Analytic Number Theory: Homework 1 (2019)

(1) Liouville's function λ is defined via $\lambda(n) = (-1)^{\Omega(n)}$, for every $n \in \mathbb{N}$. Prove that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise.} \end{cases}$$

Hint: recall that $\Omega(n)$ is the number of prime divisors of n, *multiplicity included.* In other words, $\Omega(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = a_1 + a_2 + \cdots + a_k$.

(2) Recall that $M(x) := \sum_{n \le x} \mu(n)$, and let $L(x) := \sum_{n \le x} \lambda(n)$. Notice that they are similar since $\omega(n) = \Omega(n)$ for squarefree integers so that $\mu(n) = \lambda(n)$ for them. Prove that

$$L(x) = \sum_{d^2 \le x} M\Big(\frac{x}{d^2}\Big) \qquad \text{and that} \qquad M(x) = \sum_{d^2 \le x} \mu(d) L\Big(\frac{x}{d^2}\Big).$$

(3) For $r \ge 1$, let $J_r(n)$ be the number of r-uples of integers a_j with $1 \le a_j \le n$ for $j = 1, \dots, r$, and $(a_1, \dots, a_r, n) = 1$. Prove that

$$J_r(n) = n^r \prod_{p|n} \Big(1 - \frac{1}{p^r}\Big).$$

This is *Jordan's totient* function. When $r = 1$ it coincides with Euler's function φ.
Hint: First prove that J_r is multiplicative, then compute $J_r(p^k)$.

(4) Let $M(x)$ and $L(x)$ be as in Ex. 2. Prove that

$$\forall A > 0, \quad L(x) \ll_A \frac{x}{\log^A x} \qquad \iff \qquad \forall A > 0, \quad M(x) \ll_A \frac{x}{\log^A x}$$

when $x \to \infty$.
Hint: Use identities in Exercise 2.

(5) For every integer n, let $\mathrm{rad}(n)$ be the product of all distinct primes dividing n (with $\mathrm{rad}(1) := 1$). It is called the *radical* of n. Prove that

$$\sum_{n \le x} \log(\mathrm{rad}(n)) = x \log x + O(x).$$

Hint: recall the identity $\log = 1 * \Lambda$, so that $\sum_{n \le x} \log(\mathrm{rad}(n)) = \sum_{n \le x} \sum_{d | \mathrm{rad}(n)} \Lambda(d)$.

(6) Using only Mertens' result (PNT not allowed here), prove that

$$\sum_{p \le x} \frac{\log^2 p}{p} = \frac{1}{2} \log^2 x + O(\log x).$$

(7) Let $\alpha \ge 1$. Let F_α be the Dirichlet series

$$F_\alpha(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\lceil n^\alpha \rceil^s}$$

where $\lceil x \rceil := \inf\{n \in \mathbb{Z} \colon x \le n\}$. Prove that for this series $\sigma_c = 0$ and $\sigma_a = 1/\alpha$.

(8) Let $g\colon \mathbb{N}\to\mathbb{C}$ be the arithmetical function giving the Dirichlet coefficients of the Dirichlet series $1/\zeta(2s)$.

a) Prove that $g(k^2)=\mu(k)$ for every integer k, and that $g(k)=0$ when k is not a square.

b) Prove that $|\mu(n)|=\sum_{d|n} g(d)$. Deduce that

$$\sum_{n\le x}|\mu(n)| = \sum_{\substack{n,m\\ nm\le x}} g(m) = \sum_{\substack{n,k\\ nk^2\le x}}\mu(k) = \sum_{k\le\sqrt{x}}\mu(k)\sum_{n\le x/k^2} 1 = \sum_{k\le\sqrt{x}}\mu(k)\left\lfloor\frac{x}{k^2}\right\rfloor.$$

c) Use the previous equality to deduce that

$$\sharp\{n\in\mathbb{N}\colon n\le x,\ n \text{ is squarefree}\} = \sum_{n\le x}|\mu(n)| = \frac{x}{\zeta(2)}+O(\sqrt{x}).$$

d) Deduce that for every $\theta>1/2$ there exists $x_0=x_0(\theta)$ such that

$$\{n\in\mathbb{N}\colon n\in[x,x+x^\theta],\ n \text{ is squarefree}\}\neq\emptyset \qquad \forall x\ge x_0.$$

(9) The previous exercise shows that if $\theta>1/2$, there is a squarefree integer in each interval $[x,x+x^\theta]$, if x is large enough. That argument is rooted in the general formula in Ex.8.c, and therefore it does not allow to get the claim for any $\theta\le 1/2$. The next steps will prove the claim also for smaller values of θ.

a) Let $h>0$, and observe that

$$\sharp\{n\in\mathbb{N}\colon n\in(x,x+h],\ n \text{ is squarefree}\} = \sum_{n\in(x,x+h]}|\mu(n)|.$$

b) As in Ex.8b, prove that

$$\sum_{n\in(x,x+h]}|\mu(n)| = \sum_{x<nk^2\le x+h}\mu(k),$$

and split the range to get

$$\sum_{n\in(x,x+h]}|\mu(n)| = \sum_{k\le x^{1/3}}\mu(k)\sum_{x<nk^2\le x+h}1 + \sum_{k>x^{1/3}}\mu(k)\sum_{x<nk^2\le x+h}1 =: S_1+S_2.$$

c) Prove that

$$S_1=\sum_{k\le x^{1/3}}\mu(k)\Big(\left\lfloor\frac{x+h}{k^2}\right\rfloor-\left\lfloor\frac{x}{k^2}\right\rfloor\Big) = \frac{h}{\zeta(2)}+O(h/x^{1/3})+O(x^{1/3}).$$

d) Prove that

$$\sharp\{(n,k)\in\mathbb{N}^2\colon x<nk^2\le x+h,\ k>x^{1/3}\}\le\sharp\Big\{(n,k)\in\mathbb{N}^2\colon n<x^{1/3}+\tfrac{h}{x^{2/3}},\ \sqrt{\tfrac{x}{n}}<k\le\sqrt{\tfrac{x+h}{n}}\Big\}.$$

Under the assumption $h\le\sqrt{x}$, prove that the interval $(\sqrt{\frac{x}{n}},\sqrt{\frac{x+h}{n}}]$ contains one integer, at most.

e) Assume $h\le\sqrt{x}$ and deduce that

$$S_2=O(x^{1/3}).$$

e) Hence

$$\sum_{n\in(x,x+h]}|\mu(n)| = \frac{h}{\zeta(2)}+O(x^{1/3})$$

and conclude that if $\theta>1/3$, then there is a squarefree integer in each interval $[x,x+x^\theta]$, if x is large enough.

(10) Let r be any positive integer. Let $f(n) := \frac{J_r(n)}{n^r}$ (see Ex.3 for the definition of J_r).

a) Prove that f is multiplicative.

b) Let $F(s) := \sum_{n=1}^{\infty} f(n)/n^s$ be the Dirichlet series associated with $f(n)$, and let $H(s)$ be the complex function defined in such a way that
$$F(s) = H(s)\zeta(s).$$
Prove that $H(s)$ may be written as Euler product and as Dirichlet series.

c) Working out an explicit expression for the Euler product of $H(s)$, prove that it converges absolutely for $\mathrm{Re}(s) > 1 - r$.

d) Let $h(n)$ be the sequence of numbers such that $H(s) = \sum_{n=1}^{\infty} h(n)/n^s$. From Step b) one gets that
$$f(n) = \sum_{m|n} h(m).$$
Deduce that
$$\sum_{n\le x} f(n) = \sum_{\substack{m,n\\ mn\le x}} h(m) = \sum_{m\le x} h(m) \sum_{n\le \frac{x}{m}} 1$$
so that
$$\sum_{n\le x} f(n) = \Big(\sum_{m\le x} \frac{h(m)}{m}\Big)x + O\Big(\sum_{m\le x} |h(m)|\Big).$$

e) Use Step c) to prove that
$$\sum_{m>x} \frac{h(m)}{m} \ll_\eta x^{-r+\eta} \qquad \text{and} \qquad \sum_{m\le x} |h(m)| \ll_\eta x^{1-r+\eta},$$
for every $\eta > 0$.

f) Use Steps d) and e) to deduce that
$$\sum_{n\le x} f(n) = \Big(H(1) - \sum_{m>x} \frac{h(m)}{m}\Big)x + O_\eta(x^{1-r+\eta}) = H(1)x + O_\eta(x^{1-r+\eta}).$$

g) From the representation of $H(s)$ as Euler product deduce that $H(1) = \prod_p (1 - \frac{1}{p^{r+1}}) = 1/\zeta(r+1)$.

h) Use Steps f) and g) and the partial summation formula to deduce that
$$\sum_{n\le x} J_r(n) = \frac{x^{r+1}}{(r+1)\zeta(r+1)} + O_\eta(x^{1+\eta}).$$

(11) Set $a \in \mathbb{C}$, and let $f_a\colon \mathbb{N} \to \mathbb{C}$ be the function with
$$f_a(1) := 1 \qquad \text{and} \qquad f_a(p_1^{\nu_1} p_2^{\nu_2} \cdots p_k^{\nu_k}) := (\nu_1\nu_2\cdots\nu_k)^a.$$
Following Steps a–g in Ex. 10 find a formula for $\sum_{n\le x} f_a(n)$.

(12) Let $\{p_n\}_{n\in\mathbb{N}}$ be the sequence of prime numbers, so that $p_1 = 2$, $p_2 = 3$, and so on. For every $n \in \mathbb{N}$, let $a_n := p_{p_n}$. Prove that
$$\sum_{n\colon a_n\le x} 1 = \#\{n\colon a_n \le x\} \sim \frac{x}{\log^2 x}.$$
Hint: Use PNT to deduce that $p_n \sim n\log n$, then deduce the asymptotic for a_n and compute.