# **Analytic Number Theory: Homework 1** (2021/22)

(1) Let $f\colon \mathbb{N} \to \mathbb{C}$ be an arithmetic function.

a) Prove that $f$ is $*$-invertible (i.e., invertible with respect to the Dirichlet $*$ product) if and only if $f(1) \neq 0$.

b) Prove that if $f$ is multiplicative then $f$ is $*$-invertible and $f^{-1}$ is multiplicative, too.

c) Show with some example that if $f$ is completely multiplicative then in general $f^{-1}$ is *not* completely multiplicative.

d) Characterize all functions $f$ which are invertible and such that both $f$ and $f^{-1}$ are completely multiplicative.

**Hint:** recall that the function $f$ which is identically zero satisfies the multiplicative property $f(mn) = f(m)f(n)$ for every $m, n = 1$, but it is *not* a multiplicative function, by definition.

(2) Using only Mertens' result (i.e., Eq. 1.7.7 in Notes: PNT not allowed here), prove that

$$\sum_{p\leq x} \frac{\log^k p}{p} = \frac{1}{k}\log^k x + O(\log^{k-1} x)$$

for every $k \in \mathbb{N}$, $k \geq 1$.

(3) Prove the identity

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

(4) Let $\alpha \geq 1$. Let $F_\alpha$ be the Dirichlet series

$$F_\alpha(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\lceil n^\alpha \rceil^s}$$

where $\lceil x \rceil := \inf\{n \in \mathbb{Z}\colon x \leq n\}$. Prove that for this series $\sigma_c = 0$ and $\sigma_a = 1/\alpha$.

(5) Suppose that the Dirichlet series $F(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges in some right half-plane $H$ of the complex plane. Suppose that some $a_n$ is not zero and let $\bar{n} := \min\{n\colon a_n \neq 0\}$. Prove that

$$F(s) = \frac{a_{\bar{n}} + o(1)}{\bar{n}^s} \qquad \text{as } \mathrm{Re}(s) \to +\infty,\ s \in H.$$

Use this remark to prove that if $F(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ and $G(s) := \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ are two Dirichlet series converging in some domain (not necessarily the same) define the same function in the common domain, then $a_n = b_n$ for every $n$.

(6) Let $r$ be any positive integer. An integer $n$ is called *$r$-power free* when 1 is the unique $r$-power dividing $n$. Let $\delta_r$ be the characteristic function of $r$-power free integers (thus, $\delta_r(n) = 1$ when $n$ is $r$-power free, 0 otherwise).

a) Prove that $\delta_r$ is multiplicative.

b) Let $F(s) := \sum_{n=1}^{\infty} \delta_r(n)/n^s$ be the Dirichlet series associated with $\delta_r(n)$, and let $H$ be the complex function defined in such a way that

$$F(s) = H(s)\zeta(s).$$

Prove that $H$ may be written both as Euler product and as Dirichlet series.

c) Working out an explicit expression for the Euler product of $H$, prove that it converges absolutely for $\mathrm{Re}(s) > 1/r$.

d) Let $h$ be the arithmetical function such that $H(s) = \sum_{n=1}^{\infty} h(n)/n^s$. From Step b) one gets that

$$\delta_r(n) = \sum_{m|n} h(m).$$

Deduce that

$$\sum_{n\le x} \delta_r(n) = \sum_{\substack{m,n\\ mn\le x}} h(m) = \sum_{m\le x} h(m) \sum_{n\le \frac{x}{m}} 1$$

so that

$$\sum_{n\le x} \delta_r(n) = \Big(\sum_{m\le x} \frac{h(m)}{m}\Big)x + O\Big(\sum_{m\le x} |h(m)|\Big).$$

e) Use Step c) to prove that

$$\sum_{m>x} \frac{h(m)}{m} \ll_\eta x^{-1+1/r+\eta} \qquad \text{and} \qquad \sum_{m\le x} |h(m)| \ll_\eta x^{1/r+\eta},$$

for every $\eta > 0$.

f) Use Steps d) and e) to deduce that

$$\sum_{n\le x} \delta_r(n) = \Big(H(1) - \sum_{m>x} \frac{h(m)}{m}\Big)x + O_\eta(x^{1/r+\eta}) = H(1)x + O_\eta(x^{1/r+\eta}).$$

g) From the representation of $H$ as Euler product deduce that $H(1) = \prod_p (1-\frac{1}{p^r}) = 1/\zeta(r)$.

(7) Set $a \in \mathbb{C}$, and let $f_a\colon \mathbb{N} \to \mathbb{C}$ be the function with

$$f_a(1) := 1 \qquad \text{and} \qquad f_a(p_1^{\nu_1} p_2^{\nu_2} \cdots p_k^{\nu_k}) := (\nu_1\nu_2\cdots\nu_k)^a.$$

Following Steps a–g in Ex. 6 find a formula for $\sum_{n\le x} f_a(n)$.

(8) For every integer $n$, let $\mathrm{rad}(n)$ be the product of all distinct primes dividing $n$ (with $\mathrm{rad}(1) := 1$). It is called the *radical* of $n$. Prove that

$$\sum_{n\le x} \mathrm{rad}(n) = \frac{c}{2}x^2 + O_\eta(x^{3/2+\eta})$$

for every $\eta > 0$, with $c := \prod_p (1 - \frac{1}{p(p+1)})$.

**Hint:** Use the same technology of Exercises 6/7, but with $\zeta(s-1)$ in place of $\zeta(s)$ (Why?).

(9) Devise a method to compute the correct value for the first three digits of the value of $c := \prod_p (1 - \frac{1}{p(p+1)})$.

**Hint:** Write $c$ as $\prod_{p<N} (1 - \frac{1}{p(p+1)}) \cdot \prod_{p\ge N} (1 - \frac{1}{p(p+1)})$ and estimate the second factor as $1 + R(N)$ with an explicit (and easily computed) function $R(N)$ decreasing to 0. Then fix $N$, compute the first factor, and use the estimation for the second factor in order to compute the maximum error between the true value of $c$ and the value for the first

factor. Adjust $N$ in order to have an error lower than $10^{-3}$. For this exercise you can use a software to perform the computations.

(10) Recall that

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s \int_1^{-\infty} \frac{B_1(x)}{x^{s+1}}\, \mathrm{d}x \qquad \mathrm{Re}(s) > 0.$$

a) Deduce that

$$|\zeta(s)| \geq \frac{|s+1|}{2|s-1|} - \frac{|s|}{2\,\mathrm{Re}(s)} \qquad \mathrm{Re}(s) > 0.$$

b) Set $x = u + iv$ with $u, v \in \mathbb{R}$ in previous formula, and deduce that

$$\text{if} \quad \begin{cases} v^4 + v^2(u-1)^2 - 4u^3 < 0 \\ u > 0 \end{cases} \qquad \text{then} \quad \zeta(u+iv) \neq 0.$$

Make a plot of this zero-free region.

**Optional:** Repeat the exercise using the identities

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - \frac{s}{12} - \frac{1}{2}s(s+1) \int_1^{-\infty} \frac{B_2(x)}{x^{s+2}}\, \mathrm{d}x \qquad \mathrm{Re}(s) > -1$$

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - \frac{s}{12} + \frac{1}{6}s(s+1)(s+2) \int_1^{-\infty} \frac{B_3(x)}{x^{s+3}}\, \mathrm{d}x \qquad \mathrm{Re}(s) > -2:$$

in these cases the resulting formulas are again of algebraic type in $u, v$ but quite complicated. In order to produce a graph of the zero free region, a software is probably necessary.