Analytic Number Theory: Homework 1 (2023/24)

(1) Let $f\colon \mathbb{N} \to \mathbb{C}$ be an arithmetic function.

a) Prove that f is $*$-invertible (i.e., invertible with respect to the Dirichlet $*$ product) if and only if $f(1) \neq 0$.

b) Prove that if f is multiplicative then f is $*$-invertible and f^{-1} is multiplicative, too.

c) Show with some example that if f is completely multiplicative then in general f^{-1} is *not* completely multiplicative.

d) Describe all functions f which are invertible and such that both f and f^{-1} are completely multiplicative.

(2) Let $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) := \frac{\sin(\pi z)}{\pi z}$ (extended by continuity at $z = 0$). Let

$$g(z) = \prod_{n=1}^{\infty} f(z/n).$$

Prove that the product converges absolutely and uniformly on compact sets of $\mathbb{C}$, so that g is an entire function. Prove that g is an even function, that $g(z) = 0$ if and only if $z \in \mathbb{Z}$ and that the order of $n \in \mathbb{N}$ as zero for g is $d(n) = \sum_{m|n} 1$.

(3) Let $g(s) := \sum_{n=0}^{\infty}[\zeta(n+s) - 1]$.

- Prove that the series converges absolutely and uniformly on compact subsets of $H_1 := \{s \in \mathbb{C}\colon\ \mathrm{Re}(s) > 1\}$, so that g is a holomorphic function in H_1.
- Prove that g can be extend as a meromorphic function on the full complex plane.
- Prove that the resulting function $g\colon \mathbb{C} \to \mathbb{C}$ has a simple pole at $s = 1$, at $s = 0$ and at $s = -n$ for every $n \in \mathbb{N}$, and no other poles.
- Prove that $g(2) = 1$.

(4) Let $g(s) := \prod_{n=0}^{\infty} \zeta(n+s)$.

- Prove that the product converges absolutely and uniformly on compact subsets of $H_1 := \{s \in \mathbb{C}\colon\ \mathrm{Re}(s) > 1\}$, so that g is a holomorphic function in H_1.
- Prove that g can be extend as a meromorphic function on the full complex plane.
- Prove that the resulting function $g\colon \mathbb{C} \to \mathbb{C}$ has a simple pole at $s = 1$, $s = 0$ and $s = -1$ and no other poles.

hint: for the last claim, recall that if $s \in (-\infty, 0])$, then $\zeta(s) = 0$, if and only if $s = -2n$ with $n \in \mathbb{N} = \{1, 2, 3 \ldots\}$.

(5) Using only Mertens' result (i.e., Eq. 1.7.7 in Notes: PNT not allowed here), prove that

$$\sum_{p \le x} \frac{\log^k p}{p} = \frac{1}{k} \log^k x + O(\log^{k-1} x)$$

for every $k \in \mathbb{N}$, $k \ge 1$.

(6) Let μ^{*k} be the $*$-product of k copies of μ (so that $\mu^{*2} = \mu * \mu$, $\mu^{*3} = \mu * \mu * \mu$), with $\mu^{*0} := \mathbf{1}$. Prove that μ^{*k} is multiplicative and that

$$\mu^{*k}(p^a) = \begin{cases} (-1)^a \binom{k}{a} & \text{when } 0 \le a \le k \\ 0 & \text{otherwise.} \end{cases}$$

(7) Let $\Omega\colon \mathbb{N} \to \mathbb{C}$ be the function with

$$\Omega(n) := \#\{p \text{ prime},\ k \geq 1 \colon p^k | n\}$$

(so that $\Omega(8) = 3$, while $\omega(8) = 1$). *Liouville's function* λ is the arithmetic function such that $\lambda(n) := (-1)^{\Omega(n)}$ for every n. Prove that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise.} \end{cases}$$

Deduce that for every $x > 0$,

$$\sum_{n \leq x} \lambda(n) \left\lfloor \frac{x}{n} \right\rfloor = \lfloor \sqrt{x} \rfloor .$$

(8) Prove the identities

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}, \qquad \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}, \qquad \sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^s} = \frac{\zeta(s)}{\zeta(2s)}.$$

(9) Prove that for every couple of arbitrarily fixed numbers $\tau, \nu \in \mathbb{C}$, one has

$$\sum_{n=1}^{\infty} \frac{\sigma_\tau(n)\sigma_\nu(n)}{n^s} = \frac{\zeta(s)\zeta(s-\tau)\zeta(s-\nu)\zeta(s-\tau-\nu)}{\zeta(2s-\tau-\nu)}.$$

(10) Let r be any positive integer. An integer n is called *r-power free* when 1 is the unique r-power dividing n. Let δ_r be the characteristic function of r-power free integers (thus, $\delta_r(n) = 1$ when n is r-power free, 0 otherwise).

a) Prove that δ_r is multiplicative.

b) Let $F(s) := \sum_{n=1}^{\infty} \delta_r(n)/n^s$ be the Dirichlet series associated with δ_r, and let H be the complex function defined in such a way that

$$F(s) = H(s)\zeta(s).$$

Prove that H may be written both as Euler product and as Dirichlet series.

c) Working out an explicit expression for the Euler product of H, prove that it converges absolutely for $\mathrm{Re}(s) > 1/r$.

d) Let h be the arithmetical function such that $H(s) = \sum_{n=1}^{\infty} h(n)/n^s$. From Step b) one gets that

$$\delta_r(n) = \sum_{m|n} h(m).$$

Deduce that

$$\sum_{n \leq x} \delta_r(n) = \Big(\sum_{m \leq x} \frac{h(m)}{m}\Big)x + O\Big(\sum_{m \leq x} |h(m)|\Big).$$

e) Use Step c) to prove that

$$\sum_{m > x} \frac{h(m)}{m} \ll_\varepsilon x^{-1+1/r+\varepsilon} \qquad \text{and} \qquad \sum_{m \leq x} |h(m)| \ll_\varepsilon x^{1/r+\varepsilon},$$

for every $\varepsilon > 0$.

f) Use Steps d) and e) to deduce that

$$\sum_{n\leq x} \delta_r(n) = H(1)x + O_\varepsilon(x^{1/r+\varepsilon}).$$

g) From the representation of H as Euler product deduce that $H(1) = \prod_p (1-\frac{1}{p^r}) = 1/\zeta(r)$.
Remark: the case $r = 2$ is quite famous; it proves that the number of *squarefree* integers which are $\leq x$ is $= \frac{6}{\pi^2}x + O_\varepsilon(x^{1/2+\varepsilon})$.

(11) For every positive integer n let $f(n) := \sharp\{x \in \mathbb{Z}/n\mathbb{Z}\colon x^2 = 1 \pmod{n}\}$, i.e. the number of solutions of the equation $x^2 = 1$ in the ring $\mathbb{Z}/n\mathbb{Z}$.

a) Prove that f is multiplicative.

b) Prove that $f(2) = 1$, $f(2^2) = 2$, $f(2^k) = 4$ for every $k \geq 3$.

c) Prove that $f(p^k) = 2$ when p is odd, for every k.

d) Let $F(s) := \sum_{n=1}^{\infty} f(n)/n^s$ be the Dirichlet series associated with f, and let $H(s)$ be the complex function defined in such a way that

$$F(s) = H(s)\zeta^2(s).$$

Working out an explicit expression for the Euler product of $H(s)$, prove that it converges absolutely for $\mathrm{Re}(s) > 1/2$.

e) Let h be the arithmetical function such that $H(s) = \sum_{n=1}^{\infty} h(n)/n^s$; imitating the proof of the previous step, prove that this series converges absolutely for $\mathrm{Re}(s) > 1/2$.

f) Recall that $d(n)$ is the sequence of Dirichlet's coefficients of $\zeta^2(s)$, so that from Step d) one gets that

$$f(n) = \sum_{m|n} h(m)d\Big(\frac{n}{m}\Big).$$

Deduce that

$$\sum_{n\leq x} f(n) = \sum_{\substack{m,n\\ mn\leq x}} h(m)d(n) = \sum_{m\leq x} h(m) \sum_{n\leq \frac{x}{m}} d(n).$$

g) Use Dirichlet's result on the mean value of the divisor function, in its easier form claiming that $\sum_{n\leq y} d(n) = y\log y + O(y)$, to deduce that

$$\sum_{n\leq x} f(n) = \Big(\sum_{m\leq x} \frac{h(m)}{m}\Big)x\log x - x\sum_{m\leq x} \frac{h(m)\log m}{m} + O\Big(x\sum_{m\leq x} \frac{|h(m)|}{m}\Big).$$

h) Use Steps e) and g) to deduce that

$$\sum_{n\leq x} f(n) = \Big(\sum_{m\leq x} \frac{h(m)}{m}\Big)x\log x + O(x).$$

i) Use Step e) to deduce that for every $\varepsilon > 0$,

$$\Big|\sum_{m>x} \frac{h(m)}{m}\Big| \leq \sum_{m>x} \frac{|h(m)|}{m} \ll_\varepsilon \frac{x^\varepsilon}{x^{1/2}}$$

so that Step h) gives

$$\sum_{n\leq x} f(n) = H(1)x\log x + O(x).$$

l) From the representation of H as Euler product deduce that $H(1) = \prod_p(1 - \frac{1}{p^2}) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$, and conclude that

$$\sum_{n\le x} f(n) = \frac{6}{\pi^2}x\log x + O(x).$$

Hint: For Steps b) and c) recall that $(\mathbb{Z}/p^k\mathbb{Z})^*$ is cyclic when p is odd, for every k and that its order is always even, while $(\mathbb{Z}/2^k\mathbb{Z})^*$ for $k \ge 3$ is the product of two cyclic groups of order respectively 2 and 2^{k-2}.
Remark: Using the full strength of Dirichlet's bound (i.e. $\sum_{n\le y} d(n) = y\log y + (2\gamma - 1)y + O(\sqrt{y})$), the previous argument gives

$$\sum_{n\le x} f(n) = H(1)x\log x + ((2\gamma-1)H(1) + H'(1))x + O_\varepsilon(x^{1/2+\varepsilon}).$$

(12) Let $f\colon \mathbb{N} \to \mathbb{C}$ be the function with $f(n) = 2^{\omega(n)}$ for every n. Following the argument in Ex. 11 find a formula for $\sum_{n\le x} f(n)$.

(13) Devise a method to compute the correct value for the first three digits of the number

$$c := \prod_p \Big(1 + \frac{p^2-1}{p(p+1)(p^2+1)}\Big).$$

Hint: Write c as $\prod_{p<N}(1 + \frac{p^2-1}{p(p+1)(p^2+1)}) \cdot \prod_{p\ge N}(1\frac{p^2-1}{p(p+1)(p^2+1)})$ and estimate the second factor as $1 + R(N)$ with an explicit (and easily computed) function $R(N)$ decreasing to 0. Then fix N, compute the first factor, and use the estimation for the second factor in order to compute the maximum error between the true value of c and the value for the first factor. Adjust N in order to have an error lower than 10^{-3}. For this exercise you can use a software to perform the computations.