last claim is true for every unitary representation (in particular also for the case of infinite dimensional Hilbert space), with (possibly infinite many) irreducible representations, each one being finite dimensional. This is part of the Peter–Weyl theorem. For abelian locally compact groups the analogous claim is a part of the Pontryagin duality result.

Exercise. 3.5 Let $G = U(2, \mathbb{C})$ and ρ_n be its representation as in Exercise 3.1. The following steps will prove that ρ_n is irreducible for every n (the distinction topologically/algebrically irreducible does not matter here, because the Hilbert space is finite dimensional).

1. Note that G is compact, so that ρ_n can be made unitary;
2. In the Hilbert space $\mathcal{H}_n$ take the base of monomials and notice that
$$\rho_n\Big(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}\Big)$$
is represented by a diagonal matrix in this base;
3. Notice that the diagonal entries in this matrix can be made all different for suitably chosen values of θ; conclude that if T commutes with $\rho_n\Big(\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{smallmatrix}\right)\Big)$ for every θ, then T is diagonal in that base;
4. Let $\{a_j\}_{j=0}^{n+1}$ be the constants such that $T(x^j y^{n-j}) = a_j x^j y^{n-j}$ for $j = 0, \ldots, n$. Compare
$$T\rho_n\Big(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix}\Big)x^n$$
to
$$\rho_n\Big(\begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix}\Big)Tx^n$$
and deduce that if T commutes with $\rho_n(g)$ for all $g \in G$ then it is a multiple of the identity. Apply Proposition 3.3 to conclude.

Remark: It is possible to prove that this is a full set of inequivalent topologically irreducible representations for $U(2, \mathbb{C})$.

Exercises of Chapter 2 in [RV] are instructive and should be fully studied. For a very nice introduction to the representations of compact groups and some material on representations in locally compact groups see [Ro]. For a classical treatise on representations of Lie groups see [Kn].

CHAPTER 4

Duality

4.1. Pontryagin duality

Let G be an abelian locally compact group. Let $\hat{G}$ be the set of all continuous homomorphisms $G \mapsto S^1$, where as usual S^1 denotes the set of complex numbers of absolute value 1. The elements in $\hat{G}$ are called *characters* for G, and $\hat{G}$ is also called *Pontryagin's dual* of G. It is an abelian group with respect to the pointwise product, i.e. when the product of two characters χ, η is defined as

$$\begin{aligned} \chi \cdot \eta \colon G &\longrightarrow S^1, \\ g &\mapsto (\chi \cdot \eta)(g) := \chi(g)\eta(g). \end{aligned}$$

Let K be any compact in G and U any open neighborhood of 1 in S^1; we denote $W(K, U)$ the set of characters for which the image of K is in U, i.e. the set $\{\chi \in \hat{G} \colon \chi(K) \subseteq U\}$. The sets $W(K, U)$ are a neighborhood base for the trivial character, and the compact-open topology of $\hat{G}$ is (by definition) the topology generated by the translated in each point of the family of sets $W(K, U)$. In other words, a family of characters $\{\chi_n\}_{n\in\mathbb{N}}$ converges in this topology to a character χ when for every compact K in G and every open neighborhood U of 1 in S^1 there is an integer $N = N(K, U)$ such that

$$n \geq N \qquad \Longrightarrow \qquad \forall g \in K, \quad (\chi_n\overline{\chi})(g) \in U.$$

Since the topology in S^1 is a metric topology, the compact-open topology in $\hat{G}$ coincides with the topology of uniform convergence in all compact subsets, i.e. the topology for which a family of characters $\{\chi_n\}_{n\in\mathbb{N}}$ converges to a character χ when for every compact K and every $\epsilon > 0$ there is $N = N(K, \epsilon)$ such that

$$n \geq N \qquad \Longrightarrow \qquad \|\chi_n(g) - \chi(g)\|_{\infty, K} \leq \epsilon.$$

(See [K], Theorem 11, p. 230). Moreover, if G is discrete then the compact sets in G are the finite sets, and the compact-open topology coincides with the topology of pointwise convergence.

Let $\phi \colon \mathbb{R} \to S^1$ be the covering map $\phi(x) := e^{2\pi i x}$, and for every $w > 0$, let

$$N(w) := \phi\big(\big(-\frac{w}{3}, \frac{w}{3}\big)\big),$$

the image of the open interval $(-w/3, w/3)$ via ϕ.
Moreover, for every $S \subseteq G$ and every integer m, we denote

$$S^{(m)} := \{s_1 \cdot s_2 \cdots s_m \colon s_j \in S \ \forall j = 1, \ldots, m\},$$

the set of all m-long products of elements in S.

Lemma 4.1 *Let m be any positive integer. Suppose that $U \subseteq G$ contains e, and that $\chi \colon G \to S^1$ is a group homomorphism (not necessarily continuous) such that $\chi(U^{(m)}) \subset N(1)$. Then $\chi(U) \subset N(1/m)$.*

Proof. First we prove that if $x \in S^1$ is such that $x, x^2, x^3, \ldots, x^m$ lie in $N(1)$, then $x \in N(1/m)$. The proof is by induction on m. For $m = 1$ the claim is evident. Suppose $m > 1$ and that $x, x^2, x^3, \ldots, x^m$ lie in $N(1)$. By inductive hypothesis the claim holds for $m - 1$, and hence $x \in N(1/(m-1))$. Moreover, we know that $x^m \in N(1)$, so that there exists

$y \in N(1/m)$ such that $x^m = y^m$. In other words, $x = y\zeta_m^q$ where $\zeta_m := \exp(2\pi i/m)$ (a primitive mth root of 1), and q is an integer in $0, 1, \ldots, m-1$. when $q \neq 0$, the sets $N(1/(m-1))$ and $N(1/m)\zeta_m^q$ do not intersect (take the 'worst' case $m = 2$, to convince yourself), thus the equality $x = y\zeta_m^q$ is possible only for $q = 0$, and the resulting equality $x = y$ shows that $x \in N(1/m)$, i.e. the claim.
The statement of the lemma follows easily. In fact, for every $g \in S$ and $1 \leq j \leq m$, the power g^j is a m-long product in $S^{(m)}$ (because $e \in S$, by assumption), so that $\chi(g)^j = \chi(g^j) \in N(1)$ for $j = 1, \ldots, m$ implies that $\chi(g) \in N(1/m)$, by the first part of the proof. ■

Proposition 4.1 *Let G be an abelian and locally compact group. Then:*

1. *A group homomorphism $\chi\colon G \to S^1$ is continuous (and hence a character) if and only if $\chi^{-1}(N(1))$ is a neighborhood of the identity in G;*
2. *The family $\{W(K, N(1))\}_K$, where K ranges in the set of compact subsets of G, is a neighborhood base of the trivial character for the compact-open topology of $\hat{G}$;*
3. *If G is discrete then $\hat{G}$ is compact;*
4. *If G is compact then $\hat{G}$ is discrete;*
5. *If G is locally compact, then $\hat{G}$ is locally compact, as well.*

Proof.

1. The claim is evidently necessary for the continuity of χ. We prove that it is also sufficient. Suppose that there exists $U \subseteq G$ which is an open neighborhood of e in G and such that $\chi(U) \subseteq N(1)$. Fix any positive integer k. By Proposition 1.4[1.] (iterated k times) in G there is an open set V which is a neighborhood of e and such that $V^{(2^k)} \subseteq U$. Thus $\chi(V^{(2^k)}) \subseteq N(1)$. By Lemma 4.1 we deduce that $\chi(V) \subseteq N(2^{-k})$, and this proves that χ is continuous.
2. It is sufficient to prove that for every compact K_1 and every integer m, there is a compact K such $W(K, N(1)) \subseteq W(K_1, N(1/m))$. Without loss of generality we can suppose that $e \in K_1$. Let $K := K_1^{(m)}$ (a compact, by the continuity of the product). Let $\chi \in W(K, N(1))$ and pick any $g \in K_1$. Then $\chi(g), \chi^2(g), \ldots, \chi^m(g)$ are in $N(1)$, so that $\chi(g) \in N(1/m)$ (by Lemma 4.1), proving that $\chi \in W(K_1, N(1/m))$.
3. Let G be discrete. We have already noted that in this case the compact-open topology is the pointwise topology. With this topology the set of all maps $G \to S^1$ becomes compact (by Tychonoff's theorem), and therefore also $\hat{G} = \mathrm{Hom}(G, S^1)$ (the full set of all morphisms $G \to S^1$) is compact, since it is evidently a closed subset.
4. Let G be compact. Take any $\chi \in W(G, N(1))$. then $\chi(G)$ is a compact subgroup of S^1 in $N(1)$, and $\{1\}$ is the unique such subgroup, therefore $\chi = \chi_0$, the trivial character. Thus $W(G, N(1))$ coincides with $\{\chi_0\}$. In the compact-open topology the set $W(G, N(1))$ is open, therefore we have proved that the singleton $\{\chi_0\}$ is an open set in $\hat{G}$.
5. By Part 2, to show that $\hat{G}$ is locally compact it is sufficient to prove that if K is a compact neighborhood of the identity in G, then

   $$W := W(K, \overline{N(1/4)})$$

   is a compact neighborhood of the identity in $\hat{G}$.
   Let G_0 denote the same group G but with the discrete topology. Then $\hat{G_0}$=Hom(G_0,S^1) is compact (by Part 3) and its topology (the compact-open topology) is actually the pointwise convergence topology. Let

   $$W_0 := \{\chi \in \hat{G_0}\colon \chi(K) \subseteq \overline{N(1/4)}\}.$$

It is a closed subset of $\hat{G}_0$, hence it is compact. Moreover, by Part 1 each homomorphism in W_0 is G-continuous (i.e. continuous as a map $G \to S^1$), hence $W_0 \subseteq W$. On the other hands, $W \subseteq W_0$, because every map is continuous with respect to the discrete topology. Hence

$$W = W_0.$$

Let τ_0 be the topology of W_0 induced by the one in $\hat{G}_0$ (i.e., the pointwise topology), and let τ be the topology of W induced by the one in $\hat{G}$ (i.e., the compact-open topology). Since $W = W_0$ (as sets), if we are able to prove that τ_0 is finer than τ (i.e. that each τ-open set is also τ_0-open), then the identity map

$$(W_0, \tau_0) \to (W, \tau) \colon \imath(w) = w$$

is continuous, and the compactness of W itself cames out, since W equals $\imath(W_0)$, i.e. the continuous image of the compact W_0.

Let K_1 be a compact in G and let m be a positive integer. For each $\chi \in W$, consider the subset

$$W(\chi) := \big(\chi \cdot W(K_1, N(1/m))\big) \cap W.$$

(Recall that $W(K_1, N(1/m))$ is a neighborhood of the trivial character, thus $\chi \cdot W(K_1, N(1/m))$ becomes a neighborhood of χ). Since the sets $W(\chi)$ are a base of τ-open neighborhood for χ, to conclude it is sufficient to prove that they are also τ_0-open.

Let V be an open neighborhood of the identity in G such that $V^{(2m)} \subseteq K$. Since K_1 is compact, there is a finite set F such that $K_1 \subseteq F \cdot V$. Consider

$$W_0(\chi) := \big(\chi \cdot W_0(F, N(1/(2m)))\big) \cap W,$$

where $W_0(F, N(1/(2m))$ denotes the set of homomorphisms on G_0 mapping F to $N(1/(2m))$. We claim that $W_0(\chi)$ is τ_0-open and is contained in $W(\chi)$: when proved, this will conclude the proof. The set $W_0(F, N(1/(2m))$ being clearly τ_0-open, only the inclusion needs verification.

Let $\mu \in W_0(\chi)$. Then $\mu = \chi\mu_0$ for some $\mu_0 \in \hat{G}_0$ with $\mu_0(F) \subseteq N(1/(2m))$. Note that both μ and χ are in W, by construction, thus

$$\begin{aligned} \mu_0(K) &= (\mu\chi^{-1})(K) = \mu(K) \cdot \chi^{-1}(K) = \mu(K) \cdot \overline{\chi(K)} \\ &\subseteq \overline{N(1/4)} \cdot \overline{N(1/4)} = \overline{N(1/2)} \subseteq N(1). \end{aligned}$$

By Part 1 this shows that μ_0 is continuous. Moreover, since $V^{(2m)} \subseteq K$ and since $\mu_0(V^{(2m)}) \subseteq \mu_0(K) \subseteq N(1)$, by Lemma 4.1 we get that $\mu_0(V) \subseteq N(1/(2m))$. Thus we further have

$$\mu_0(K_1) \subseteq \mu_0(F) \cdot \mu_0(V) \subseteq N(1/(2m)) \cdot N(1/(2m)) = N(1/m)$$

proving that μ_0 is in $W(K_1, N(1/m))$, so that μ is in $W(\chi)$.

■

4.2. Functions of positive type

Let G be a locally compact group. Pick a left invariant Haar measure μ in G, and let $L^p(G)$ with $p \in [1, +\infty]$ be the usual spaces of (class of equivalence of) measurable functions $G \to \mathbb{C}$, such that

$$\|f\|_p := \Big(\int_G |f(g)|^p \,\mathrm{d}\mu(g)\Big)^{1/p} < +\infty$$

for $p \in [1, +\infty)$, and

$$\|f\|_\infty := \inf\big\{\alpha \in \mathbb{R} \colon \mu(\{g \in G \colon |f(g)| \geq \alpha\}) = 0\big\} < \infty$$

(the essential sup is finite) for $p = +\infty$.
We demand to specialized books for standard results (completeness, local convexity, duality, measure product and so on); we simply note that essentially everything which is known for $\mathbb{R}$ holds true also in the present case since locally compact groups are union of σ-compact spaces and Haar measures are Radon measures.
Let $\phi \in L^\infty(G)$ be a function such that

$$\phi(g^{-1}) = \overline{\phi(g)} \qquad \forall g \in G \tag{4.1}$$

(well, actually for almost all $g \in G$). With such a function we can define a scalar product in $\mathcal{C}_c(G)$, setting

$$\langle f_1, f_2 \rangle_\phi := \int_{G\times G} \phi(s^{-1}t) f_1(s) \overline{f_2(t)} \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t),$$

where in $G \times G$ we have introduced the product measure. It is well defined, with

$$\begin{aligned} |\langle f_1, f_2 \rangle_\phi| &\le \|\phi\|_\infty \|f_1\|_{L^1} \|f_2\|_{L_1} \\ &\le \|\phi\|_\infty \|f_1\|_\infty \|f_2\|_\infty \mu(\mathrm{supp}(f_1)) \mu(\mathrm{supp}(f_2)). \end{aligned} \tag{4.2}$$

It is evidently sesquilinear in its arguments, and it is also a hermitian scalar product, i.e. it satisfies the identity

$$\langle f_1, f_2 \rangle_\phi = \overline{\langle f_2, f_1 \rangle_\phi}$$

as a consequence of the assumption (4.1) on ϕ.
We are interested into those functions ϕ for which the scalar product is positive, i.e. such that

$$\langle f, f \rangle_\phi \ge 0 \qquad \forall f \in \mathcal{C}_c(G).$$

These functions are called *of positive type*. Note that we are *not* requiring that $\langle \cdot, \cdot \rangle_\phi$ is *definite* positive (i.e. that it is zero only on the zero function).
Indeed, our interest for this set of functions comes from two remarks:

- All characters are of positive type. In fact, for a character χ the identity $\chi(g^{-1}) = \overline{\chi(g)}$ is evident, and the positivity holds since
$$\begin{aligned} \langle f, f \rangle_\chi &= \int_{G\times G} \chi(s^{-1}t) f(s) \overline{f(t)} \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) \\ &= \int_{G\times G} \overline{\chi(s)} f(s) \chi(t) \overline{f(t)} \, \mathrm{d}\mu(s) \, \mathrm{d}\mu(t) = \Big| \int_G \overline{\chi(s)} f(s) \, \mathrm{d}\mu(s) \Big|^2. \end{aligned}$$
- The set of functions of positive type is a convex set, i.e. if ϕ, ψ are of positive type, then $\lambda\phi + (1-\lambda)\psi$ is of positive type for every $\lambda \in [0,1]$, too.

The following steps will characterize the characters as the extremal point of a suitable convex subset of all functions of positive type: in this way (via the Krein–Milman theorem) we will see that the characters are a sufficiently large family to allow the Fourier transform to have a unique inverse (in a proper sense). However, the proof is not easy and we will need a lot of theory to reach our purpose.
The positivity assumption produces a Cauchy-Schwartz-type inequality[1]

$$|\langle f, h \rangle_\phi| \le \|f\|_\phi \cdot \|h\|_\phi,$$

[1] In fact, if ϕ is positive then $\langle \lambda f + h, \lambda f + h \rangle_\phi \ge 0$ for every $f, h \in \mathcal{C}_c(G)$ and every $\lambda \in \mathbb{C}$. Thus

$$0 \le |\lambda|^2 \langle f, f \rangle_\phi + 2\,\mathrm{Re}(\lambda \langle f, h \rangle_\phi) + \langle h, h \rangle_\phi.$$

If $\langle f, f \rangle_\phi \ne 0$ we can set $\lambda = -\frac{\overline{\langle f,h \rangle_\phi}}{\langle f,f \rangle_\phi}$ yielding

$$0 \le \frac{|\langle f, h \rangle_\phi|^2}{\langle f, f \rangle_\phi} - 2\frac{|\langle f, h \rangle_\phi|^2}{\langle f, f \rangle_\phi} + \langle h, h \rangle_\phi = -\frac{|\langle f, h \rangle_\phi|^2}{\langle f, f \rangle_\phi} + \langle h, h \rangle_\phi,$$

where

$$\|f\|_\phi := (\langle f, f\rangle_\phi)^{1/2} = \Big[\int_{G\times G} \phi(s^{-1}t)f(s)\overline{f(t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\Big]^{1/2}.$$

Note that $\|\cdot\|_\phi$ it *not* a norm, in spite of its notation, since it is possibile to have $\|f\|_\phi = 0$ also with a nonzero function f. However, an immediate consequence of the Cauchy–Schwarz inequality is that $\|f+g\|_\phi \le \|f\|_\phi + \|g\|_\phi$[2] so that the set of functions

$$D_\phi := \{f \in \mathcal{C}_c(G) \colon \|f\|_\phi = 0\}$$

is a closed subspace of $\mathcal{C}_c(G)$[3]. The usual arguments show that $\|\cdot\|_\phi$ becomes a true norm on the quotient space $\mathcal{C}_c(G)/D_\phi$, which we convert into a Hilbert space $\mathcal{H}_\phi$ by completing the norm.

The left action $L_g \colon \mathcal{H}_\phi \to \mathcal{H}_\phi$, defined by $(L_g f)(s) := f(g^{-1}s)$ actually defines a representation of G into $\mathcal{H}_\phi$:

$$\begin{aligned} L \colon G &\to \mathrm{Aut}(\mathcal{H}_\phi)\\ g &\mapsto L_g. \end{aligned}$$

It is unitary: it is sufficient to prove it for elements in $\mathcal{C}_c(G)$, and a direct computation shows that

$$\begin{aligned} \langle L_g f, L_g f\rangle_\phi &= \int_{G\times G} \phi(s^{-1}t)(L_g f)(s)\overline{(L_g f)(t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\ &= \int_{G\times G} \phi(s^{-1}t)f(g^{-1}s)\overline{f(g^{-1}t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\ &= \int_{G\times G} \phi((g^{-1}s)^{-1}(g^{-1}t))f(g^{-1}s)\overline{f(g^{-1}t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\ &= \int_{G\times G} \phi(s^{-1}t)f(s)\overline{f(t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) = \langle f, f\rangle_\phi \end{aligned}$$

(recall that μ is *left* invariant, by assumption).

It is also a topological representation. In fact, if (s_α, f_α) is a sequence in $G\times\mathcal{H}_\phi$ converging to (s, f), then

$$\|L_s f - L_{s_\alpha} f_\alpha\|_\phi \le \|L_{s_\alpha} f - L_{s_\alpha} f_\alpha\|_\phi + \|L_s f - L_{s_\alpha} f\|_\phi = \|f - f_\alpha\|_\phi + \|L_{ss_\alpha^{-1}} f - f\|_\phi$$

(because L is unitary). The first term can be made arbitrarily small because $f_\alpha \to f$, by assumption; the second term goes to zero because $s_\alpha \to s$ and $L_{ss_\alpha^{-1}} f \to f$ in sup norm when $f \in \mathcal{C}_c(G)$ (by uniform continuity on compact domains), so that this is true also in $L^p(G)$ with $p \in [1, +\infty)$[4]. In particular it is true in $L^1(G)$ and hence also in $\|\cdot\|_\phi$ norm, because ϕ is essentially bounded, by (4.2).

---

which is the claim. Suppose $\langle f, f\rangle_\phi = 0$. Then if $\langle f, h\rangle_\phi \ne 0$ we could take $\lambda = -\eta\frac{\langle h,h\rangle_\phi}{2\langle f,h\rangle_\phi}$ with any $\eta \in \mathbb{R}$, getting a contradiction when $\eta > 1$. Thus, in case $\langle f, f\rangle_\phi = 0$ also $\langle f, h\rangle_\phi = 0$, and the inequality holds in this case, too.

[2]Because

$$\begin{aligned} \|f+h\|_\phi^2 =& \langle f+h, f+h\rangle_\phi = \langle f, f\rangle_\phi + 2\,\mathrm{Re}(\langle f, h\rangle_\phi) + \langle h, h\rangle_\phi\\ =& \|f\|_\phi^2 + 2\,\mathrm{Re}(\langle f, h\rangle_\phi) + \|h\|_\phi^2 \le \|f\|_\phi^2 + 2\|f\|_\phi\|h\|_\phi + \|h\|_\phi^2\\ =& (\|f\|_\phi + \|h\|_\phi)^2. \end{aligned}$$

[3]Closed, because $\|\cdot\|_\phi$ is a continuous map.

[4]In fact, let K be the support of f. Then the support of $L_{ss_\alpha^{-1}} f$ is $ss_\alpha^{-1}K$. Let U be a compact neighborhood of e: it exists because G is locally compact. Hence ss_α^{-1} is eventually in U and the support of $L_{ss_\alpha^{-1}} f$ is eventually in UK, which is a compact set. Hence the support of $L_{ss_\alpha^{-1}} f - f$ is eventually in $UK \cup K$, and we have

$$\|L_{ss_\alpha^{-1}} f - f\|_{L^p(G)}^p = \int_G |L_{ss_\alpha^{-1}} f(t) - f(t)|^p\,\mathrm{d}\mu(t) \le \|L_{ss_\alpha^{-1}} f - f\|_\infty^p \mu(UK \cup K) \to 0.$$

Let f, g be a pair of complex Borel functions on G. Formally, their convolution is the new function

$$(f * g)(t) := \int_G f(s)g(s^{-1}t)\,\mathrm{d}\mu(s) = \int_G f(ts)g(s^{-1})\,\mathrm{d}\mu(s)$$

(for the second equality recall that μ is by assumption a left-invariant Haar measure), but this applies only when the integral is well defined: This happens for sure when $f \in \mathcal{C}_c(G)$ and $g \in L^\infty(G)$, and in this case the resulting function $f * g$ is continuous[5].

Remark. 4.1 Note that (at least for the moment) we are not assuming that G is abelian. As a consequence $f * g$ and $g * f$ are not necessarily the same function, because

$$(g * f)(t) = \int_G g(s)f(s^{-1}t)\,\mathrm{d}\mu(s) = \int_G f(st)g(s^{-1})\,\frac{d\mu(s)}{\Delta(s)}$$

(recall Formula (1.2)) which in general is not equal to $(f * g)(t)$, unless $f(st) = f(ts)\Delta(s)$. Note that this happens for every f only when G is abelian. □

Let ϕ be a function of positive type, and let $f, h \in \mathcal{C}_c(G)$. Then (via the usual Fubini–Tonelli theorems) we have the identity

$$\begin{aligned}\langle f, h\rangle_\phi &= \int_{G\times G} \phi(s^{-1}t)f(s)\overline{h(t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\ &= \int_G \Big[\int_G f(s)\phi(s^{-1}t)\,\mathrm{d}\mu(s)\Big]\overline{h(t)}\,\mathrm{d}\mu(t)\\ &= \int_G (f*\phi)(t)\overline{h(t)}\,\mathrm{d}\mu(t).\end{aligned}$$

In particular, suppose that $\|f\|_\phi = 0$. Then, by Cauchy–Schwarz inequality we know that $\langle f, h\rangle_\phi = 0$ for every $h \in \mathcal{C}_c(G)$, and the previous formula gives

$$\int_G (f*\phi)(t)\overline{h(t)}\,\mathrm{d}\mu(t) = 0 \qquad \text{for all } h \in \mathcal{C}_c(G).$$

We know that $f * \phi$ is continuous, therefore we deduce that $(f * \phi)(t) = 0$ for every $t \in G$.

The following proposition shows a very useful representation formula for functions of positive type.

Proposition 4.2 *Let ϕ be a function of positive type on G. Then there exists an element $x_\phi \in \mathcal{H}_\phi$ such that*

$$\phi(s) = \langle x_\phi, L_s x_\phi\rangle_\phi \qquad \text{for a.e. } s \in G.$$

Proof. Let $f \in \mathcal{C}_c(G)$. Note that $f * \phi$ is a well defined continuous function, and define

$$\Phi(f) := (f*\phi)(e) = \int_G f(s)\phi(s^{-1})\,\mathrm{d}\mu(s),$$

This proves the convergence of $L_{ss_\alpha^{-1}}f$ to f in $L^p(G)$ norm, when $f \in \mathcal{C}_c(G)$. For a generic $g \in L^p(G)$ (with $p \in [0, +\infty)$) the claim from the density of $\mathcal{C}_c(G)$ as subset of $L^p(G)$, and the inequality

$$\begin{aligned}\|L_{ss_\alpha^{-1}}g - g\|_{L^p(G)} &\le \|L_{ss_\alpha^{-1}}g - L_{ss_\alpha^{-1}}f\|_{L^p(G)} + \|L_{ss_\alpha^{-1}}f - f\|_{L^p(G)} + \|f-g\|_{L^p(G)}\\ &= \|L_{ss_\alpha^{-1}}f - f\|_{L^p(G)} + 2\|f-g\|_{L^p(G)}.\end{aligned}$$

[5]In fact, let t_α be a family of elements in G converging to t. Then

$$\begin{aligned}|(f*g)(t_\alpha) - (f*g)(t)| &\le \int_G |f(t_\alpha s) - f(ts)|\cdot|g(s^{-1})|\,\mathrm{d}\mu(s) \le \|g\|_\infty \int_G |f(t_\alpha s) - f(ts)|\,\mathrm{d}\mu(s)\\ &= \|g\|_\infty \cdot \|L_{t_\alpha^{-1}}f - L_{t^{-1}}f\|_{L^1(G)} = \|g\|_\infty \cdot \|L_{tt_\alpha^{-1}}f - f\|_{L^1(G)},\end{aligned}$$

and we can repeat the argument in Footnote 4.

which is a linear functional on $\mathcal{C}_c(G)$. Let $f \in D_\phi$, i.e. suppose that $\|f\|_\phi = 0$. Then we have noticed that $f * \phi$ is identically zero, in particular, $\Phi(f) = 0$. This proves that Φ is well defined on the quotient space $\mathcal{C}_c(G)/D_\phi$. Now we prove that Φ can be extended as a continuous linear functional on $\mathcal{H}_\phi$. In fact, let $\{\alpha\}$ be an index set for the collection of all open neighborhood V_α of e in G. Since G is Hausdorff, $\bigcap_\alpha V_\alpha = \{e\}$, and if we write $\alpha \leq \beta$ when $V_\beta \subseteq V_\alpha$, then the set of indexes becomes a direct set. By Urysohn's lemma for every α there is a function $g_\alpha \colon G \to [0, +\infty)$ which is continuous and compactly supported on V_α, and such that

$$\int_G g_\alpha(t)\,\mathrm{d}\mu(t) = 1.$$

This family defines a corresponding family of functionals on $\mathcal{C}_c(G)$:

$$f \mapsto \int_G f(t) g_\alpha(t)\,\mathrm{d}\mu(t).$$

For every fixed f the limit in α of all these values is $f(e)$, thus the family of functionals converges weakly to the Dirac distribution in e. This is true also for functions f which are in $\mathcal{C}(G)$ (i.e. which not necessarily have a compact support). In fact, let K be a compact neighborhood of e and let $V_\beta := \mathring{K}$ (the open part of K is an open neighborhood of e). By Urysohn lemma, let $h \colon G \to \mathbb{R}$ be a continuous function supported in K and which is 1 in V_β. Then

$$\int_G f(t) g_\alpha(t)\,\mathrm{d}\mu(t) = \int_G f(t) h(t) g_\alpha(t)\,\mathrm{d}\mu(t)$$

when $\alpha \geq \beta$ (because g_α is supported in V_α and $V_\alpha \subseteq V_\beta$ for $\alpha \geq \beta$), thus

$$\lim_\alpha \int_G f(t) g_\alpha(t)\,\mathrm{d}\mu(t) = \lim_\alpha \int_G f(t) h(t) g_\alpha(t)\,\mathrm{d}\mu(t) = f(e)h(e) = f(e).$$

Therefore, we have

$$\begin{aligned}\Phi(f) = (f * \phi)(e) &= \lim_\alpha \int_G (f * \phi)(t) g_\alpha(t)\,\mathrm{d}\mu(t) \\ &= \lim_\alpha \int_{G\times G} \phi(s^{-1}t) f(s) g_\alpha(t)\,\mathrm{d}\mu(s) d\mu(t) = \lim_\alpha \langle f, g_\alpha \rangle_\phi.\end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\langle f, g_\alpha\rangle_\phi| &\leq \|f\|_\phi \|g_\alpha\|_\phi && \text{(Cauchy–Schwarz)} \\ &\leq \|f\|_\phi \|\phi\|_\infty \|g_\alpha\|_{L^1} && \text{(comparison of norms, Eq. (4.2))} \\ &= \|\phi\|_\infty \|f\|_\phi && (g_\alpha \text{ is nonnegative and } \|g_\alpha\|_{L_1} = \textstyle\int_G g_\alpha\,\mathrm{d}\mu = 1). \end{aligned}$$

Thus the previous equality shows that

$$|\Phi(f)| \leq \|\phi\|_\infty \|f\|_\phi,$$

i.e. Φ is a continuous functional on $\mathcal{C}_c(G)/D_\phi$. This implies that it can be extended in a unique way to a continuous functional on $\mathcal{H}_\phi$, that we denote Φ, as well. Let $x_\phi \in \mathcal{H}_\phi$ be the element such that $\Phi(f) = \langle f, x_\phi \rangle_\phi$ (by Riesz isomorphism theorem).
Let $\xi \in \mathcal{C}_c(G)$, then

$$\langle \xi, x_\phi \rangle_\phi = \Phi(\xi) = (\xi * \phi)(e) = \int_G \phi(t^{-1}) \xi(t)\,\mathrm{d}\mu(t) = \int_G \overline{\phi(t)} \xi(t)\,\mathrm{d}\mu(t), \tag{4.3}$$

where in the last step we have used (4.1). Substituting ξ with $L_{s^{-1}}\xi$ in this formula, and recalling that L_s is unitary in $\mathcal{H}_\phi$, we get

$$\langle \xi, L_s x_\phi \rangle_\phi = \langle L_{s^{-1}} \xi, x_\phi \rangle_\phi = \int_G \overline{\phi(t)} \xi(st)\,\mathrm{d}\mu(t) = \int_G \overline{\phi(s^{-1}t)} \xi(t)\,\mathrm{d}\mu(t). \tag{4.4}$$

Let h be another function in $\mathcal{C}_c(G)$, then

$$
\begin{aligned}
\langle \xi, h\rangle_\phi &= \int_{G\times G} \phi(s^{-1}t)\xi(s)\overline{h(t)}\,\mathrm{d}\mu(s)d\mu(t)\\
&= \int_G \Big[\int_G \phi(s^{-1}t)\xi(s)\,\mathrm{d}\mu(s)\Big]\overline{h(t)}\,\mathrm{d}\mu(t)\\
&= \int_G \Big[\int_G \overline{\phi(t^{-1}s)}\xi(s)\,\mathrm{d}\mu(s)\Big]\overline{h(t)}\,\mathrm{d}\mu(t)\\
&= \int_G \langle \xi, L_t x_\phi\rangle_\phi \overline{h(t)}\,\mathrm{d}\mu(t), \qquad (4.5)
\end{aligned}
$$

where in the last step we have used (4.4). By density, this formula holds also for every $\xi \in \mathcal{H}_\phi$. Setting $\xi = x_\phi$ (and recalling (4.3)) yields

$$
\begin{aligned}
\int_G \langle x_\phi, L_t x_\phi\rangle_\phi \overline{h(t)}\,\mathrm{d}\mu(t) &= \langle x_\phi, h\rangle_\phi = \overline{\langle h, x_\phi\rangle_\phi} = \overline{\Phi(h)}\\
&= \overline{\int_G \overline{\phi(t)} h(t)\,\mathrm{d}\mu(t)} = \int_G \phi(t)\overline{h(t)}\,\mathrm{d}\mu(t).
\end{aligned}
$$

Since $h \in \mathcal{C}_c(G)$ is arbitrary, we conclude that $\phi(t) = \langle x_\phi, L_t x_\phi\rangle_\phi$ for almost all $t \in G$. ■

Remark. 4.2 Note that the equality (4.5) shows that the $\mathbb{C}$-span of $L_t x_\phi$ when t ranges in G is dense in $\mathcal{H}_\phi$. In fact, the formula shows that if ξ is $\langle\cdot,\cdot\rangle_\phi$-orthogonal to all $L_t x_\phi$, then $\langle \xi, h\rangle_\phi = 0$ for every $h \in \mathcal{C}_c(G)$. When the formula is extended to $\xi \in \mathcal{H}_\phi$, this shows that ξ is the zero element in $\mathcal{H}_\phi$. □

Exercise. 4.1 Let $(\rho, \mathcal{H})$ be any unitary representation of a locally compact group G. Take any $v \in \mathcal{H}$, and set

$$\phi\colon G \to \mathbb{C}, \quad \phi(g) := \langle v, \rho(g)v\rangle.$$

Prove that ϕ is of positive type.

Corollary 4.1 *Let $\phi \in L^\infty(G)$ be a function of positive type. Then ϕ is a continuous functions almost everywhere. Moreover, suppose that ϕ is continuous, then*

- $\phi(e) = \|x_\phi\|_\phi^2 \geq 0$;
- $|\phi(s)| \leq \|x_\phi\|_\phi \|L_s x_\phi\|_\phi = \|x_\phi\|_\phi^2 = \phi(e)$.

Proof. By Proposition 4.2 we know that $\phi(s) = \langle x_\phi, L_s x_\phi\rangle_\phi$ for a.e. $s \in G$. The function $s \mapsto \langle x_\phi, L_s x_\phi\rangle_\phi$ is continuous, hence the first claim follows. The other facts are immediate consequence of this identity (plus the Cauchy–Schwarz inequality and the fact that L_s is a unitary representation in $\mathcal{H}_\phi$). ■

4.3. Elementary functions

Let

$$\mathcal{P}(G) := \{\phi \in \mathcal{C}(G) \cap L^\infty(G)\colon \phi \text{ of positive type}, \|\phi\|_\infty \leq 1\},$$

which is a convex set. Note that elements in $\mathcal{P}$ are continuous, so that for them the condition $\|\phi\|_\infty \leq 1$ is equivalent to the assumption $\phi(e) \leq 1$ (by Corollary 4.1). We also introduce

$$\mathcal{E}(G) := \{\text{extreme points of } \mathcal{P}(G)\},$$

where *extreme* is in the sense of convex sets, i.e. $\phi \in \mathcal{E}(G)$ cannot be written as $\lambda\phi_1 + (1 - \lambda)\phi_2$ with $\phi_1, \phi_2 \in \mathcal{P}(G)$, $\phi_1 \neq \phi_2$ and $\lambda \in (0, 1)$.
We notice that the null function belongs to $\mathcal{E}(G)$[6]. All other functions in $\mathcal{E}(G)$ are called

[6]Suppose the contrary, i.e. that $0 = \lambda\phi_1 + (1 - \lambda)\phi_2$ with $\phi_1, \phi_2 \in \mathcal{P}$ and $\phi_1 \neq \phi_2$. Then $0 = \lambda\phi_1(e) + (1 - \lambda)\phi_2(e)$, but $\phi_j(e) \geq 0$, hence $\phi_1(e) = \phi_2(e) = 0$ implying that both ϕ_j are the null function (By Corollary 4.1). In particular $\phi_1 = \phi_2$, which is a contradiction.

elementary functions, by definition. Note that $\|\phi\|_\infty = \phi(e) = 1$ for elementary functions; In fact, $\phi(e) \neq 0$ (otherwise ϕ is the null function), and assume that $\phi(e) < 1$. Then $\phi/\phi(e)$ is in $\mathcal{P}(G)$ and the equality $\phi = \phi(e) \cdot \frac{\phi}{\phi(e)} + (1 - \phi(e)) \cdot 0$ shows that ϕ is not an extreme point.

Let $\phi \in \mathcal{P}(G)$, with $\phi(e) = 1$. Then the following properties are equivalent:

i) ϕ is extremal, so that cannot be written as $\lambda\phi_1 + (1-\lambda)\phi_2$ with $\phi_1 \neq \phi_2 \in \mathcal{P}(G)$ and $\lambda \in (0, 1)$,

ii) if $\phi = \phi_1 + \phi_2$ with $\phi_1, \phi_2 \in \mathcal{P}(G)$, then there exists $\lambda \in [0, 1]$ such that $\phi_1 = \lambda\phi$.

In fact:

i)$\Rightarrow$ ii). Suppose $\phi = \phi_1 + \phi_2$ with $\phi_1, \phi_2 \in \mathcal{P}(G)$. Evaluating the equality at e we deduce that $1 = \phi(e) = \phi_1(e) + \phi_2(e)$, with $\phi_1(e), \phi_2(e) \in [0, 1]$ (because ϕ_j are in $\mathcal{P}$). If $\phi_1(e), \phi_2(e) \in (0, 1)$, then

$$\phi = \phi_1(e)\frac{\phi_1}{\phi_1(e)} + \phi_2(e)\frac{\phi_2}{\phi_2(e)},$$

with $\frac{\phi_j}{\phi_j(e)} \in \mathcal{P}$. Assumption i) implies that $\frac{\phi_1}{\phi_1(e)} = \frac{\phi_2}{\phi_2(e)} = \phi$. Therefore $\phi_j = \phi_j(e)\phi$ for $j = 1, 2$, and ii) is proved. The same conclusion holds also in case $\phi_1(e) \in \{0, 1\}$ or $\phi_2(e) \in \{0, 1\}$, trivially.

ii)$\Rightarrow$ i). Suppose $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$ with $\phi_1, \phi_2 \in \mathcal{P}(G)$ and $\lambda \in (0, 1)$. Then $1 = \phi(e) = \lambda\phi_1(e) + (1-\lambda)\phi_2(e) \leq 1$, with equality if and only if $\phi_1(e) = \phi_2(e) = 1$. Suppose that this is the case. Let $\psi_1 := \lambda\phi_1$, which is in $\mathcal{P}(G)$ since $\psi_1(e) = \lambda\phi_1(e) = \lambda \leq 1$. The same for $\psi_2 := (1-\lambda)\phi_2$. Assumption ii) implies that there exists $\alpha \in [0, 1]$ such that $\lambda\phi_1 = \psi_1 = \alpha\phi$. Evaluating at e this equality gives $\lambda = \alpha$, so that $\phi_1 = \phi$. The same happens to $\phi_2 = \phi$, so that in particular $\phi_1 = \phi_2$.

Theorem 4.1 *Let ϕ be a continuous function of positive type on G, with $\phi(e) = 1$. Then $\phi \in \mathcal{E}(G)$ if and only if the unitary representation $s \mapsto L_s$ in $\mathcal{H}_\phi$ is topologically irreducible.*

Proof. $\Rightarrow$). The claim is evident if ϕ is the null function (because in this case $\mathcal{H}_\phi$ is the trivial space $\{0\}$). Assume that ϕ is an elementary function. Let W be a closed subspace of $\mathcal{H}_\phi$ which is L_s invariant for every $s \in G$, let $W^\perp$ be its orthogonal complement and let P_W be the orthogonal projector to W. Since the representation is unitary, also $W^\perp$ is invariant, and P_W commutes with L_s for every s. Thus, to prove the claim it is sufficient to prove that the unique orthogonal projectors commuting with L_s for every $s \in G$ are the zero or the identity map. Let A be any orthogonal projector. Then $A^2 = A$ and $A = A^*$, and also $1_{\mathcal{H}_\phi} - A$ is an orthogonal projector. Suppose that A commutes with L_s for every s. Thus for every $s \in G$ we have

$$\begin{aligned}
\phi(s) &= \langle x_\phi, L_s x_\phi \rangle_\phi \\
&= \langle Ax_\phi, L_s x_\phi \rangle_\phi + \langle (1_{\mathcal{H}_\phi} - A)x_\phi, L_s x_\phi \rangle_\phi \\
&= \langle Ax_\phi, AL_s x_\phi \rangle_\phi + \langle (1_{\mathcal{H}_\phi} - A)x_\phi, (1_{\mathcal{H}_\phi} - A)L_s x_\phi \rangle_\phi \qquad \text{since } A^*A = A^2 = A \\
&= \langle Ax_\phi, L_s Ax_\phi \rangle_\phi + \langle (1_{\mathcal{H}_\phi} - A)x_\phi, L_s(1_{\mathcal{H}_\phi} - A)x_\phi \rangle_\phi.
\end{aligned}$$

This identity gives ϕ as sum of tho functions of positive type (see Exercise 4.1). Since ϕ is extreme, there is λ such that

$$\langle Ax_\phi, L_s x_\phi \rangle_\phi = \langle Ax_\phi, L_s Ax_\phi \rangle_\phi = \lambda \langle x_\phi, L_s x_\phi \rangle_\phi \qquad \forall s \in G,$$

i.e.

$$\langle (A - \lambda 1_{\mathcal{H}_\phi})x_\phi, L_s x_\phi \rangle_\phi = 0 \qquad \forall s \in G.$$

By Remark 4.2 we know that $\{L_s x_\phi \colon s \in G\}$ spans a dense subset of $\mathcal{H}_\phi$, hence we deduce that $(A - \lambda 1_{\mathcal{H}_\phi})x_\phi = 0$, i.e. $Ax_\phi = \lambda x_\phi$. But A is a projector, hence $\lambda = 0$ or 1. Changing A with $1_{\mathcal{H}_\phi} - A$, if necessary, we can assume that $\lambda = 1$, i.e. that $Ax_\phi = x_\phi$. Since A

commutes with L_s, we deduce that $AL_s x_\phi = L_s A x_\phi = L_s x_\phi$, and hence $A = 1_{\mathcal{H}_\phi}$ (again because $L_s x_\phi$ spans a dense subset of $\mathcal{H}_\phi$).
$\Leftarrow$). Suppose that the representation $(L, \mathcal{H}_\phi)$ is irreducible, and that $\phi = \phi_1 + \phi_2$ with ϕ_1, ϕ_2 both in $\mathcal{P}(G)$. Then for every $f \in \mathcal{C}_c(G)$

$$\langle f, f\rangle_{\phi_1} + \langle f, f\rangle_{\phi_2} = \langle f, f\rangle_\phi$$

so that

$$\langle f, f\rangle_{\phi_1} \leq \langle f, f\rangle_\phi. \tag{4.6}$$

This proves that if f is degenerate with respect to ϕ, then it is degenerate also with respect to ϕ_1, i.e. that $D_\phi \subseteq D_{\phi_1}$. This inclusion produces a surjection $j\colon \mathcal{H}_\phi \twoheadrightarrow \mathcal{H}_{\phi_1}$. The map $\langle j(\cdot), j(\cdot)\rangle_{\phi_1}$ defines a positive hermitian form on $\mathcal{H}_\phi$, and hence there exists a continuous and positive endomorphism A on $\mathcal{H}_\phi$ such that

$$\langle j(\xi), j(\psi)\rangle_{\phi_1} = \langle A\xi, \psi\rangle_\phi \qquad \forall \xi, \psi \in \mathcal{H}_\phi. \tag{4.7}$$

This formula also shows that A is self-adjoint, because it implies that

$$\langle A\xi, \psi\rangle_\phi = \langle j(\xi), j(\psi)\rangle_{\phi_1} = \overline{\langle j(\psi), j(\xi)\rangle_{\phi_1}} = \overline{\langle A\psi, \xi\rangle_\phi} = \langle \xi, A\psi\rangle_\phi.$$

We verify that A commutes with L_s for every s. In fact, by (4.7), the fact that L is unitary, and the fact that j commutes with L_s, we get, for every $s \in G$ and every $\xi, \psi \in \mathcal{H}_\phi$:

$$\begin{aligned}\langle AL_s\xi, \psi\rangle_\phi &= \langle j(L_s\xi), j(\psi)\rangle_{\phi_1} = \langle L_s j(\xi), j(\psi)\rangle_{\phi_1} = \langle j(\xi), L_{s^{-1}} j(\psi)\rangle_{\phi_1} \\ &= \langle j(\xi), j(L_{s^{-1}}\psi)\rangle_{\phi_1} = \langle A\xi, L_{s^{-1}}\psi\rangle_\phi = \langle L_s A\xi, \psi\rangle_\phi.\end{aligned}$$

This means that $\langle (AL_s - L_s A)\xi, \psi\rangle_\phi = 0$ for every $\xi, \psi \in \mathcal{H}_\phi$ so that $AL_s - L_s A = 0$. Schur's lemma (Theorem 3.1[2]) implies that $A = \lambda 1_{\mathcal{H}_\phi}$ for some $\lambda \in \mathbb{C}$, thus (4.7) becomes

$$\langle j(\xi), j(\psi)\rangle_{\phi_1} = \lambda\langle \xi, \psi\rangle_\phi, \qquad \forall \xi, \psi \in \mathcal{H}_\phi.$$

Setting $\xi = \psi$ we deduce that λ is real and in $[0, 1]$ (it is ≥ 0 because $\langle j(\cdot), j(\cdot)\rangle_{\phi_1}$ is positive, and ≤ 1 by (4.6)). Let $f, g \in \mathcal{C}_c(G)$, and let ξ, ψ their images in $\mathcal{H}_\phi$. Then the previous equality becomes

$$\int_{G\times G} \phi_1(s^{-1}t) f(s)\overline{g(t)}\, \mathrm{d}\mu(s)\, \mathrm{d}\mu(t) = \lambda \int_{G\times G} \phi(s^{-1}t) f(s)\overline{g(t)}\, \mathrm{d}\mu(s)\, \mathrm{d}\mu(t).$$

Since f, g are arbitrary this implies that $\phi_1(s) = \lambda\phi(s)$ for almost every $s \in G$, and hence for every s, because they are continuous. ■

Theorem 4.2 *Let G be an abelian and locally compact group. Then the set of the elementary functions of G coincides with the set of (continuous) characters.*

Proof. We have already verified that the characters are functions of positive type, and for them the condition $\chi(e) = 1$ is evident. Thus, by Theorem 4.1 the claim will follow immediately if we are able to prove that the following conditions are equivalent:

i. $\phi(e) = 1$ and the representation of G in $\mathcal{H}_\phi$ is irreducible;
ii. ϕ is a character on G.

(ii.) $\Longrightarrow$ (i.) Suppose that ϕ is a character. Then

$$\langle f, f\rangle_\phi = \Big|\int_G \overline{\phi(s)} f(s)\, \mathrm{d}\mu(s)\Big|^2$$

proving that D_ϕ coincides with the set of all functions $f \in \mathcal{C}_c(G)$ such that $\int_G \overline{\phi(s)} f(s)\, \mathrm{d}\mu(s) = 0$. Thus, the dimension of $\mathcal{C}_c(G)/D_\phi(G)$ is one[7], and $\mathcal{H}_\phi$ itself is one-dimensional.

[7]Pick any $h \in \mathcal{C}_c(G)$, with $\int_G h(s)\, \mathrm{d}\mu(s) = 1$. Set $\alpha := \int_G \overline{\phi(s)} f(s)\, \mathrm{d}\mu(s)$. Then

$$\int_G \overline{\phi(s)}\big[f(s) - \alpha\phi(s)h(s)\big]\, \mathrm{d}\mu(s) = \alpha - \alpha\int_G h(s)\, \mathrm{d}\mu(s) = 0,$$

Therefore the representation is irreducible.
(i.) $\Longrightarrow$ (ii.) Assume that the representation L of G in $\mathcal{H}_\phi$ is irreducible. Then $\mathcal{H}_\phi$ is one-dimensional (by Proposition 3.4). We know that $x_\phi \in \mathcal{H}_\phi$ is not zero (otherwise $\phi(s) = \langle x_\phi, L_s x_\phi\rangle_\phi$ would be identically 0, contradicting the assumption that $\phi(e) = 1$). Thus $\mathcal{H}_\phi = \mathbb{C}x_\phi$, and there exists $\lambda\colon G \to \mathbb{C}$ such that $L_s x_\phi = \lambda(s)x_\phi$. The function λ is a morphism of groups, since L is a morphism. It is also a continuous map, since the representation L is a topological representation, therefore $\lambda \in \hat{G}$. Finally, we have

$$\phi(s) = \langle x_\phi, L_s x_\phi\rangle_\phi = \langle x_\phi, \lambda(s)x_\phi\rangle_\phi = \overline{\lambda(s)}\langle x_\phi, x_\phi\rangle_\phi = \overline{\lambda(s)}\phi(e) = \overline{\lambda(s)},$$

and therefore ϕ is a character. ■

4.4. The Fourier inversion formula

Let G be a locally compact and *abelian* group. Fix μ be any Haar measure on G: it is both left and right invariant, because G is commutative. Let $f \in L^1(G)$; by definition the function

$$\hat{f}\colon \hat{G} \to \mathbb{C}, \qquad \hat{f}(\chi) := \int_G \overline{\chi(g)} f(g)\,\mathrm{d}\mu(g)$$

is its *Fourier transform*. Note that it is well defined, because $|\chi(g)| = 1$ for every g, and that $\hat{f}$ is bounded, since $\|\hat{f}\|_\infty \leq \|f\|_{L^1(G)}$.

Let $V(G)$ denote the complex span of the continuous functions of positive type (so that it is $\langle \mathcal{P}(G)\rangle_{\mathbb{C}}$), and let

$$V^1(G) := V(G) \cap L^1(G).$$

We know that also $\hat{G}$ is a locally compact abelian group. Thus, also $\hat{G}$ support a Haar measure. The following theorem states the main result of this section: it is possible to fix the Haar measure in $\hat{G}$ in such a way that it is well related with the Haar measure in G, on a suitable set of functions.

Theorem 4.3 (Fourier inversion formula) *It is possible to fix the Haar measure* $\mathrm{d}\hat{\mu}$ *in* $\hat{G}$ *in such a way that if* $f \in V^1(G)$ *then* $\hat{f} \in V^1(\hat{G})$*, and*

$$f(y) = \int_{\hat{G}} \hat{f}(\chi)\chi(y)\,\mathrm{d}\hat{\mu}(\chi) \qquad \forall y \in G.$$

Moreover, the Fourier transform $f \mapsto \hat{f}$ *is an isometric bijection of* $V^1(G)$ *into* $V^1(\hat{G})$*.*

The proof of this result will need some work, actually, but this claim in itself is not completely satisfying since the formulation of the duality in $V^1(G)$ space is a bit artificial. A more conventional choice would be the space $L^2(G) \cap L^1(G)$, and then extended by density to a map in $L^2(G)$. We will need some other tools to move from the theorem on $V^1(G)$ to a theorem claiming the same conclusions on these more usual spaces.

Firstly, we need the following statement analyzing the structure of $L^1(G)$ with respect to the convolution.

Proposition 4.3 *Let* $*$ *denote the convolution product. Then* $L^1(G)$ *is a commutative and associative algebra with respect to the pointwise sum and the* $*$ *product. Moreover,*

$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \cdot \|g\|_{L^1} \qquad \forall f, g \in L^1(G),$$

so that it is a Banach algebra.

---

and this proves that $f - \alpha\phi h$ is in D_ϕ, or, which is the same, that f is represented by $\alpha\phi h$ in the quotient $\mathcal{C}_c(G)/D_\phi$. This proves that the quotient $\mathcal{C}_c(G)/D_\phi$ is generated by the function ϕh, and hence it is one-dimensional.

Proof. We have already noted that the convolution is commutative for abelian groups. The proof that $f * g$ is a measurable function is the unique point needing some care, but can be performed with the same strategy for the usual case where $G = \mathbb{R}^n$. The upper bound for the norm and the associativity of the product are elementary consequence of the formal properties of the product (plus Fubini–Tonelli theorems). ■

Remark. 4.3 In general $L^1(G)$ is not a unital algebra, i.e. it does not contain a function g such that $f * g = f$ for every $f \in L^1(G)$. In fact, it can be proved that such a function g exists if and only if G is discrete (and in that case the Haar measure is the function counting points and the function which is 1 in e and 0 otherwise becomes the unit in $L^1(G)$). This fact represents a difficulty, because we need to apply to $L^1(G)$ several tools we have introduced in Chapter 2 (spectrum, Gelfand transform, and so on) for unital algebras. However, there is a canonic way to inject any algebra into a unital algebra via an isometry of Banach algebras (see Exercise 2.1): This will allow us to recover some properties for $L^1(G)$ from the analogous property of its canonical unital extension. □

To proceed we need a different characterization of the compact-open topology of $\hat{G}$. As a first step we prove that the Fourier transform of $L^1(G)$ functions are continuous maps.

Lemma 4.2 *Let $f \in L^1(G)$. Then $\hat{f}$ is a continuous map on $\hat{G}$.*

Proof. Haar measure is a Radon measure, by design. As a consequence also $|f(s)|\,\mathrm{d}\mu(s)$ is a Radon measure. In particular it is inner regular and finite (because $f \in L^1(G)$, by assumption), so that for every $\epsilon > 0$ there exists a compact $K = K(\epsilon)$ such that

$$\int_{K^c} |f(s)|\,\mathrm{d}\mu(s) \leq \epsilon.$$

Let $\{\chi_\alpha\}_\alpha$ be any direct system of characters in $\hat{G}$ converging to a given character χ. Fix $\epsilon > 0$. The compact-open topology coincides with the uniform on compact convergence topology. Thus, let K be the compact with $\int_{K^c} |f(s)|\,\mathrm{d}\mu(s) \leq \epsilon$ and let α be large enough so that $\|\chi_\beta - \chi\|_{\infty,K} \leq \epsilon$ for every $\beta \geq \alpha$. Then

$$\begin{aligned} |\hat{f}(\chi_\beta) - \hat{f}(\chi)| &\leq \int_G |f(s)| \cdot |\chi_\beta(s) - \chi(s)|\,\mathrm{d}\mu(s) \\ &= \int_K |f(s)| \cdot |\chi_\beta(s) - \chi(s)|\,\mathrm{d}\mu(s) + \int_{K^c} |f(s)| \cdot |\chi_\beta(s) - \chi(s)|\,\mathrm{d}\mu(s). \end{aligned}$$

By assumption, $|\chi_\beta(s) - \chi(s)|$ is bounded by ϵ in K if $\beta \geq \alpha$, and by 2 in general, thus

$$\leq \epsilon \int_K |f(s)|\,\mathrm{d}\mu(s) + 2\int_{K^c} |f(s)|\,\mathrm{d}\mu(s).$$

Moreover, the integral of $|f|$ is bounded by $\|f\|_{L^1}$ in K and by ϵ in K^c, hence

$$\leq \epsilon\big(\|f\|_{L^1} + 2\big),$$

which proves the claim. ■

Let

$$\hat{A} := \{\hat{f} \colon f \in L^1(G)\},$$

the set of Fourier transforms of $L^1(G)$ functions. In $\hat{G}$ we can introduce also a (formally) different topology, namely the *transform topology*, t-topology in brief, which is the weak-* topology induced by the set $\hat{A}$; in this topology a family of characters χ converges to χ_0 if and only if $\hat{f}(\chi) \to \hat{f}(\chi_0)$ for every $\hat{f} \in \hat{A}$. By its definition, this topology is the coarser topology such that all functions in $\hat{A}$ become continuous. Lemma 4.2 shows that this topology is weaker than the compact-open topology, i.e. that every set in $\hat{G}$ which is t-open is also open in the compact-open topology[8]. We want to prove that actually these topologies are the same. To this purpose we need a second lemma.

Lemma 4.3 *Let $G \times \hat{G}$ have the product topology of the assumed topology in G and the t-topology in $\hat{G}$. Then the map*

$$\begin{aligned} G \times \hat{G} &\longrightarrow \mathbb{C}, \\ (y, \chi) &\mapsto \chi(y) \end{aligned}$$

is continuous.

Proof. Let f be any map in $L^1(G)$. Firstly we prove the continuity of the map

$$\begin{aligned} G \times \hat{G} &\longrightarrow \mathbb{C}, \\ (y, \chi) &\mapsto \widehat{L_y f}(\chi). \end{aligned}$$

Choose any $(y_0, \chi_0) \in G \times \hat{G}$. For very $\epsilon > 0$ there is an open neighborhood U of y_0 such that $\|L_y f - L_{y_0} f\|_{L^1} \leq \epsilon$[9]. Hence

$$|\widehat{L_y f}(\chi) - \widehat{L_{y_0} f}(\chi)| \leq \|L_y f - L_{y_0} f\|_{L^1} \leq \epsilon \qquad \forall \chi \in \hat{G},$$

when y belongs to this set. Moreover, by definition of t-topology, there is a t-open neighborhood V of χ_0 such that

$$|\widehat{L_{y_0} f}(\chi) - \widehat{L_{y_0} f}(\chi_0)| \leq \epsilon \qquad \forall \chi \in V.$$

Therefore, if $(y, \chi) \in U \times V$ then

$$|\widehat{L_y f}(\chi) - \widehat{L_{y_0} f}(\chi_0)| \leq |\widehat{L_y f}(\chi) - \widehat{L_{y_0} f}(\chi)| + |\widehat{L_{y_0} f}(\chi) - \widehat{L_{y_0} f}(\chi_0)| \leq 2\epsilon,$$

which proves the claim.
Now, we note the following equality:

$$\chi(y)\widehat{L_y f}(\chi) = \int_G (L_y f)(s)\chi(y)\overline{\chi(s)}\, \mathrm{d}\mu(s) = \int_G f(y^{-1}s)\overline{\chi(y^{-1}s)}\, \mathrm{d}\mu(s) = \hat{f}(\chi).$$

Pick again any couple (y_0, χ_0) in $G \times \hat{G}$, and let f_0 be any function such that $\hat{f}_0(\chi_0) \neq 0$ (such a function exists, for example take $f = h\overline{\chi_0}$ where h is any function in $\mathcal{C}_c(G)$ with $\int_G h\, \mathrm{d}\mu \neq 0$). Then $\widehat{L_{y_0} f_0}(\chi_0) \neq 0$ (by the previous formula). By the first part of the proof, the value of $\widehat{L_y f_0}(\chi)$ is non zero in a suitable open neighborhood of (y_0, χ_0), so that the map

$$(y, \chi) \mapsto \frac{\hat{f}_0(\chi)}{\widehat{L_y f_0}(\chi)} = \chi(y)$$

is well defined there, and hence continuous in $U \times V$ (because quotient of continuous maps). Repeating this argument in every point we conclude that the map is continuous everywhere. ■

[8] This is probably my fault, but I have always some hesitation about the truth of this claim. I reproduce here its simple proof. Consider the following diagram:

$$\begin{array}{ccc} (\hat{G}, \text{c.-open top.}) & \overset{i}{\hookrightarrow} & (\hat{G}, \text{t-top.}) \\ & \underset{\hat{f}}{\searrow} & \downarrow \hat{f} \\ & & \mathbb{C} \end{array}$$

where i is the identity map. Let O be an open set in the t-topology. Then, directly from its definition, there is a function $f \in L^1(G)$ and an open set $V \subseteq \mathbb{C}$ such that $O = \hat{f}^{-1}(V)$. By Lemma 4.2, $\hat{f}\colon (\hat{G}, \text{com.-open top.}) \to \mathbb{C}$ is continuous, hence $O = \hat{f}^{-1}(V)$ is an open set in the comp.-open topology. This proves that each open set in t-topology is also an open set in comp.-open topology. In particular the inclusion i map is itself continuous.

[9] We have already used this property for functions in $\mathcal{C}_c(G)$, see Footnote 5 in Section 4.2. The claim extends to all $L^1(G)$ because $\mathcal{C}_c(G)$ is dense in this space.

Theorem 4.4 *Compact-open topology and t-topology in $\hat{G}$ are the same.*

Proof. It is sufficient to prove that every set $W(K, V)$ (with K compact in G and V open neighborhood of 1 in S^1) is a neighborhood of the trivial character also with respect to the t-topology. This proves that the compact-open topology is contained in the t-topology, and the claim follows because we already know the opposite inclusion.
By Proposition 4.1 it is sufficient to prove the claim when $V = N(1)$. Let K be a compact subset in G and let χ_0 be the trivial character; evidently $\chi_0 \in W(K, N(1))$. By Lemma 4.3, for every $y_0 \in K$ there is an open neighborhood U of y_0 in G and a t-open neighborhood V of χ_0 in $\hat{G}$ such that $\chi(y) \in N(1)$ for all $(y, \chi) \in U \times V$. The compact set K is covered by finitely many $U_1, \dots, U_r$. Let $V_1, \dots, V_r$ be the corresponding t-open neighborhoods in $\hat{G}$. Then $\mathcal{V} := \bigcap_{j=1}^{r} V_j$ is a t-open set which is contained in $W(K, N(1))$. This proves that $W(K, N(1))$ is a neighborhood of the trivial character also in the t-topology. ■

Proposition 4.4 *Let B denote the Banach algebra $L^1(G)$, and let $\hat{B}$ be the set of complex characters of B (as algebra). For every $\chi \in \hat{G}$ and $f \in B$, define*

$$\hat{\nu}_\chi(f) := \hat{f}(\chi) = \int_G f(s)\overline{\chi(s)}\, \mathrm{d}\mu(s).$$

Then ν_χ is in $\hat{B}$, and the map

$$\begin{aligned} \hat{\nu} \colon \hat{G} &\longrightarrow \hat{B}, \\ \chi &\mapsto \hat{\nu}_\chi \end{aligned}$$

is a bijection. It is a homeomorphism of topological spaces when in $\hat{G}$ we consider the compact-open topology and in $\hat{B}$ we take the Gelfand topology.

Recall that a character of an algebra is by definition a nonzero map. Note that the proposition states the validity of the formula

$$\widehat{f * g}(\chi) = \hat{\nu}_\chi(f * g) = \hat{\nu}_\chi(f)\hat{\nu}_\chi(g) = \hat{f}(\chi)\hat{g}(\chi)$$

for every χ, i.e. that $\widehat{f * g} = \hat{f}\hat{g}$ (so that the Fourier transform converts the convolution into the pointwise product).

Proof. Every $\hat{\nu}_\chi$ is evidently linear on B, and not identically 0 (because for every χ there is a function $f \in B$ with $\hat{f}(\chi) \neq 0$). A routine computation suffices to prove the multiplicative property of $\hat{\nu}_\chi$ (recall that G is abelian, thus the Haar measure is left-right invariant):

$$\begin{aligned} \hat{\nu}_\chi(f * g) &= \int_G (f * g)(s)\overline{\chi(s)}\, \mathrm{d}\mu(s) \\ &= \int_G \int_G f(t)g(st^{-1})\, \mathrm{d}\mu(t)\overline{\chi(s)}\, \mathrm{d}\mu(s) \\ &= \int_G \int_G f(t)g(st^{-1})\, \mathrm{d}\mu(t)\overline{\chi(t)\chi(st^{-1})}\, \mathrm{d}\mu(s) \\ &= \int_G f(t)\overline{\chi(t)}\Big[\int_G g(st^{-1})\overline{\chi(st^{-1})}\, \mathrm{d}\mu(t)\Big]\, \mathrm{d}\mu(s) \\ &= \int_G f(t)\overline{\chi(t)}\, \mathrm{d}\mu(s)\ \hat{g}(\chi) = \hat{f}(\chi)\hat{g}(\chi). \end{aligned}$$

Let χ, χ' be two characters. Suppose that $\hat{\nu}_\chi = \hat{\nu}_{\chi'}$, i.e. that $\hat{f}(\chi) = \hat{\nu}_\chi(f) = \hat{\nu}_{\chi'}(f) = \hat{f}(\chi')$ for every $f \in B$. Then $\chi(s) = \chi'(s)$ for a.e. $s \in G$. Since they are continuous, they are equal everywhere. This proves that $\hat{\nu}$ is injective.
Let ψ be an element in $\hat{B}$, i.e. a character of the algebra $L^1(G)$. Then it is automatically

continuous (by Proposition 2.3[iii]) [10]. In particular it is a continuous and linear morphism of the Banach linear space $L^1(G)$, and its norm is finite[11]. The classical duality result in $L^p(G)$ spaces shows that there exists $\varphi \in L^\infty(G)$ such that

$$\psi(f) = \int_G f(s)\varphi(s)\,\mathrm{d}\mu(s) \qquad \forall f \in L^1(G),$$

with $\|\varphi\|_\infty = \|\psi\|$. Moreover, ψ is multiplicative, thus (Fubini–Tonelli result applied several times, here)

$$\begin{aligned}\psi(f)\int_G g(t)\varphi(t)\,\mathrm{d}\mu(t) &= \psi(f)\psi(g) = \psi(f*g) = \psi(g*f)\\ &= \int_G (g*f)(s)\varphi(s)\,\mathrm{d}\mu(s) = \int_G\int_G g(t)f(t^{-1}s)\,\mathrm{d}\mu(t)\varphi(s)\,\mathrm{d}\mu(s)\\ &= \int_G g(t)\Big[\int_G (L_t f)(s)\varphi(s)\,\mathrm{d}\mu(s)\Big]\,\mathrm{d}\mu(t) = \int_G \psi(L_t f)g(t)\,\mathrm{d}\mu(t).\end{aligned}$$

In this identity g is arbitrary, therefore

$$\psi(f)\varphi(t) = \psi(L_t f) \qquad \text{for a.e. } t \in G.$$

The function appearing to the right side of this formula is continuous in t and we can select f so that $\psi(f) \neq 0$, hence changing φ in a null set at most, we can assume that φ itself is continuous and that the equality holds everywhere. Then, applying the identity three times we conclude that

$$\psi(f)\varphi(st) = \psi(L_{st}f) = \psi(L_s L_t f) = \varphi(s)\psi(L_t f) = \varphi(s)\varphi(t)\psi(f)$$

for every $s, t \in G$, proving that φ is multiplicative. The map $s \mapsto |\varphi(s)|$ is a multiplicative map $G \to [0, +\infty)$ whose range is in $[0, \|\psi\|_1]$, because $\|\varphi\|_\infty = \|\psi\|_1$. The unique multiplicative subgroup in $\mathbb{R}$ which is also bounded is $\{1\}$, hence $|\varphi(s)|$ is identically 1. This completes the proof that φ is a character, i.e. that $\hat{\nu}$ is a surjection.
Note that the Gelfand topology in $\hat{B}$ is only another name for the weak-* topology, which in this setting is the topology producing the pointwise convergence. Thus, a sequence of characters in $\hat{B}$ converges if and only if and only if their values in each element $f \in B$ converges. We have just verified that the characters in $\hat{B}$ are all of the type $\hat{\nu}_\chi$ for some χ, so that they converge if and only if $\hat{\nu}_\chi(f)$ converges for every f; i.e. if and only if $\hat{f}(\chi)$ converges for every f; this is exactly the notion of t-topology we have introduced before on $\hat{G}$. This proves that $\hat{B}$ (with Gelfand topology) and $\hat{G}$, with t-topology, are homeomorphic. According to Theorem 4.4 t-topology and compact-open topology in $\hat{G}$ are the same topology, so that also the last claim follows. ■

Proposition 4.5 *The set $\mathcal{P}(G)$ is a compact subset of $L^\infty(G)$ with respect to the weak-* topology.*

Proof. According to the weak-* topology, a sequence of elements $\{\phi_\alpha\}_\alpha \in L^\infty(G)$ converges if and only if $\phi_\alpha(f) := \int_G f(s)\phi_\alpha(s)\,\mathrm{d}\mu(s)$ converges, for every $f \in L^1(G)$ (here we are using the elements in $L^\infty(G)$ to produce functionals on $L^1(G)$). The set $\mathcal{P}(G)$ is bounded by 1, as subset of $L^\infty(G)$. It is also weak-* closed. In fact, if the sequence $\{\phi_\alpha\}_\alpha \in \mathcal{P}(G)$ converges (weak-*) to some $\phi \in L^\infty(G)$, then by definition

$$\lim_\alpha \int_G \phi_\alpha(s)h(s)\,\mathrm{d}\mu(s) = \int_G \phi(s)h(s)\,\mathrm{d}\mu(s) \qquad \forall h \in L^1(G). \tag{4.8}$$

[10] Actually, Proposition 2.3 assumes that the Banach algebra is commutative and unital. The Banach algebra B is for sure abelian, but not always unital. In order to apply that proposition to the present case we need to pass through the canonical banach algebra containing B, as we have told in the introduction of this section.

[11] It is bounded by 1, by Proposition 2.3 again, but we do not need this point here.

First we prove that ϕ satisfies (4.1). In fact, let $h \in L^1(G)$, and set $\tilde{h} \in L^1(G)$, with $\tilde{h}(s) := \overline{h(s^{-1})}$ (for a.e. $s \in G$). Then each ϕ_α satisfies (4.1), and using (4.8) twice, with $\tilde{h}$ and with h, we get

$$\begin{aligned}\int_G \overline{\phi(s^{-1})} h(s)\,\mathrm{d}\mu(s) &= \overline{\int_G \phi(s^{-1}) \overline{h(s)}\,\mathrm{d}\mu(s)} = \overline{\int_G \phi(s)\tilde{h}(s)\,\mathrm{d}\mu(s)} \\ &= \overline{\lim_\alpha \int_G \phi_\alpha(s)\tilde{h}(s)\,\mathrm{d}\mu(s)} = \lim_\alpha \int_G \overline{\phi_\alpha(s)\tilde{h}(s)}\,\mathrm{d}\mu(s) \\ &= \lim_\alpha \int_G \overline{\phi_\alpha(s^{-1})} h(s)\,\mathrm{d}\mu(s) = \lim_\alpha \int_G \phi_\alpha(s) h(s)\,\mathrm{d}\mu(s) \\ &= \int_G \phi(s) h(s)\,\mathrm{d}\mu(s).\end{aligned}$$

Since this equality holds for every $h \in L^1(G)$, we conclude that $\overline{\phi(s^{-1})} = \phi(s)$ holds for a.e. s, i.e. that ϕ itself satisfies (4.1).
Moreover, Let $f \in \mathcal{C}_c(G)$ and set $\tilde{f}$ as before. Then $\tilde{f} * \overline{f} \in L^1(G)$, and for every $\psi \in L^\infty(G)$ we get

$$\begin{aligned}\int_G \psi(s)(\tilde{f} * \overline{f})(s)\,\mathrm{d}\mu(s) &= \int_G \psi(s)\Big[\int_G \tilde{f}(u)\overline{f(su^{-1})}\,\mathrm{d}\mu(u)\Big]\,\mathrm{d}\mu(s) \\ &= \int_{G\times G} \psi(s) f(u^{-1})\overline{f(su^{-1})}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(u) \\ &= \int_{G\times G} \psi(uv) f(u^{-1})\overline{f(v)}\,\mathrm{d}\mu(v)\,\mathrm{d}\mu(u) \\ &= \int_{G\times G} \psi(u^{-1}v) f(u)\overline{f(v)}\,\mathrm{d}\mu(v)\,\mathrm{d}\mu(u) = \langle f, f\rangle_\psi\end{aligned}$$

(in the last two steps we have first set $su^{-1} =: v$ and then changed $u \mapsto u^{-1}$). Applying this computation to ϕ and to each ϕ_α, by (4.8) with $h := \tilde{f} * \overline{f}$ we conclude that

$$\langle f, f\rangle_\phi = \lim_\alpha \langle f, f\rangle_{\phi_\alpha} \qquad \forall f \in \mathcal{C}_c(G).$$

This proves that ϕ is of positive type, because each $\langle f, f\rangle_{\phi_\alpha} \geq 0$, and we already know that ϕ has the property (4.1). The function ϕ is almost everywhere equal to a continuous function, by Proposition 4.2, and changing its definition in a null set we can consider it as a continuous function. It is also bounded by 1, because every ϕ_α is bounded by 1. These facts prove that $\mathcal{P}(G)$ is a weak-* closed subset in the unitary ball of $L^\infty(G)$, hence it is weak-* compact by Alaoglu's theorem. ∎

Proposition 4.6 *The set $\hat{A} := \{\hat{f} : f \in L^1(G)\}$ is a separating, self-adjoint and dense subalgebra of $\mathcal{C}_0(\hat{G})$. (For a definition of $\mathcal{C}_0(\hat{G})$ see Section 2.3).*

Proof. Let $f \in L^1(G)$. Let $\tilde{f} : G \to \mathbb{C}$, $\tilde{f}(s) := \overline{f(s^{-1})}$. Then $\tilde{f} \in L^1(G)$, and

$$\begin{aligned}\hat{\tilde{f}}(\chi) &= \int_G \tilde{f}(s)\overline{\chi(s)}\,\mathrm{d}\mu(s) = \int_G \overline{f(s^{-1})\chi(s)}\,\mathrm{d}\mu(s) \\ &= \overline{\int_G f(s^{-1})\chi(s)\,\mathrm{d}\mu(s)} = \overline{\int_G f(s)\overline{\chi(s)}\,\mathrm{d}\mu(s)} = \overline{\hat{f}}(\chi),\end{aligned}$$

which proves that $\hat{A}$ is self-adjoint. It is also separating. In fact, suppose we have two characters $\chi \neq \psi \in \hat{G}$. Then $\chi - \psi$ is a non-identically zero in a suitable open set, and we can always find a function f in $\mathcal{C}_c(G)$ with $\int_G f(s)\overline{(\chi - \psi)(s)}\,\mathrm{d}\mu(s) \neq 0$, hence with $\hat{f}(\chi) \neq \hat{f}(\psi)$ (for example, take $f := (\chi - \psi)h$ where $h \in G \to [0, +\infty)$ is continuous with compact support and the open part of the support intersects the open set where $\chi \neq \psi$).

Proposition 4.4 identifies $\hat{G}$ with $\hat{B}$, using the map $\hat{\nu}$. So that each map $\hat{f}\colon \hat{G} \to \mathbb{C}$ becomes a map on $\hat{B}$, setting $\hat{f}(\hat{\nu}_\chi) := \hat{f}(\chi)$.

$$\begin{array}{ccc} \hat{G} & \xrightarrow{\ \hat{\nu}\ } & \hat{B} \\ {\scriptstyle \hat{f}}\big\downarrow & \swarrow & \\ \mathbb{C} & & \end{array}$$

The map generated in this way is continuous, because $\hat{f}$ is continuous on $\hat{G}$ (by Lemma 4.2) and $\hat{\nu}$ is a homeomorphism (again by Proposition 4.4). This shows that $\hat{A}$ is a separating and self-adjoint subalgebra of $\mathcal{C}(\hat{B})$. If G is discrete, then B (i.e. $L^1(G)$) is unital, and in this case Gelfand's theory shows that $\hat{B}$ is compact so that $\hat{A}$ is dense in $\mathcal{C}(\hat{B}) = \mathcal{C}_0(\hat{B})$ by Stone–Weierstrass' theorem.

Suppose that G is not discrete, so that B is not unital. Then $\hat{B}$ is no more closed: in fact it coincides with the set of linear functionals on B (i.e. $L^1(G)$) associated with the characters of G[12]. Thus the elements in $\hat{B}$ are still bounded by 1 (because the norm of the functional equals the sup norm of the character, i.e. 1), but it is no more a weak-* closed set. However, if $\phi \in L^\infty(G)$ is the weak-* limit of characters, then it is in any case bounded by 1 and multiplicative. It is also in $\mathcal{P}(G)$ (by Proposition 4.6, because the characters are in $\mathcal{P}(G)$). In particular it is continuous. Suppose that it is zero in some point s_0, then it zero everywhere, because $\phi(s) = \phi(ss_0^{-1}s_0) = \phi(ss_0^{-1})\phi(s_0) = 0$. This fact proves that either ϕ is a character, or it is identically zero. In other words, we have proved that $\hat{B}' := \hat{B} \cup \{0\}$ is the weak-* closure of $\hat{B}$. The set $\hat{B}'$ is also bounded by 1, hence it is weak-* compact (by Alaoglu theorem). Note that $\hat{B}'$ can be identified (as topological space) with the one-point compactification of $\hat{B}$.
If we set $\hat{f}(0) := 0$, then $\hat{f}$ becomes a continuous function in $\mathcal{C}(\hat{B}')$ which is zero in 0, i.e. an element of $\mathcal{C}_0(\hat{B})$[13]. Moreover, for every character $\chi \in \hat{G}$, there is an f such that $\hat{f}(\hat{\nu}_\chi) = \hat{f}(\chi) \neq 0$ (take $f = h\chi$ with any suitable $h \in \mathcal{C}_c(G)$, and $\int_G h(s)\,\mathrm{d}\mu(s) = 1$). Thus all assumptions of Corollary 2.2 are satisfied, proving that $\hat{A}$ is dense in $\mathcal{C}_0(\hat{B})$ (and hence in $\mathcal{C}_0(\hat{G})$). ■

4.5. The Fourier transform of character measures

Let $\hat{\sigma}$ be any complex Radon on $\hat{G}$ which is also a finite measure, i.e. such that $|\hat{\sigma}|(\hat{G}) < +\infty$[14]. Then, for every $s \in G$ we set

$$\begin{aligned} T_{\hat{\sigma}}\colon G &\longrightarrow \mathbb{C}, \\ s &\mapsto T_{\hat{\sigma}}(s) := \textstyle\int_{\hat{G}} \chi(s)\,\mathrm{d}\hat{\sigma}(\chi), \end{aligned}$$

which is called the *Fourier transform of the measure* $\hat{\sigma}$. By assumption it is bounded, being $|T_{\hat{\sigma}}(\chi)| \le \int_{\hat{G}} |\,\mathrm{d}\hat{\sigma}(\chi)| = |\hat{\sigma}|(\hat{G})$. It is also a continuous function (see Exercise 4.2).

Exercise. 4.2 Let $\hat{\sigma}$ be any complex Radon measure on $\hat{G}$. The following argument proves that $T_{\hat{\sigma}}\colon G \to \mathbb{C}$ is a continuous function.

i. Pick any compact neighborhood K of e in G and any compact neighborhood $\hat{K}$ of χ_0 (the trivial character) in $\hat{G}$. Prove that the map $K \times \hat{K} \to \mathbb{C}$, $(s, \chi) \mapsto \chi(s)$ is

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[12] By Proposition 4.4 each element in $\hat{B}$ is of the type $\tilde{\nu}_\chi$ for some χ, and its value on f is $\int_G f(s)\overline{\chi(s)}\,\mathrm{d}\mu(s)$.

[13] The continuity comes from an application of the dominated convergence theorem: if $\hat{\nu}_{\chi_\alpha}$ is a net of elements in $\hat{B}$ converging to 0 in the weak-* topology, and hence pointwise, then $\hat{f}(\hat{\nu}_{\chi_\alpha}) = \hat{f}(\chi_\alpha) = \int_G f(s)\overline{\chi_\alpha(s)}\,\mathrm{d}\mu(s)$ goes to zero, because the family $f\overline{\chi_\alpha}$ is dominated by $f \in L^1(G)$.

[14] *Complex* Radon measure means that $\hat{\sigma}(E) \in \mathbb{C}$, not necessarily in $[0, +\infty]$. Recall that every finite complex Radon measure can be written as $\sum_{k=0}^{3} i^k \hat{\sigma}_{i^k}$ where each $\hat{\sigma}_{i^k}$ is a positive and finite Radon measure.

uniformly continuous, and deduce that for every $\epsilon > 0$ there are open neighborhoods U of e in K and $\hat{U}$ of χ_0 in $\hat{K}$ such that

$$st^{-1} \in U, \quad \chi\eta^{-1} \in \hat{U} \qquad \Longrightarrow |\chi(s) - \eta(t)| \le \epsilon.$$

Pick $t = e$ (and $\eta = \chi$), and deduce that

$$s \in U, \quad \chi \in \hat{K} \qquad \Longrightarrow |\chi(s) - 1| \le \epsilon.$$

ii. Fix $\epsilon > 0$, and let $\hat{K}$ be a compact in $\hat{G}$ such that $|\hat{\sigma}|(K^c) \le \epsilon$: such a compact exists because $|\hat{\sigma}|$ is a Radon (in particular it is inner regular) and finite measure. Let U be as in Step [i.], and note that for $st^{-1} \in U$ we have

$$\begin{aligned} |T_{\hat{\sigma}}(s) - T_{\hat{\sigma}}(t)| &\le \int_{\hat{K}} |\chi(s) - \chi(t)| \,\mathrm{d}|\hat{\sigma}|(\chi) + \int_{\hat{K}^c} |\chi(s) - \chi(t)| \,\mathrm{d}|\hat{\sigma}|(\chi) \\ &\le \epsilon(|\hat{\sigma}|(\hat{G}) + 2). \end{aligned}$$

An application of the Fubini-Tonelli theorems shows that for every $f \in L^1(G)$,

$$\begin{aligned} \int_G \overline{f(s)} T_{\hat{\sigma}}(s) \,\mathrm{d}\mu(s) &= \int_G \overline{f(s)} \int_{\hat{G}} \chi(s) \,\mathrm{d}\hat{\sigma}(\chi) \,\mathrm{d}\mu(s) \\ &= \int_{\hat{G}} \int_G \overline{f(s)} \chi(s) \,\mathrm{d}\mu(s) \,\mathrm{d}\hat{\sigma}(\chi) = \int_{\hat{G}} \overline{\hat{f}}(\chi) \,\mathrm{d}\hat{\sigma}(\chi) \end{aligned} \tag{4.9}$$

(the computation is correct since $\hat{f}$ is bounded and $\overline{f(s)}\chi(s)$ is measurable by Lemma 4.3). The following proposition shows that the measure is completely determined by its transform.

Proposition 4.7 *If $T_{\hat{\sigma}}(s) = 0$ for every s, then $\hat{\sigma} = 0$.*

Proof. In fact, if $T_{\hat{\sigma}}(s) = 0$ identically, then

$$\int_{\hat{G}} \overline{\hat{f}}(\chi) \,\mathrm{d}\hat{\sigma}(\chi) = 0$$

for every $f \in L^1(G)$, by (4.9). According to Proposition 4.6 the set of Fourier transforms of functions in $L^1(G)$ is a dense subset of $\mathcal{C}_0(\hat{G})$, therefore

$$\int_{\hat{G}} g(\chi) \,\mathrm{d}\hat{\sigma}(\chi) = 0$$

for every $g \in \mathcal{C}_0(\hat{G})$, and hence $\hat{\sigma} = 0$[15]. ■

The following surprising connection was discovered by Bochner.

Theorem 4.5 (Bochner) *The set $\mathcal{P}(G)$ coincides with the set of the Fourier transforms of all* positive *Radon measures on $\hat{G}$ with total mass ≤ 1.*

Proof. Let

$$M := \{T_{\hat{\sigma}} \colon \hat{\sigma} \text{ positive Radon measure, } \hat{\sigma}(G) \le 1\}.$$

We already know that $T_{\hat{\sigma}}$ is a continuous function, with $\|T_{\hat{\sigma}}\|_\infty = \hat{\sigma}(G) \le 1$. Directly from it definition we see that

$$\overline{T_{\hat{\sigma}}(s^{-1})} = \overline{\int_{\hat{G}} \chi(s^{-1}) \,\mathrm{d}\hat{\sigma}(\chi)} = \int_{\hat{G}} \overline{\chi(s^{-1})} \,\mathrm{d}\hat{\sigma}(\chi) = \int_{\hat{G}} \chi(s) \,\mathrm{d}\hat{\sigma}(\chi) = T_{\hat{\sigma}}(s),$$

[15]Recall that the set of finite Radon measures is the topological dual of the space $\mathcal{C}_0(\hat{G})$ (see [HR1], p. 170), with the norm of the functional associated to the measure $\hat{\sigma}$ which is always equal to $|\hat{\sigma}|(\hat{G})$; if you are not aware of this result, you can try a direct proof: if the Radon measure is not identically zero, then there is a compact $\hat{K} \subseteq \hat{G}$ with non zero measure and contained into an open set $\hat{U}$ with $|\hat{\sigma}(\hat{U}\backslash\hat{K})| < |\hat{\sigma}(\hat{K})|$. Select a nonnegative and continuous function $g \colon \hat{G} \to [0,1]$ which is 1 in K and 0 outside U (Urysohn lemma) and show that $\int_G g(\chi) \,\mathrm{d}\hat{\sigma}(\chi) \ne 0$. This contradicts the assumption.

which proves that $T_{\hat{\sigma}}$ has the property (4.1).
Moreover, for every $f \in \mathcal{C}_c(G)$ we have (by Fubini-Tonelli in $G \times G \times \hat{G}$ with product measure $\mu(s)\mu(t)\hat{\sigma}(\chi)$)

$$\begin{aligned}\int_{G\times G} T_{\hat{\sigma}}(s^{-1}t)f(s)\overline{f(t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) &= \int_{G\times G}\Big[\int_{\hat{G}}\chi(s^{-1}t)\,\mathrm{d}\hat{\sigma}(\chi)\Big]f(s)\overline{f(t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\ &= \int_{\hat{G}}\Big[\int_{G\times G}\chi(s^{-1}t)f(s)\overline{f(t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\Big]\,\mathrm{d}\hat{\sigma}(\chi)\\ &= \int_{\hat{G}}\langle f, f\rangle_\chi\,\mathrm{d}\hat{\sigma}(\chi).\end{aligned}$$

We know that $\langle f, f\rangle_\chi \geq 0$ for every f, thus the computation makes evident that $T_{\hat{\sigma}}$ is of positive type. This proves that $T_{\hat{\sigma}} \in \mathcal{P}(G)$ for every $\hat{\sigma}$, i.e. that $M \subseteq \mathcal{P}(G)$.
On the other hand, when $\hat{\sigma}$ is the Dirac measure δ_η supported at a character η, we have

$$T_{\delta_\eta}(s) = \int_{\hat{G}}\chi(s)\,\mathrm{d}\delta_\eta(\chi) = \eta(s),$$

which proves that M contains the characters. It is also a convex set, because the set of positive Radon measures with total mass ≤ 1 is itself a convex set and T is linear in the measure. This proves that M is a convex subset of $\mathcal{P}(G)$ containing the characters (and the null function, of course). By Theorem 4.2 these are the extremal points on $\mathcal{P}(G)$, which is a weak-* compact by Proposition 4.5; thus whether we are able to prove that M is weak-* closed, then we can conclude that $M = \mathcal{P}(G)$ by Krein–Milman's theorem.
Suppose we have any net $\{T_{\hat{\sigma}_\alpha}\}_\alpha$ weak-* converging to some $\phi \in L^\infty(G)$. By definition, this means that

$$\lim_\alpha \int_G T_{\hat{\sigma}_\alpha}(s)\overline{h(s)}\,\mathrm{d}\mu(s) = \int_G \phi(s)\overline{h(s)}\,\mathrm{d}\mu(s) \qquad \forall h \in L^1(G).$$

According to (4.9), this means that

$$\lim_\alpha \int_{\hat{G}} \overline{\hat{h}(\chi)}\,\mathrm{d}\hat{\sigma}_\alpha(\chi) = \int_G \phi(s)\overline{h(s)}\,\mathrm{d}\mu(s) \qquad \forall h \in L^1(G);$$

in particular the limit of $\int_{\hat{G}} \overline{\hat{h}(\chi)}\,\mathrm{d}\hat{\sigma}_\alpha(\chi)$ exists for every $h \in L^1(G)$. Since the Fourier transform of $L^1(G)$ functions is dense in $\mathcal{C}_0(\hat{G})$ (by Proposition 4.6), we conclude that actually the limit of $\int_{\hat{G}} k(\chi)\,\mathrm{d}\hat{\sigma}_\alpha(\chi)$ exists for every $k \in \mathcal{C}_0(\hat{G})$. Since complex Radon measures on $\hat{G}$ are the topological dual of $\mathcal{C}_0(\hat{G})$ (see [HR1], p. 170), this proves that there exists a complex Radon measure $\hat{\sigma}$ such that $\lim_\alpha \int_{\hat{G}} k(\chi)\,\mathrm{d}\hat{\sigma}_\alpha(\chi) = \int_{\hat{G}} k(\chi)\,\mathrm{d}\hat{\sigma}(\chi)$, for every k. The measure $\hat{\sigma}$ has total mass ≤ 1, because each $\hat{\sigma}_\alpha$ does, and actually it is a positive measure, because each $\hat{\sigma}_\alpha$ does. Therefore $T_{\hat{\sigma}}$ is well defined, and when k is the Fourier transform of any $h \in L^1(G)$, this equality becomes

$$\lim_\alpha \int_G T_{\hat{\sigma}_\alpha}(s)\overline{h(s)}\,\mathrm{d}\mu(s) = \int_G T_{\hat{\sigma}}(s)\overline{h(s)}\,\mathrm{d}\mu(s) \qquad \forall h \in L^1(G),$$

by (4.9). This shows that $\{T_{\hat{\sigma}_\alpha}\}_\alpha$ weak-* converges to $T_{\hat{\sigma}}$. This proves that M is weak-* closed. ∎

Let $f \in V(G)$, so that it is a $\mathbb{C}$-linear combination of functions of positive type. According to Theorem 4.5 there is a complex Radon measure $\hat{\sigma}_f$ with finite total mass (i.e. $|\hat{\sigma}_f|(\hat{G}) < +\infty$), such that f is the Fourier transform of $\hat{\sigma}_f$, i.e.

$$f(s) = \int_{\hat{G}} \chi(s)\,\mathrm{d}\hat{\sigma}_f(\chi) \qquad \forall s \in G. \tag{4.10}$$

The following lemma shows that the association $f \mapsto \hat{\sigma}_f$ satisfies a kind of duality.

Lemma 4.4 *Let $f, g \in V^1(G) := V(G) \cap L^1(G)$. Then*

$$\hat{g}\ \mathrm{d}\hat{\sigma}_f = \hat{f}\ \mathrm{d}\hat{\sigma}_g$$

(equality as measures).

Proof. Proposition 4.7 shows that the measures are uniquely determined by their Fourier transform, hence it is sufficient to verify that their Fourier transform are equal. We have

$$\begin{aligned} T_{\hat{g}\,\mathrm{d}\hat{\sigma}_f}(s) &= \int_{\hat{G}} \chi(s)\hat{g}(\chi)\,\mathrm{d}\hat{\sigma}_f(\chi) \\ &= \int_{\hat{G}} \chi(s)\Big[\int_G g(t)\overline{\chi(t)}\,\mathrm{d}\mu(t)\Big]\,\mathrm{d}\hat{\sigma}_f(\chi) \\ &= \int_G g(t)\Big[\int_{\hat{G}} \chi(st^{-1})\,\mathrm{d}\hat{\sigma}_f(\chi)\Big]\,\mathrm{d}\mu(t) \\ &= \int_G g(t)f(st^{-1})\,\mathrm{d}\mu(t) = (g * f)(s). \end{aligned}$$

In our setting this is equal to $(f * g)(s)$, and the same computation proves that it is $T_{\hat{f}\,\mathrm{d}\hat{\sigma}_g}(s)$. ∎

We introduce two further sets. Let $\mathcal{C}_b(\hat{G}) \subseteq \mathcal{C}(\hat{G})$ be the subset of bounded functions. Let $\mathcal{F}$ be the set of functions $\phi \in \mathcal{C}_b(\hat{G})$ for which there exists some complex Radon measure $\hat{\tau}_\phi$ on $\hat{G}$ such that the equality

$$\phi\ \mathrm{d}\hat{\sigma}_f = \hat{f}\ \mathrm{d}\hat{\tau}_\phi \tag{4.11}$$

holds for all $f \in V^1(G)$, as measures. Lemma 4.4 shows that $\mathcal{F}$ contains the Fourier transform of $V^1(G)$ functions: in that case the measure $\hat{\tau}_{\hat{g}}$ associated to $\hat{g}$ is actually $\hat{\sigma}_g$.

Lemma 4.5 *the set $\mathcal{F}$ has the following properties:*

i. if $\phi \in \mathcal{F}$, then the associated measure $\hat{\tau}_\phi$ is unique;

ii. if $\phi \in \mathcal{F}$ is positive, then the associated measure $\hat{\tau}_\phi$ is positive;

iii. the set $\mathcal{F}$ is a module over the $\mathcal{C}_b(\hat{G})$, with the map $\phi \mapsto \hat{\tau}_\phi$ as morphism of modules. In particular,

$$\hat{\tau}_{\phi+\gamma} = \hat{\tau}_\phi + \hat{\tau}_\gamma, \qquad \hat{\tau}_{a\phi} = a\hat{\tau}_\phi$$

for every $a \in \mathcal{C}_b(\hat{G})$ and $\phi, \gamma \in \mathcal{F}$;

iv. let $\eta \in \hat{G}$ and let $L_\eta\colon \mathcal{C}_b(\hat{G}) \to \mathcal{C}_b(\hat{G})$ with $(L_\eta\phi)(\chi) := \phi(\eta^{-1}\chi)$ for every $\chi \in \hat{G}$. Then $L_\eta(\mathcal{F}) \subseteq \mathcal{F}$, with

$$\hat{\tau}_{L_\eta\phi} = L_\eta\hat{\tau}_\phi$$

where for any measure $\hat{\tau}$ on $\hat{G}$, the measure $L_\eta\hat{\tau}$ is defined as the one such that $(L_\eta\hat{\tau})(E) := \hat{\tau}(\eta^{-1}E)$ for every measurable set E.

v. $\mathcal{C}_c(\hat{G}) \subseteq \mathcal{F}$.

Proof.

i.-ii. There is a net $\{f_\alpha\}_\alpha$ of functions in $V^1(G)$ whose Fourier transform converges to the constant function 1 uniformly on the compact sets (See [RV], Exr. 12 p. 129, or Exercise 4.3 here below), and whose Radon measure $\hat{\sigma}_{f_\alpha}$ is a positive measure. Using this family in the role of f in (4.11), we conclude that

$$\lim_\alpha \phi\ \mathrm{d}\hat{\sigma}_{f_\alpha} = \mathrm{d}\hat{\tau}_\phi.$$

This equality shows that $\mathrm{d}\hat{\tau}_\phi$ is completely determined by ϕ. This formula also shows that the measure is positive when ϕ is positive.

iii. This is an easy computation;

iv. Let $\eta \in \hat{G}$ and $f \in V^1(G)$. Then the definition of $\hat{\tau}_f$ gives

$$(\eta f)(s) = \int_{\hat{G}} (\eta\chi)(s)\,\mathrm{d}\hat{\tau}_f(\chi) = \int_{\hat{G}} \chi(s)\,\mathrm{d}\hat{\tau}_f(\eta^{-1}\chi) = \int_{\hat{G}} \chi(s)\,\mathrm{d}(L_\eta\hat{\tau}_f)(\chi)$$

proving that $L_\eta\hat{\tau}_f = \hat{\tau}_{\eta f}$.
Let $h \in \mathcal{C}_c(\hat{G})$ and $f \in V^1(\hat{G})$. Then

$$\begin{aligned}\int_{\hat{G}} h(\chi)(L_\eta\phi)(\chi)\,\mathrm{d}\hat{\tau}_f(\chi) &= \int_{\hat{G}} h(\chi)\phi(\eta^{-1}\chi)\,\mathrm{d}\hat{\tau}_f(\chi) = \int_{\hat{G}} h(\eta\chi)\phi(\chi)\,\mathrm{d}\hat{\tau}_f(\eta\chi)\\ &= \int_{\hat{G}} h(\eta\chi)\phi(\chi)\,\mathrm{d}\big(L_{\eta^{-1}}\hat{\tau}_f\big)(\chi) = \int_{\hat{G}} h(\eta\chi)\phi(\chi)\,\mathrm{d}\big(\hat{\tau}_{\eta^{-1}f}\big)(\chi)\\ &= \int_{\hat{G}} h(\eta\chi)\widehat{\eta^{-1}f}(\chi)\,\mathrm{d}\big(\hat{\tau}_\phi\big)(\chi) = \int_{\hat{G}} h(\eta\chi)\hat{f}(\eta\chi)\,\mathrm{d}\big(\hat{\tau}_\phi\big)(\chi)\\ &= \int_{\hat{G}} h(\chi)\hat{f}(\chi)\,\mathrm{d}\big(L_\eta\hat{\tau}_\phi\big)(\chi)\end{aligned}$$

which proves the claim.

v. Let $\gamma \in \mathcal{C}_c(\hat{G})$. Let $\hat{K}$ be a compact in $\hat{G}$ containing the support of γ. Let $f \in V^1(G)$ be a function whose Fourier transform stays away from zero in $\hat{K}$: such a function exists because we know that there exists a net of such functions whose Fourier transforms converge (uniformly on compact sets) to the constant 1 function (See [RV], Exr. 12 p. 129, or Exercise 4.3 here below). Then $a := \gamma/\hat{f}$ is a well defined function in $\mathcal{C}_c(\hat{G}) \subseteq \mathcal{C}_b(\hat{G})$; $\hat{f}$ is in $\mathcal{F}$ (because it is the Fourier transform of $f \in V^1(G)$), hence $\gamma = a\hat{f}$ is in $\mathcal{F}$, as well (by property iii.).

■

Exercise. 4.3 Let $\{K_\alpha\}_\alpha$ be a net of compact neighborhoods of e in G, with $\cap_\alpha K_\alpha = \{e\}$ (it exists, by local compactness). Let $g_\alpha\colon G \to [0, +\infty)$ be any continuous function supported in K_α, with $\int_G g_\alpha(s)\,\mathrm{d}\mu(s) = 1$.

i. Pick any compact neighborhood K of e in G and any compact neighborhood $\hat{K}$ of χ_0 (the trivial character) in $\hat{G}$. Prove that the map $K \times \hat{K} \to \mathbb{C}$, $(s, \chi) \mapsto \chi(s)$ is uniformly continuous, and deduce that for every $\epsilon > 0$ there are open neighborhoods U of e in K and $\hat{U}$ of χ_0 in $\hat{K}$ such that

$$st^{-1} \in U, \quad \chi\eta^{-1} \in \hat{U} \qquad \Longrightarrow |\chi(s) - \eta(t)| \le \epsilon.$$

Pick $t = e$ (and $\eta = \chi$), and deduce that

$$s \in U, \quad \chi \in \hat{K} \qquad \Longrightarrow |\chi(s) - 1| \le \epsilon.$$

ii. Use the previous step to conclude that $\{\hat{g}_\alpha\}_\alpha$ converges to the constant function 1, uniformly on compact subsets of $\hat{G}$.

iii. Let $f_\alpha := g_\alpha * \tilde{g}_\alpha$, where $\tilde{g}_\alpha(s) := \overline{g_\alpha(s^{-1})}$. Verify that $f_\alpha \in \mathcal{C}_c(G)$, and that it is of positive type, so that $f_\alpha \in V^1(G)$.

iv. Verify that $\hat{f}_\alpha = |\hat{g}_\alpha|^2$; deduce that $\{f_\alpha\}_\alpha$ is a net of functions in $V^1(G)$ whose Fourier transform converges to 1 uniformly on compact sets. Note that the Radon measure $\hat{\sigma}_{f_\alpha}$ is a positive Radon measure.

4.6. Proof of the Fourier inversion formula

According to the property [v.] of the previous lemma the set $\mathcal{C}_c(\hat{G})$ is a subset of $\mathcal{F}$. This allows to associate to each function γ in $\mathcal{C}_c(\hat{G})$ the unique (by [i.]) measure $\hat{\tau}_\gamma$. This shows that the map

$$\begin{array}{rl} \mathcal{C}_c(\hat{G}) & \xrightarrow{\Lambda} \mathbb{C} \\ \gamma & \mapsto \Lambda(\gamma) := \int_{\hat{G}} 1 \,\mathrm{d}\hat{\tau}_\gamma(\chi) \end{array}$$

is well defined. This map is a linear functional (by [iii.]) on $\mathcal{C}_c(\hat{G})$, which is positive for positive argument (by [ii.]).[16] Take any $f \in \mathcal{P}(G) \cap L^1(G)$, not identically zero. Then the measure $\tau_{\hat{f}}$ is $\hat{\sigma}_f$, which is nonzero by (4.10) and positive by Bochner's theorem. Let $\hat{K}$ be a compact set in $\hat{G}$ whose $\hat{\sigma}_f$ measure is not zero (it exists, since $\hat{\sigma}_f$ is a nonzero Radon measure). Take any $a \in \mathcal{C}_c(\hat{K}) \subseteq \mathcal{C}_c(\hat{G})$. Then $a\hat{f} \in \mathcal{C}_c(\hat{G})$ and $a \,\mathrm{d}\hat{\sigma}_f$ is its measure. Thus

$$\Lambda(a\hat{f}) = \int_{\hat{G}} 1 \,\mathrm{d}\hat{\tau}_{a\hat{f}}(\chi) = \int_{\hat{G}} a(\chi) \,\mathrm{d}\hat{\sigma}_f(\chi).$$

The sup value of this integral when a ranges in $\mathcal{C}_c(\hat{K}) \cap \{a\colon a(\chi) \in [0,1]\, \forall \chi \in \hat{G}\}$ is $\hat{\sigma}_f(\hat{K})$, which is strictly positive by our assumption on $\hat{K}$. This proves that Λ is not identically zero. It is also L_η-invariant for every character η, because equality [iv.] yields:

$$\begin{aligned} \Lambda(L_\eta\gamma) &= \int_{\hat{G}} 1 \,\mathrm{d}\hat{\tau}_{L_\eta\gamma}(\chi) = \int_{\hat{G}} 1 \,\mathrm{d}L_\eta\hat{\tau}_\gamma(\chi) = \int_{\hat{G}} 1 \,\mathrm{d}\hat{\tau}_\gamma(\eta^{-1}\chi) \\ &= \int_{\hat{G}} (L_\eta 1)(\chi) \,\mathrm{d}\hat{\tau}_\gamma(\chi) = \int_{\hat{G}} 1 \,\mathrm{d}\hat{\tau}_\gamma(\chi) = \Lambda(\gamma). \end{aligned}$$

This proves that the measure associated with Λ is actually a Haar measure on $\hat{G}$. We adopt this measure as standard measure $\hat{\mu}$ on $\hat{G}$. Thus we can write

$$\int_{\hat{G}} 1 \,\mathrm{d}\hat{\tau}_\gamma(\chi) = \Lambda(\gamma) = \int_{\hat{G}} \gamma(\chi) \,\mathrm{d}\hat{\mu}(\chi), \qquad \forall \gamma \in \mathcal{C}_c(\hat{G}).$$

Let $\phi \in \mathcal{F}$ and $a \in \mathcal{C}_c(\hat{G})$. Then $a\phi \in \mathcal{C}_c(\hat{G})$ as well (because $\mathcal{F} \subseteq \mathcal{C}_b(\hat{G})$, by design), and the previous identity shows that

$$\int_{\hat{G}} a(\chi)\phi(\chi) \,\mathrm{d}\hat{\mu}(\chi) = \Lambda(a\phi) = \int_{\hat{G}} 1 \,\mathrm{d}\hat{\tau}_{(a\phi)}(\chi) = \int_{\hat{G}} a(\chi) \,\mathrm{d}\hat{\tau}_\phi(\chi)$$

(using [iii.]). This shows that $\phi \,\mathrm{d}\hat{\mu} = \mathrm{d}\hat{\tau}_\phi$, which describes the measure associated with ϕ in terms of the Haar measure and ϕ itself. Then, for every $f \in V^1(G)$,

$$\hat{f} \,\mathrm{d}\hat{\mu} = \mathrm{d}\hat{\tau}_{\hat{f}} = \mathrm{d}\hat{\sigma}_f,$$

and hence, by (4.10), we get

$$f(s) = \int_{\hat{G}} \chi(s) \,\mathrm{d}\hat{\sigma}_f(\chi) = \int_{\hat{G}} \hat{f}(\chi)\chi(s) \,\mathrm{d}\hat{\mu}(\chi).$$

This proves the formula in Theorem 4.3.

The following corollary gives a first nontrivial result toward the proof of the second part of Theorem 4.3 identifying $V^1(G)$ with $V^1(\hat{G})$.

Corollary 4.2 *Let $f \in L^1(G)$. Then*

i. if f is continuous and of positive type, then $\hat{f}$ is positive and integrable; in particular $\int_G f(s) \,\mathrm{d}\mu(s) \geq 0$;

ii. if f is positive then $\hat{f}$ is of positive type.

Thus the Fourier transform defines an injection of $V^1(G)$ into $V^1(\hat{G})$.

Proof.

[16] Recall that *positive* does not mean that it *strictly* positive, in particular it is possible that Λ is the null functional. The computation following this footnote shows that this is not the case.

i. By hypothesis f is continuous and of positive type, then the Radon measure $\hat{\sigma}_f$ which is associated to f is finite and positive (by Bochner's theorem). The previous computation shows that in terms of the Haar measure $\hat{\mu}$ we have selected in $\hat{G}$ one has

$$\mathrm{d}\hat{\sigma}_f = \hat{f}\ \mathrm{d}\hat{\mu}.$$

Since $\hat{\sigma}_f$ and $\hat{\mu}$ are both positive and Radon measures, we conclude that $\hat{f}$ is positive. It is also integrable, since $\mathrm{d}\hat{\sigma}_f$ is a finite measure.

ii. This is an easy computation: firstly we note that

$$\overline{\hat{f}(\chi^{-1})} = \overline{\int_G f(s)\overline{\chi^{-1}(s)}\,\mathrm{d}\mu(s)} = \overline{\int_G f(s)\chi(s)\,\mathrm{d}\mu(s)} = \int_G f(s)\overline{\chi(s)}\,\mathrm{d}\mu(s) = \hat{f}(\chi)$$

(in the last steps we have used the assumption that $f(s) \in \mathbb{R}$), proving that $\hat{f}$ satisfies the identity (4.1). Moreover, for every function $h \in \mathcal{C}_c(\hat{G})$ we have

$$\begin{aligned}
&\int_{\hat{G}\times\hat{G}} \hat{f}(\chi\psi^{-1})h(\chi)\overline{h(\psi)}\,\mathrm{d}\hat{\mu}(\chi)\,\mathrm{d}\hat{\mu}(\psi)\\
&\quad= \int_{\hat{G}\times\hat{G}} \Big[\int_G f(s)\overline{(\chi\psi^{-1})(s)}\,\mathrm{d}\mu(s)\Big]h(\chi)\overline{h(\psi)}\,\mathrm{d}\hat{\mu}(\chi)\,\mathrm{d}\hat{\mu}(\psi)\\
&\quad= \int_G f(s)\Big[\int_{\hat{G}\times\hat{G}} \overline{\chi(s)}\psi(s)h(\chi)\overline{h(\psi)}\,\mathrm{d}\hat{\mu}(\chi)\,\mathrm{d}\hat{\mu}(\psi)\Big]\,\mathrm{d}\mu(s)\\
&\quad= \int_G f(s)\Big|\int_{\hat{G}} \overline{\chi(s)}h(\chi)\,\mathrm{d}\hat{\mu}(\chi)\Big|^2\,\mathrm{d}\mu(s) \geq 0
\end{aligned}$$

(the last step holds since $f(s) \geq 0$ for every $s \in G$).

Let $f \in V^1(G)$. It is a $\mathbb{C}$-linear combination of functions in $\mathcal{P}(G)$, and each function in $\mathcal{P}(G)$ is a sum of continuous and positive functions in $\mathcal{P}(G)$[17]. By [ii.] we conclude that $\hat{f}$ (which is a continuous and bounded function) is also a $\mathbb{C}$-linear combination of functions in $\mathcal{P}(\hat{G})$, i.e. it is a function in $V(\hat{G})$. It is also in $L^1(\hat{G})$, applying to the same decomposition the conclusions in [i.]. This proves that the Fourier transform maps $V^1(G) \to V^1(\hat{G})$. ■

4.7. Pontryagin Duality

Once again, let G be a locally compact abelian group. Then also $\hat{G}$ is a locally compact abelian group, so that $\hat{G}$ itself may be considered as the stem group for a new leap moving from $\hat{G}$ towards is dual: $\hat{\hat{G}}$. There is a natural map from G to $\hat{\hat{G}}$ given by the evaluation map:

$$\begin{aligned}
\alpha\colon G &\longrightarrow \hat{\hat{G}},\\
y &\mapsto \alpha(y)\colon \hat{G} \longrightarrow \mathbb{C}, \quad \alpha(y)(\chi) := \chi(y).
\end{aligned}$$

The famous result of Pontryagin states that this procedure generates all elements in $\hat{\hat{G}}$, in a continuous way.

Theorem 4.6 *The map $\alpha\colon G \to \hat{\hat{G}}$ is an isomorphism of topological groups, so that G and $\hat{G}$ are mutually dual.*

Its proof will interact with the remaining part of Theorem 4.3 we still have to prove. The following lemma gives a first glimpse in the subject.

Lemma 4.6 *The map α is injective.*

[17] This is a consequence of the decomposition $z = \sum_{k=0}^{3} i^k \max(0, \mathrm{Re}(i^k\overline{z}))$, which is true for every $z \in \mathbb{C}$, and the fact that $\max(0, \mathrm{Re}(i^k\overline{f}(s)))$ is a continuous and nonnegative function of s whenever f is continuous.

Proof. In other words, we have to prove that $\hat{G}$ separates the points of G, and since G is a group, it is sufficient to prove that $\hat{G}$ separates z from e, when $z \neq e$. Suppose that this is not true, i.e. that $z \neq e$ but that $\chi(z) = 1$ for all $\chi \in \hat{G}$. Let f be any function in $L^1(G)$; then

$$\hat{f}(\chi) = \chi(z)\widehat{L_z f}(\chi) = \widehat{L_z f}(\chi), \qquad \forall \chi \in \hat{G}.$$

If further we assume that $f \in V(G)$, then by inversion formula we deduce that $f = L_z f$, for every $f \in V^1(G)$[18]. This is impossible: in fact, let U be an open neighborhood of e such that $Uz \cap U = \emptyset$: such an open set exists, because G is a Hausdorff space. It is possible to produce a nonzero function f in $V^1(G)$ supported in U[19]: for such a function the equality $f = L_z f$ is impossible. ■

Let $\hat{K}$ be a compact neighborhood of the identity in $\hat{G}$, and V be an open neighborhood of 1 in S^1. By construction, the sets

$$W(\hat{K}, V) := \{\psi \in \hat{\hat{G}} \colon \psi(\chi) \subseteq V \text{ for all } \chi \in \hat{K}\}$$

and their translates give a base for the topology of $\hat{\hat{G}}$. Lemma 4.6 allows to consider G as a subset of $\hat{\hat{G}}$; with this identification we can consider

$$W_G(\hat{K}, V) := W(\hat{K}, V) \cap \alpha(G)$$

as a subset of ($\alpha(G)$, and hence of) G. We use these sets to prove that α is a homeomorphism onto its image.

Proposition 4.8 *The sets $W_G(\hat{K}, V)$ and their translates are a base for the topology of G. As a consequence, the map α is bicontinuous, so that it defines a homeomorphism on its image.*

Proof. Let U be an open neighborhood of e in G, and let g be a function in $V^1(G)$ supported in U and which is of positive type, with $g(e) = 1$ (such a function exists, see the argument we have used in the footnote of the proof of Lemma 4.6). Then $\int_{\hat{G}} \hat{g}(\chi)\, \mathrm{d}\hat{\mu}(\chi) = g(e) = 1$ (by inversion formula), and $\hat{g}$ is positive (by Corollary 4.2[i]). Let $\epsilon > 0$, and let $\hat{K}$ be a compact in $\hat{G}$ such that $\int_{\hat{K}^c} \hat{g}(\chi)\, \mathrm{d}\hat{\mu}(\chi) \leq \epsilon$: such a compact exists, since $\hat{g}\, \mathrm{d}\hat{\mu}$ is a Radon (hence inner regular) measure on $\hat{G}$. Then

$$\begin{aligned} |g(s) - 1| = |g(s) - g(e)| &\leq \int_{\hat{K}} \hat{g}(\chi)|\chi(s) - 1|\, \mathrm{d}\hat{\mu}(\chi) + \int_{\hat{K}^c} \hat{g}(\chi)|\chi(s) - 1|\, \mathrm{d}\hat{\mu}(\chi) \\ &\leq \int_{\hat{K}} \hat{g}(\chi)|\chi(s) - 1|\, \mathrm{d}\hat{\mu}(\chi) + 2\epsilon. \end{aligned}$$

If we choose $V = \{\omega \in \mathbb{C} \colon |\omega - 1| < \epsilon\}$, then also the remaining integral is bounded by ϵ whenever $s \in W_G(\hat{K}, V)$, so that

$$|g(s)| \geq 1 - 3\epsilon.$$

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[18] Note that we cannot easily deduce the claim from the injectivity property of the Fourier map in $V^1(G)$ claimed in Corollary 4.2, because we have not proved that $L_z f \in V^1(G)$ when $f \in V^1(G)$. However, for such an f we have the equality

$$f(s) = \int_{\hat{G}} \hat{f}(\chi)\chi(s)\, \mathrm{d}\hat{\mu}(\chi).$$

Setting $s \mapsto z^{-1}s$ in this equality we get

$$L_z f(s) = f(z^{-1}s) = \int_{\hat{G}} \hat{f}(\chi)\chi(z^{-1}s)\, \mathrm{d}\hat{\mu}(\chi). = \int_{\hat{G}} \hat{f}(\chi)\overline{\chi(z)}\chi(s)\, \mathrm{d}\hat{\mu}(\chi). = \int_{\hat{G}} \widehat{L_z f}(\chi)\chi(s)\, \mathrm{d}\hat{\mu}(\chi).$$

These equalities show that the equality $\hat{f}(\chi) = \widehat{L_z f}(\chi)$ implies the equality $f = L_z f$.

[19] Take V be an open and symmetric neighborhood of e such that $V^2 \subseteq U$. Let K be a compact and symmetric neighborhood of e contained in V. Let g be a nonzero continuous function supported in K, and take $f := g * \tilde{g}$. Then f is not zero, it is supported in $KK^{-1} = K^2 \subseteq V^2 \subseteq U$ and is of positive type, so that $f \in V^1(G)$.

By construction U contains the support of g, so that we have proved that $W_G(\hat{K}, V) \subseteq U$. On the other hand, let $\hat{K}$ be any compact neighborhood of the identity in $\hat{G}$. Let K be a compact neighborhood of e in G. The map $K \times \hat{K} \to \mathbb{C}$ such that $(y, \chi) \mapsto \chi(y)$ is uniformly continuous, hence for every open neighborhood V of 1 in S^1, there are open neighborhoods U, and $\hat{U}$ such that

$$st^{-1} \in U, \quad \chi\eta^{-1} \in \hat{U} \quad \implies \quad \chi(s)\eta^{-1}(t) \in V.$$

Choosing $t = e$ this equality shows that

$$s \in U, \quad \chi \in \hat{K} \quad \implies \quad \chi(s) \in V.$$

In other words, it proves that $U \subseteq W_G(\hat{K}, V)$. This completes the proof of the equivalence of the topologies. The fact that α is a homeomorphism onto its image is immediate. ■

Corollary 4.3 *The image of α is closed in $\hat{\hat{G}}$.*

Proof. Let $H := \overline{\alpha(G)}$, the closure of $\alpha(G)$ in $\hat{\hat{G}}$. Then H is a closed subgroup in $\hat{\hat{G}}$, and $\alpha(G)$ is dense in H. The image $\alpha(G)$ is also locally compact in the subspace topology of $\hat{\hat{G}}$ (because G is locally compact and α is a homeomorphism), and hence also in the subspace topology of H (because H is closed). By a standard argument of general topology[20] this implies that $\alpha(G)$ is open in H. But $\alpha(G)$ is a subgroup of H, hence it is also closed in H (see Proposition 1.4[5.]). Since H is closed in $\hat{\hat{G}}$, we conclude that $\alpha(G)$ coincides with H, and hence $\alpha(G)$ is closed in $\hat{\hat{G}}$. ■

4.8. Plancherel theorem

Let $f \in L^1(G)$ and let $\tilde{f}(s) := \overline{f(s^{-1})}$. We know that $\hat{\tilde{f}} = \overline{\hat{f}}$. Let $g := f * \tilde{f}$. Then $g \in L^1(G)$ and $\hat{g} = |\hat{f}|^2$. If f belongs also to $L^2(G)$, then g is also continuous[21], bounded[22] and of positive type[23]. Hence g is in $V^1(G)$, and the inversion formula gives

$$\int_G |f(s)|^2 \, \mathrm{d}\mu(s) = g(e) = \int_{\hat{G}} \hat{g}(\chi) \, \mathrm{d}\hat{\mu}(\chi) = \int_{\hat{G}} |\hat{f}(\chi)|^2 \, \mathrm{d}\hat{\mu}(\chi).$$

[20] The argument is the following. Let H be a Hausdorff topological space, and let $X \subseteq H$ be a dense and locally compact subspace. Then X is open in H.

Proof. In fact, pick any $a \in X$. Let K be a compact neighborhood of a in X: compactness here is with respect to the subspace topology of X as subset of H. Note that K exists, by the locally compactness assumed for X. K is compact also as subset of H (because the inclusion of X in H is a continuous map when the subspace topology is used in X). In particular it is closed, since H is Hausdorff.
We have the inclusions $a \in \mathring{K} \subseteq K \subseteq X$. Let B be the open set in H such that $\mathring{K} = X \cap B$. Then $a \in B$. The intersection $B \cap K^c$ is empty; in fact, it is an open set such that $X \cap (B \cap K^c) = (X \cap B) \cap K^c = \mathring{K} \cap K^c = \emptyset$. Thus, in case $B \cap K^c \neq \emptyset$ we get a contradiction with the assumed density of X in H. As a consequence, $B = B \cap K$, or, which is the same, $B \subseteq K$. Since $K \subseteq X$, the set B is an open set such that $a \in B \subseteq X$. This proves that X is open in H. ■

[21] In fact,

$$\begin{aligned}|g(s) - g(t)| &= \Big| \int_G (f(su^{-1}) - f(tu^{-1}))\tilde{f}(u) \, \mathrm{d}\mu(u) \Big| \leq \|L_{s^{-1}}f - L_{t^{-1}}f\|_{L^2(G)} \cdot \|\tilde{f}\|_{L^2(G)} \\ &= \|L_{ts^{-1}}f - f\|_{L^2(G)} \cdot \|f\|_{L^2(G)}\end{aligned}$$

and the first term goes to zero when t goes to s (along any net) (see Footnote 4).

[22] In fact

$$|g(s)| = \Big| \int_G f(su^{-1})\tilde{f}(u) \, \mathrm{d}\mu(u) \Big| \leq \|L_{s^{-1}}f\|_{L^2(G)} \cdot \|\tilde{f}\|_{L^2(G)} = \|f\|^2_{L^2(G)} \qquad \forall s \in G.$$

[23] This is a direct computation: the definition $g = f * \tilde{f}$ gives $g(s) = \int_G f(su)\overline{f(u)} \, \mathrm{d}\mu(u)$ so that

$$g(s^{-1}) = \int_G f(s^{-1}u)\overline{f(u)} \, \mathrm{d}\mu(u) = \int_G f(u)\overline{f(su)} \, \mathrm{d}\mu(u) = \overline{\int_G \overline{f(u)} f(su) \, \mathrm{d}\mu(u)} = \overline{g(s)}$$

This proves that the Fourier transform induces a map

$$\begin{aligned} L^1(G)\cap L^2(G) &\longrightarrow L^2(\hat{G}),\\ f &\mapsto \hat{f} \end{aligned}$$

which is an isometry onto its image. Let

$$\hat{A}_{1,2} := \{\hat{f} \colon f \in L^1(G)\cap L^2(G)\}$$

which we consider as a subspace of $L^2(\hat{G})$, via the previous map.

Lemma 4.7 *$\hat{A}_{1,2}$ is a dense subspace of the Hilbert space $L^2(\hat{G})$.*

Proof. It is sufficient to prove that 0 is the only element in $L^2(\hat{G})$ which is orthogonal to $\hat{A}_{1,2}$. Assume that $g \in L^2(\hat{G})$ is orthogonal to $\hat{A}_{1,2}$. Pick any $f \in L^1(G)\cap L^2(G)$, and notice that the equality $(\alpha(s)\hat{f})(\chi) = \chi(s)\hat{f}(\chi) = \widehat{L_s f}(\chi)$ for all $\chi \in \hat{G}$ shows that $\alpha(s)\hat{f} \in \hat{A}_{1,2}$, for every $s \in G$. Therefore, the supposed orthogonality implies that

$$\int_{\hat{G}} g(\chi)\overline{\hat{f}(\chi)}\chi(s)\,\mathrm{d}\hat{\mu}(\chi) = 0 \qquad \forall s \in G.$$

This formula shows that the Fourier transform of the measure $g\overline{\hat{f}}\,\mathrm{d}\hat{\mu}$ is zero. Note that the product $g\overline{\hat{f}}$ is in $L^1(\hat{G})$ (because $\hat{f}, g \in L^2(\hat{G})$), hence the measure is a complex Radon measure on $\hat{G}$, and the previous identity states that its Fourier transform is zero. By Proposition 4.7, we conclude that $g\overline{\hat{f}}$ is a.e. zero. This happens for every f with $\hat{f} \in \hat{A}_{1,2}$. Suppose that g is not a.e. null. Then it is nonzero on a set having positive measure, which we can take compact (by inner regularity). It is always possible to produce a function $f \in L^1(G)\cap L^2(G)$ such that $\hat{f}$ is not zero in a given compact set[24]. This contradicts the fact that $g\overline{\hat{f}}$ is 0 almost everywhere. Hence g is zero a.e. and the claim is proved. ■

Theorem 4.7 (Plancherel) *Let G be an abelian locally compact group. Then the Fourier transform can be extend to an isometry of Hilbert spaces from $L^2(G)$ onto $L^2(\hat{G})$.*

Proof. The claim follows immediately from Lemma 4.7 and the fact that $L^1(G)\cap L^2(G)$ is dense in $L^2(G)$ (for example because $\mathcal{C}_c(G)$ is dense both in $L^1(G)$ and in $L^2(G)$). ■

Corollary 4.4 (Parseval's identity) *For all $f, g \in L^2(G)$, we have*

$$\int_G f(s)\overline{g(s)}\,\mathrm{d}\mu(s) = \int_{\hat{G}} \hat{f}(\chi)\overline{\hat{g}(\chi)}\,\mathrm{d}\hat{\mu}(\chi).$$

Proof. An isometry is necessarily unitary. ■

Corollary 4.5 *Let $f, g \in L^2(G)$, and set $h := fg$. Note that $h \in L^1(G)$. Then $\hat{h} = \hat{f} * \hat{g}$.*

Proof. Let $\eta \in \hat{G}$. The Fourier transform of $\overline{g}\eta$ is $\overline{\hat{g}(\eta\chi^{-1})}$, and by Pereseval's identity in Corollary 4.4 we get

$$\hat{h}(\eta) = \int_G f(s)g(s)\overline{\eta(s)}\,\mathrm{d}\mu(s) = \int_G f(s)\overline{\overline{g(s)}\eta(s)}\,\mathrm{d}\mu(s)$$

---

proving (4.1), and

$$\begin{aligned} \int_{G\times G} g(s^{-1}t)h(s)\overline{h(t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) &= \int_{G\times G\times G} f(s^{-1}tu)\overline{f(u)}h(s)\overline{h(t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(u)\\ &= \int_{G\times G\times G} f(tu)\overline{f(su)}h(s)\overline{h(t)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(u)\\ &= \int_G \Big[\int_G f(tu)\overline{h(t)}\,\mathrm{d}\mu(t)\Big]^2 \mathrm{d}\mu(u) \geq 0 \end{aligned}$$

(Fubini–Tonelli used here, and the change $u \mapsto su$ in the second line).

[24] For example, the sequence of functions we have proposed in Ex. 4.3 are in $\mathcal{C}_c(G)$ (hence in $L^1(G)\cap L^2(G)$), and their Fourier transform converges to the constant 1 uniformly on the compact sets.

$$= \int_{\hat{G}} \hat{f}(\chi)\hat{g}(\eta\chi^{-1})\,\mathrm{d}\hat{\mu}(\chi) = (\hat{f} * \hat{g})(\eta).$$

■

Corollary 4.6 *The ring $\hat{A}$ of Fourier transforms of $L^1(G)$ functions coincides with the set of convolution products of functions in $L^2(\hat{G})$.*

Proof. Let $h \in L^1(G)$. Then $h = r|r|$, where

$$r(s) := \begin{cases} h(s)/\sqrt{|h(s)|} & \text{if } h(s) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and $r \in L^2(G)$. Corollary 4.5 shows that $\hat{h}$ is the convolution of the Fourier transforms of r and $|r|$, and both transforms belong to $L^2(\hat{G})$ (because $r \in L^2(G)$). On the other hand, the convolution of two functions in $L^2(\hat{G})$ has the form $\hat{f} * \hat{g}$ for suitable $f, g \in L^2(G)$, by Plancherel theorem. Hence it is the Fourier transform of fg, which is in $L^1(G)$. This proves that $\hat{f} * \hat{g} \in \hat{A}$. ■

Proposition 4.9 *Let $\hat{U}$ be any nonempty open subset of $\hat{G}$. Then there exists a nonzero function $\hat{f} \in \hat{A}$ with support in $\hat{U}$.*

Proof. The measure of $\hat{U}$ is positive. By inner regularity, there is a compact set $\hat{K} \subseteq \hat{U}$ with positive measure. There is an open neighborhood $\hat{V}$ of the identity in $\hat{G}$, such that $\hat{V}\hat{K} \subseteq \hat{U}$ (apply Lemma 1.1 to the locally compact group $\hat{G}$). We can further assume that $\hat{V}$ is finite measured (for example intersecting $\hat{V}$ with an open subset of a compact neighborhood of e). Define f' as the convolution of the characteristic functions of $\hat{K}$ and $\hat{V}$, respectively. Then it is in $\hat{A}$, by Corollary 4.6 (the characteristic functions are in $L^2(\hat{G})$), hence there exists a function f in $L^1(G)$ such that $f' = \hat{f}$. Its support is in $\hat{K}\hat{V}$, and hence in $\hat{U}$. Moreover,

$$\int_{\hat{G}} f'(\chi)\,\mathrm{d}\hat{\mu}(\chi) = \int_{\hat{G}} (\delta_{\hat{K}} * \delta_{\hat{V}})(\chi)\,\mathrm{d}\hat{\mu}(\chi) = \hat{\mu}(\hat{K})\hat{\mu}(\hat{V}) > 0,$$

proving that $\hat{f}$ is nonzero on a set of positive measure. ■

4.9. Proof of Pontryagin theorem

Corollary 4.3 states that $\alpha(G)$ is closed in $\hat{\hat{G}}$. Suppose that it is not $\hat{\hat{G}}$. Then the open set $\hat{\hat{U}} := \alpha(G)^c$ is not empty. By Proposition 4.9 (applied to the group $\hat{G}$) there exists a function $\phi \in L^1(\hat{G})$ whose Fourier transform $\hat{\phi}$ is nonzero and is supported in $\hat{\hat{U}}$. In particular it is zero in $\alpha(G)$. Let $\hat{\eta} \in \hat{\hat{G}}$. Then

$$\hat{\phi}(\hat{\eta}) = \int_{\hat{G}} \phi(\chi)\overline{\hat{\eta}(\chi)}\,\mathrm{d}\hat{\mu}(\chi).$$

The assumption that $\hat{\phi}$ vanishes on $\alpha(G)$ means that

$$0 = \hat{\phi}(\alpha(s)) = \int_{\hat{G}} \phi(\chi)\overline{\alpha(s)(\chi)}\,\mathrm{d}\hat{\mu}(\chi) = \int_{\hat{G}} \phi(\chi)\chi(s^{-1})\,\mathrm{d}\hat{\mu}(\chi) \qquad \forall s \in G.$$

By assumption $\phi \in L^1(\hat{G})$, thus $\phi\,\mathrm{d}\hat{\mu}$ is a complex Radon measure on $L^1(\hat{G})$ and the previous identity states that its Fourier transform is zero. By Proposition 4.7, this implies that $\phi = 0$ almost everywhere, so that $\hat{\phi} = 0$. This contradicts the assumptions on ϕ showing that the assumptions are impossible. This proves the claim.

4.10. Proof of the second part of Theorem 4.3

We complete the proof of Theorem 4.3, showing that the Fourier transform gives a bijection of $V^1(G)$ with $V^1(\hat{G})$. We already know that it is injective. Let $F \in V^1(\hat{G})$, and set

$$f(s) := \int_{\hat{G}} F(\chi)\chi(s)\,\mathrm{d}\hat{\mu}(\chi).$$

The identification of $\hat{\hat{G}}$ with G shows that $f(s) = \hat{F}(s^{-1})$. Hence $f \in V^1(G)$, by Corollary 4.2 (because $\hat{F}$ is in $V^1(\hat{\hat{G}})$ which is isomorphic to $V^1(G)$). We identify $\hat{\hat{G}}$ with G (via the map α). In particular, the dual measure $\mathrm{d}\hat{\hat{\mu}}$ (i.e., the Haar measure in $\hat{\hat{G}}$ which is able to produce the inversion Fourier transform for $\hat{G} \to \hat{\hat{G}}$), is a constant multiple of the Haar measure in G. This happens because both are Haar measures, which are uniquely determined up to a positive constant. In a formula, we have

$$\mathrm{d}\hat{\hat{\mu}} = c\,\mathrm{d}\mu$$

for some positive constant c. Then, by the Fourier inversion formula for F we get

$$\begin{aligned} F(\chi) &= \int_{\hat{\hat{G}}} \hat{F}(\alpha(s))\alpha(s)(\chi)\,\mathrm{d}\hat{\hat{\mu}}(\alpha(s)) \\ &= c\int_G \hat{F}(s)\chi(s)\,\mathrm{d}\mu(s) \\ &= c\int_G f(s^{-1})\chi(s)\,\mathrm{d}\mu(s) \\ &= \int_G cf(s)\overline{\chi(s)}\,\mathrm{d}\mu(s) \end{aligned}$$

which shows that F is the Fourier transform of cf. Hence $V^1(\hat{G})$ is contained into the image of $V^1(G)$, and the claim is proved. Finally, we can prove that $c = 1$ by applying this computation to functions in $V^1(G)$ and comparing the original Fourier transform formula and the one we get using $\hat{\hat{G}}$. This proves that measures $\mathrm{d}\mu$ and $\mathrm{d}\hat{\mu}$ are mutually dual.

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