Analytic Number Theory: Homework 2

(1) Let S_k be the set of integers which are divisible only by k distinct primes, i.e.

$$S_k := \{n\colon \exists p_1 \ldots, p_k \in \mathcal{P},\ \exists \nu_1, \ldots, \nu_k \in \mathbb{N}, n = p_1^{\nu_1} \cdots p_k^{\nu_k}\}.$$

A more compact notation for the same set is $S_k := \{n\colon \omega(n) = k\}$. From the Prime Number Theorem we know that $\#\{S_1 \cap [0,x]\} \sim \pi(x) \sim \frac{x}{\log x}$. Landau was able to prove that for every k,

$$\#\{S_k \cap [0,x]\} \sim_k \frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!} \quad \text{as} \quad x \to \infty.$$

(The result is *not* uniform in k). The following argument proves this formula for $k = 2$.

a) Prove that $S_2 \cap [0,x] = \bigcup_{a,b}\{p, q \in \mathcal{P}\colon p < q,\ p^a q^b \le x\}$.

b) Prove that if $a, b \ge 2$, then

$$\#\{p, q \in \mathcal{P}\colon p < q,\ p^a q^b \le x\} \le \#\{m, n \in \mathbb{N}\colon m^2 n^2 \le x\} \ll x^{1/2}\log x.$$

c) Prove that if $(a, b) = (2, 1)$ or $(1, 2)$, then

$$\#\{p, q \in \mathcal{P}\colon p < q,\ p^a q^b \le x\} \le \#\{m \in \mathbb{N}, q \in \mathcal{P}\colon m^2 q \le x\} \ll \frac{x}{\log x}.$$

d) Deduce that $\displaystyle\sum_{\substack{a,b \\ (a,b)\neq(1,1)}} \#\{p, q \in \mathcal{P}\colon p < q,\ p^a q^b \le x\} \ll \frac{x}{\log x}$.

e) Prove that $\#\{p, q \in \mathcal{P}\colon p < q,\ pq \le x\} = \frac{1}{2}\#\{p, q \in \mathcal{P}\colon pq \le x\} + O(\sqrt{x})$.

f) Prove that

$$\#\{p, q \in \mathcal{P}\colon pq \le x\} \sim 2\frac{x \log\log x}{\log x}.$$

g) Conclude.

Hint: You can deal with Step f) by proving that

$$\begin{aligned}
\#\{p, q \in \mathcal{P}\colon pq \le x\} &= \sum_{2 \le p \le x/2} \pi\Big(\frac{x}{p}\Big) = \sum_{2 \le p \le x/2} \frac{x/p}{\log(x/p)} + \text{ error} \\
&= -\int_2^{x/2} \frac{\mathrm{d}}{\mathrm{d}p}\Big[\frac{x/p}{\log(x/p)}\Big]\frac{p}{\log p}\,\mathrm{d}p + \text{ error} \\
&= \int_2^{x/2} \frac{x\,\mathrm{d}p}{p\log p \log(x/p)} + \text{ error} = 2\frac{x\log\log x}{\log x} + \text{ error}.
\end{aligned}$$

where "error" denotes a remaining term that has to be estimated and which will be smaller than the main term.

Integrals $\int_2^{x/2} \frac{x\,\mathrm{d}p}{p \log p \log^j(x/p)}$ with any j (here you have $j = 1$ and $j = 2$) can be computed via the change of variable $p = e^z$.

(2) **Optional:** Prove Landau's asymptotic relation we have mentioned in previous exercise, for every k.

Hint: The proof is by induction on k, and is similar to what you have done for Ex. 1, but now there are a lot of minor details that make the computation quite long and complicated. For example, the integrals are no more explicitly computable and to get their value one needs to introduce the series representation of $\frac{1}{\log(x/p)} = \frac{1}{\log x}\sum_{\ell=0}^{\infty}\left(\frac{\log p}{\log x}\right)^{\ell}$, which converges uniformly for $p \in [2, x/2]$. Try this exercise only if you are strongly motivated or you have a lot of spare time!

(3) Let $f\colon \mathbb{N} \to \mathbb{C}$ be a multiplicative function which is supported on squarefree integers. Suppose that there exists $\alpha > 0$ such that

$$f(p) = 1 + O(p^{-\alpha}) \qquad \text{as} \qquad p \to \infty.$$

Let $\theta := \min(1/2, \alpha)$ and let $c := \prod_p \big(1 + f(p)/p\big)\big(1 - 1/p\big)$. Prove that

$$\sum_{n\le x} f(n) = c\,x + O_\eta(x^{1-\theta+\eta}) \qquad \text{as} \qquad x \to \infty,$$

for every $\eta > 0$.
Hint: Let $F(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$, write this function as Euler product and follow Exercises 6/7/8 in Homework 1.

(4) Let $f\colon \mathbb{N} \to \mathbb{C}$ be a multiplicative function which is supported on squarefree integers. Suppose that there exists $\alpha > 0$ such that

$$f(p) = \frac{1}{p}(1 + O(p^{-\alpha})) \qquad \text{as} \qquad p \to \infty.$$

Let $\theta := \min(1/2, \alpha)$ and let $c := \prod_p \big(1 + f(p)\big)\big(1 - 1/p\big)$. Prove that there exists a constant d such that

$$\sum_{n\le x} f(n) = c\log x + d + O_\eta(x^{-\theta+\eta}) \qquad \text{as} \qquad x \to \infty,$$

for every $\eta > 0$.
Hint: Notice that the function $g\colon \mathbb{N} \to \mathbb{C}$ with $g(n) = nf(n)$ for every n satisfies the hypothesis in Exercise 1. Write $\sum_{n\le x} f(n)$ as $\sum_{n\le x} nf(n)\cdot\frac{1}{n}$ and apply the partial summation formula.
Notice that $c = 1$ when $f(n) = \mu(n)^2/\varphi(n)$ (hence this computation improves Lemma 3.1 of the notes).

(5) Let χ be a real character modulo q, $\chi \ne \chi_0$. The following steps give an alternative proof of the fact that $L(1,\chi) \ne 0$.
a) Let $f(n) := (1 * \chi)(n) = \sum_{d\mid n}\chi(d)$. Prove that f is multiplicative, that $f(n) \ge 0$ for every n, and that $f(n^2) \ge 1$ for every n.

b) Let $F(z) := \sum_{n=1}^{\infty} f(n)e^{-n/z}$. Prove that F is well defined for all $z \ge 1$, and that

$$F(z) \ge \sum_{n\le z/2} f(n)e^{-n/z} \gg \sum_{n\le z/2} f(n) \gg \sqrt{z} \qquad \text{as } z \to \infty.$$

c) Show that

$$F(z) = \sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\chi(n)e^{-mn/z} = \sum_{n=1}^{+\infty}\frac{\chi(n)}{e^{n/z}-1} = z\sum_{n=1}^{+\infty}\frac{\chi(n)}{n} + \sum_{n=1}^{\infty}\chi(n)\Big[\frac{1}{e^{n/z}-1} - \frac{z}{n}\Big].$$

d) Apply the partial summation formula to conclude that

$$\Big|\sum_{n=1}^{+\infty}\chi(n)\Big[\frac{1}{e^{n/z}-1} - \frac{z}{n}\Big]\Big| \ll_q \int_1^{+\infty}\Big|\Big(\frac{1}{e^{x/z}-1} - \frac{z}{x}\Big)'\Big|\,\mathrm{d}x \ll_q 1 \qquad \text{as } z \to \infty.$$

e) From Steps b), c) and d) deduce that

$$\sqrt{z} \ll F(z) = zL(1,\chi) + O_q(1) \qquad \text{as } z \to \infty,$$

and conclude that $L(1,\chi) \neq 0$.

Hint: For Step d recall that $A(x) := \sum_{n \leq x} \chi(n)$ is a bounded function, and prove that $\frac{1}{e^y - 1} - \frac{1}{y}$ is a negative and increasing function in $y \geq 0$. Thus $\int_1^{+\infty} |(\frac{1}{e^{x/z}-1} - \frac{z}{x})'| \, dx = \int_1^{+\infty} (\frac{1}{e^{x/z}-1} - \frac{z}{x})' \, dx = z - \frac{1}{e^{1/z}-1}$ which stays bounded as $z \to \infty$ (Prove it!).

(6) Let k be a positive even integer greater than 1. Show that the number of primes $p \leq x$ such that $kp + 1$ is also prime is

$$\ll \frac{x}{\log^2 x} \prod_{p|k} \Big(1 + \frac{1}{p}\Big)$$

uniformly in k (i.e. the implicit constant is independent of k (and of x, of course).

Hint: Follow the proof of Theorem 3.4 in the notes (no need to repeat all computations for Proposition 3.3).

(7) Set $D \in \mathbb{N}$, $D \geq 2$. Let

$$S_D := \{n \in \mathbb{N} \colon \exists p \in \mathcal{P},\ k \in \mathbb{N} \text{ s.t. } n = p + D^k\}$$

(i.e. the of integers which can be represented as a sum of a prime and a D power). Prove that

$$\liminf_{x \to \infty} \frac{1}{x} \#(S_D \cap [0,x]) > 0.$$

Hint: Follow the proof of Romanoff's theorem in the notes.