Analytic Number Theory: Homework 2

(1) Let $f\colon \mathbb{N} \to \mathbb{C}$ be a multiplicative function which is supported on squarefree integers. Suppose that there exists $\alpha > 0$ such that

$$f(p) = 1 + O(p^{-\alpha}) \qquad \text{as} \qquad p \to \infty.$$

Let $\theta := \min(1/2, \alpha)$ and let $c := \prod_p \big(1 + f(p)/p\big)\big(1 - 1/p\big)$. Prove that

$$\sum_{n \le x} f(n) = c\,x + O_\eta(x^{1-\theta+\eta}) \qquad \text{as} \qquad x \to \infty,$$

for every $\eta > 0$.
Hint: Let $F(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$, write this function as Euler product and follow Exercise 10 in Homework 1.

(2) Let $f\colon \mathbb{N} \to \mathbb{C}$ be a multiplicative function which is supported on squarefree integers. Suppose that there exists $\alpha > 0$ such that

$$f(p) = \frac{1}{p}(1 + O(p^{-\alpha})) \qquad \text{as} \qquad p \to \infty.$$

Let $\theta := \min(1/2, \alpha)$ and let $c := \prod_p \big(1 + f(p)\big)\big(1 - 1/p\big)$. Prove that there exists a constant d such that

$$\sum_{n \le x} f(n) = c \log x + d + O_\eta(x^{-\theta+\eta}) \qquad \text{as} \qquad x \to \infty,$$

for every $\eta > 0$.
Hint: Notice that the function $g\colon \mathbb{N} \to \mathbb{C}$ with $g(n) = nf(n)$ for every n satisfies the hypothesis in Exercise 1. Write $\sum_{n \le x} f(n)$ as $\sum_{n \le x} nf(n) \cdot \frac{1}{n}$ and apply the partial summation formula.
Notice that $c = 1$ when $f(n) = \mu(n)^2/\varphi(n)$ (hence this computation improves Lemma 3.1 of the notes).

(3) Prove that for every couple of coprime integers a, q, there is a constant $\gamma_{a,q}$ such that

$$\prod_{\substack{p \le x \\ p = a \ (\mathrm{mod}\ q)}} \left(1 - \frac{1}{p}\right)^{-1} \sim e^{\gamma_{a,q}} (\log x)^{1/\varphi(q)} \qquad \text{as} \qquad x \to +\infty.$$

Prove that

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} \gamma_{a,q} = \gamma + \sum_{p \mid q} \log(1 - 1/p),$$

where γ is Euler–Mascheroni constant.
Hint: Use what we have proved for prime numbers in linear progressions.

(4) Let χ be a real character modulo q, $\chi \ne \chi_0$. The following steps give an alternative proof of the fact that $L(1, \chi) \ne 0$.

a) Let $f(n) := (1 * \chi)(n) = \sum_{d \mid n} \chi(d)$. Prove that f is multiplicative, that $f(n) \ge 0$ for every n, and that $f(n^2) \ge 1$ for every n.

b) Let $F(z) := \sum_{n=1}^{\infty} f(n)e^{-n/z}$. Prove that F is well defined for all $z \ge 1$, and that

$$F(z) \ge \sum_{n \le z/2} f(n)e^{-n/z} \gg \sum_{n \le z/2} f(n) \gg \sqrt{z} \qquad \text{as } z \to \infty.$$

c) Show that

$$F(z)=\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\chi(n)e^{-mn/z}=\sum_{n=1}^{+\infty}\frac{\chi(n)}{e^{n/z}-1}=z\sum_{n=1}^{+\infty}\frac{\chi(n)}{n}+\sum_{n=1}^{\infty}\chi(n)\Big[\frac{1}{e^{n/z}-1}-\frac{z}{n}\Big].$$

d) Apply the partial summation formula to conclude that

$$\Big|\sum_{n=1}^{+\infty}\chi(n)\Big[\frac{1}{e^{n/z}-1}-\frac{z}{n}\Big]\Big|\ll_q\int_1^{+\infty}\Big|\Big(\frac{1}{e^{x/z}-1}-\frac{z}{x}\Big)'\Big|\,\mathrm{d}x\ll_q 1\qquad\text{as } z\to\infty.$$

e) From Steps b), c) and d) deduce that

$$\sqrt{z}\ll F(z)=zL(1,\chi)+O_q(1)\qquad\text{as } z\to\infty,$$

and conclude that $L(1,\chi)\neq 0$.

Hint: For Step d recall that $A(x):=\sum_{n\le x}\chi(n)$ is a bounded function, and prove that $\frac{1}{e^y-1}-\frac{1}{y}$ is a negative and increasing function in $y\ge 0$. Thus $\int_1^{+\infty}|(\frac{1}{e^{x/z}-1}-\frac{z}{x})'|\,\mathrm{d}x=\int_1^{+\infty}(\frac{1}{e^{x/z}-1}-\frac{z}{x})'\,\mathrm{d}x=z-\frac{1}{e^{1/z}-1}$ which stays bounded as $z\to\infty$ (Prove it!).

(5) Let G be a finite group with n elements.

a) Suppose G cyclic, and let g be a generator. Let ζ be any primitive nth root of the unity. For each $j=1,\dots,n$ let $\chi_j(g):=\zeta^j$ and extend χ_j multiplicatively to G (i.e. set $\chi_j(g^2):=\chi_j(g)^2$,$\chi_j(g^3):=\chi_j(g)^3$,$\dots$). Prove that each χ_j is a character for G and that $\{\chi_j\}_{j=1}^n$ is the full set of characters.

b) Let G and χ_j be as in Step a. Show that $\hat{G}$ is cyclic, and that χ_j is a generator of $\hat{G}$ if and only if j is coprime with n.

c) Let G_1,G_2 be finite abelian groups. Prove that $\widehat{G_1\times G_2}$ and $\hat{G}_1\times\hat{G}_2$ are (canonically) isomorphic. Deduce that $\hat{G}$ is (not canonically) isomorphic to G, for every finite abelian group G.

(6) Fix $q\in\mathbb{N}$. Let $N_2(q)$ be the number of Dirichlet character modulo q which are real. Prove that

$$N_2(q)=2^{\omega_2(q)}\cdot 2^{\omega_{\mathrm{odd}}(q)}\qquad\text{where}\qquad \omega_2(q):=\begin{cases}0 & \text{if } 2\nmid q\\ 0 & \text{if } 2\|q\\ 1 & \text{if } 2^2\|q\\ 2 & \text{if } 2^3|q\end{cases}$$

and $\omega_{\mathrm{odd}}(q)$ is the number of odd primes dividing q.

Hint: Use Ex. 5 to conclude that $N_2(q)$ is also the number of solutions of $x^2=1$ in $(\mathbb{Z}/q\mathbb{Z})^*$. Then recall the cyclic decomposition of $(\mathbb{Z}/p^\nu\mathbb{Z})^*$ when p is a prime (see for example Th. 2.4.4.6 p. 33, in Fine and Rosenberg *Number theory*, Birkhäuser 2007).

(7) Let q be an odd prime. Let $G:=(\mathbb{Z}/q\mathbb{Z})^*$.

a) Prove that every finite group which is contained in the multiplicative group of a field is always cyclic. Deduce that G is cyclic.

b) Prove that there are only two real Dirichlet characters modulo q, i.e. characters χ such that $\chi^2=\chi_0$.

c) Let $\chi_{2,q}$ be the real character modulo q which is not χ_0: compute the values of $\chi_{2,q}(n)$ for $n=1,\dots,q$ when $q=3,5,7,11,13,17$.

d) Using the results in Step c show that

$$L(1,\chi_{2,q}) = \sum_{n=1}^{+\infty} \frac{\chi_{2,q}(n)}{n} > 0$$

for $q = 3, 5, 7, 11, 13, 17$.

Hint: For Step d, use Step c to prove that $\sum_{k=1}^{q} \frac{\chi_{2,q}(k)}{k+q\ell} > 0$ for every $\ell \in \mathbb{N}$, then deduce the claim.

(8) Let q be an odd prime. This exercise gives two classical formulas for $L(1,\chi_{2,q})$, which are due to Dirichlet.

a) Prove that

$$\frac{1}{q}\sum_{\ell=0}^{q-1} e^{\frac{2\pi i k}{q}\ell} = \begin{cases} 1 & \text{when } k = 0 \pmod q \\ 0 & \text{otherwise} \end{cases}$$

This identity is a special case of the orthogonal relation for the characters of the group $\mathbb{Z}/q\mathbb{Z}$, but a direct proof is probably easier.

b) Let $z \in \mathbb{C}$, with $|z| = 1$ and $z \neq 1$. Prove that

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{z^n}{n} = -\log(1-z)$$

where log is the holomorphic extension to $\mathbb{C}\backslash\{(-\infty,0]\}$ of the real logarithm. (If you are not familiar with complex analysis, note that actually we will need this equality only for the case where z is a qth roots of 1, and that in this case the conclusion follows from an application of the summation by parts formula).

c) Let $\chi_{2,q}$ be the real and non trivial character modulo q. Note that

$$L(1,\chi_{2,q}) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{\chi_{2,q}(n)}{n} = \lim_{N\to\infty} \sum_{a=1}^{q} \chi_{2,q}(a) \sum_{\substack{n=1 \\ n=a\,(q)}}^{N} \frac{1}{n}$$

$$= \frac{1}{q} \lim_{N\to\infty} \sum_{\ell=0}^{q-1}\sum_{a=1}^{q} \chi_{2,q}(a) \sum_{n=1}^{N} \frac{e^{\frac{2\pi i(n-a)}{q}\ell}}{n}.$$

d) Prove that the term with $\ell = 0$ does not contribute in the previous sum (here the condition $\chi_{2,q} \neq \chi_0$ is used). Deduce that

$$L(1,\chi_{2,q}) = \frac{-1}{q}\sum_{\ell=1}^{q-1}\sum_{a=1}^{q} \chi_{2,q}(a) e^{\frac{-2\pi i a}{q}\ell} \log(1 - e^{\frac{2\pi i}{q}\ell}).$$

e) Let $\tau_q := \sum_{a=1}^{q} \chi_{2,q}(a) e^{\frac{2\pi i a}{q}}$. Then, for every ℓ which is coprime to q, prove that

$$\sum_{a=1}^{q} \chi_{2,q}(a) e^{\frac{-2\pi i a}{q}\ell} = \chi_{2,q}(-\ell)\tau_q.$$

Thus,

$$L(1,\chi_{2,q}) = \frac{-\tau_q}{q}\sum_{\ell=1}^{q-1} \chi_{2,q}(-\ell)\log(1 - e^{\frac{2\pi i}{q}\ell}).$$

This is already a formula giving $L(1,\chi_{2,q})$ as a finite sum of explicitly computable numbers, but it can be further simplified.

f) Using the standard properties of the logarithm, deduce that

$$L(1,\chi_{2,q}) = \frac{-\tau_q}{q}\sum_{\ell=1}^{q-1}\chi_{2,q}(-\ell)\Big[\log\Big(\sin\Big(\frac{\pi\ell}{q}\Big)\Big) + \log(-2ie^{\frac{\pi i}{q}\ell})\Big]$$

g) Use the identity $\sum_{\ell=1}^{q-1}\chi_{2,q}(-\ell) = 0$ to deduce that

$$L(1,\chi_{2,q}) = \frac{-\tau_q}{q}\sum_{\ell=1}^{q-1}\chi_{2,q}(-\ell)\Big[\log\Big(\sin\Big(\frac{\pi\ell}{q}\Big)\Big) + \frac{\pi i}{q}\ell\Big].$$

h) Suppose that $\chi_{2,q}$ is an even character, i.e. that $\chi_{2,q}(-1) = 1$: prove that

$$\sum_{\ell=1}^{q-1}\chi_{2,q}(-\ell)\ell = 0$$

so that in this case

$$L(1,\chi_{2,q}) = \frac{-\tau_q}{q}\sum_{\ell=1}^{q-1}\chi_{2,q}(\ell)\log\Big(\sin\Big(\frac{\pi\ell}{q}\Big)\Big).$$

i) Suppose that $\chi_{2,q}$ is an odd character, i.e. that $\chi_{2,q}(-1) = -1$: prove that

$$\sum_{\ell=1}^{q-1}\chi_{2,q}(-\ell)\log\Big(\sin\Big(\frac{\pi\ell}{q}\Big)\Big) = 0$$

so that in this case

$$L(1,\chi_{2,q}) = \frac{i\pi\tau_q}{q^2}\sum_{\ell=1}^{q-1}\chi_{2,q}(\ell)\ell.$$

j) Test the formulas with $q = 5$ and $q = 7$.
(The value of $L(1,\chi_{2,q})$ approximated within two decimal digits may be computed following Step d, and using a software package for the computations).

(9) Let q be an odd prime. Prove that $\chi_{2,q}$ is even when $q = 1 \pmod 4$ and is odd when $q = 3 \pmod 4$.

(10) Let q be an odd prime. Let τ_q as in the previous exercise; it is called the *Gauss quadratic sum* for the prime q. Prove that $|\tau_q| = \sqrt{q}$.
Remark: This is a standard but not elementary computation. You can find it in every text on number theory. A more complicated result states that $\tau_q = \sqrt{q}$ when $\chi_{2,q}$ is even, and $\tau_q = i\sqrt{q}$ when $\chi_{2,q}$ is odd, but needs deeper tools. (A proof of Dirichlet based on Poisson formula is in Davenport *Multiplicative number theory* GTM 74, Springer 2000 pp. 12–16. The original Gauss' argument is reproduced in Berndt, Evans and Williams *Gauss and Jacobi sums*, John Wiley & Sons Inc. 1998 pp. 18–24, and a different proof of Schur based on the matrix theory is reproduced in the same book in exercises 26–29 pp. 47–48).

(11) Let q be a prime. Let $S_q := \sum_{\ell=1}^{q-1} \chi_{2,q}(\ell)\ell$. Note that S_q is an integer. Prove that S_q is always divisible by q when $q > 3$.

(12) Let q be an odd prime. We know that $L(1, \chi_{2,q}) \neq 0$; deduce that $L(1, \chi_{2,q}) > 0$. Collecting the previous exercises conclude that

$$L(1, \chi_{2,q}) \geq \frac{\pi}{\sqrt{q}}$$

when $\chi_{2,q}(-1) = -1$ and $q > 3$.
Remark: This bound is called the *trivial lower bound* for $L(1, \chi_{2,q})$, but actually it is not trivial at all. Any similar lower bound in terms of q for the case of an even real character would be rich of important consequences, but its search has been essentially fruitless up to now.

(13) Let $q > 3$ be a prime which is congruent to 3 modulo 4. Let $S_q = \sum_{\ell=1}^{q-1} \chi_{2,q}(\ell)\ell$. Compute S_q for $q = 7, 11, 19, 23, 31$. Computations show that $S_q < 0$; collecting the results in the previous exercises (and the remark in Ex. 10), prove that this is true in general.
Remark: The celebrated Dirichlet's class number formula says that $-S_q/q$ is the class number of the quadratic field $\mathbb{Q}[\sqrt{-q}]$ for every prime $q = 3 \pmod 4$, $q > 3$.

(14) Let a_n be any real sequence. Let $A_1(x) := \sum_{n \leq x} a_n$ (with $A_1(x) = 0$ for $x < 1$).
a) Using the partial summation formula prove that

$$\sum_{n=1}^{L} \frac{a_n}{n^s} = \frac{A_1(L)}{L^s} - \sum_{n=1}^{L-1} A_1(n)\Big(\frac{1}{(n+1)^s} - \frac{1}{n^s}\Big)$$

for every integer L.

b) Set $A_2(x) := \sum_{n \leq x} A_1(x)$. Iterating the previous formula prove that

$$\sum_{n=1}^{L} \frac{a_n}{n^s} = \frac{A_1(L)}{L^s} - A_2(L-1)\Big(\frac{1}{L^s} - \frac{1}{(L-1)^s}\Big) + \sum_{n=1}^{L-2} A_2(n)\Big(\frac{1}{(n+2)^s} - \frac{2}{(n+1)^s} + \frac{1}{n^s}\Big)$$

c) Suppose $A_1(x)$ bounded. From the previous formulas deduce that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s \int_1^{+\infty} \frac{A_1(x)}{x^{s+1}}\,\mathrm{d}x \qquad \forall s \in \mathbb{C},\ \mathrm{Re}s > 0;$$

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = s(s+1) \int_1^{+\infty} \int_x^{x+1} \frac{A_2(x)}{u^{s+2}}\,\mathrm{d}u\,\mathrm{d}x \qquad \forall s \in \mathbb{C},\ \mathrm{Re}s > 0.$$

d) Let q be an odd prime and $\chi_{2,q}$ be the nontrivial quadratic character modulo q. For each $q \leq 41$ prove that $L(s, \chi_{2,q}) > 0$ when $s \in \mathbb{R}$, $s > 0$.

Hint: For Step d apply formulas in Step c with $a_n = \chi_{2,q}(n)$ and show that for all q in the list the sequence $A_1(n)$ (or at least the sequence $A_2(n)$) is non negative.
Remark: Chowla conjectured that $L(s, \chi_{2,q}) > 0$ in $(0, +\infty)$ for every q (so that, in particular, $L(1/2, \chi_{2,q}) > 0$). If proved, this conjecture would have extraordinary consequences (for example it rules out the Siegel zero). The approach illustrated in the exercise can be generalized introducing sequences $A_3(x) := \sum_{n \leq x} A_2(n)$, $A_4(x) := \sum_{n \leq x} A_3(n)$ and so on, but it is not at all evident that for every given q there is some index k for which A_k is nonnegative. The conjecture is wildly open.