Analytic Number Theory: Homework 3

(1) Recall that if $n = \prod_p p^{\nu_p}$ is the decomposition of the integer n as product of distinct prime powers, then $\omega(n) := \#\{p\colon \nu_p \neq 0\}$ and $\Omega(n) = \sum_p \nu_p$. Prove that there exist positive constants c, c' such that

$$\sum_{n\leq x} 2^{\omega(n)} = (c + o(1))x\log x \quad \text{and} \quad \sum_{n\leq x} 2^{\Omega(n)} = (c' + o(1))x\log^2 x, \qquad \text{as } x \to \infty.$$

Hint: Note that ω and Ω are additive functions, so that $2^{\omega(\cdot)}$ and $2^{\Omega(\cdot)}$ are multiplicative.

(2) (continued) Describe and algorithm computing c and c' with any arbitrarily chosen precision. Test your formula giving their value with an error lower than 10^{-3}.

(3) Recall that an integer n is *squarefull* when $p|n$ implies that $p^2|n$ for every prime p.
  (a) Prove that every squarefull n can be written in a unique way as m^3d^2 where m is squarefree.
  (b) Let $\pi_{\text{full}}(x)$ be the set of squarefull integers lower than x. Using the identity

$$\pi_{\text{full}}(x) = \sum_{\substack{m,d \\ m^3d^2<x \\ m \text{ squarefree}}} 1 = \sum_{\substack{m<x^{1/3} \\ m \text{ squarefree}}} \sum_{d<\sqrt{x/m^3}} 1$$

  prove that

$$\pi_{\text{full}}(x) \sim c\sqrt{x}, \qquad \text{with} \qquad c = \sum_{\substack{m=1 \\ m \text{ squarefree}}}^{\infty} \frac{1}{m^{3/2}}.$$

  (c) Prove that $c = \zeta(3/2)/\zeta(3)$.

Note: $\sqrt{x}$ is the number of squares lower than x, hence the result can also be written as

$$\frac{\pi_{\text{full}}(x)}{\#\{\text{squares} \leq x\}} \sim \frac{\zeta(3/2)}{\zeta(3)}.$$

When written in this way, the conclusion shows that the number of squarefull integers in first instance is driven by the squares, the other numbers contributing only to the constant.

(4) Let k be a positive even integer greater than 1. Show that the number of primes $p \leq x$ such that $kp + 1$ is also prime is

$$\ll \frac{x}{\log^2 x} \prod_{p|k} \Big(1 + \frac{1}{p}\Big)$$

uniformly in k (i.e. the implicit constant is independent of k (and of x, of course).
Hint: Follow the proof of Theorem 3.4 in the notes.

(5) Set $D \in \mathbb{N}$, $D \geq 2$. Let

$$S_D := \{n \in \mathbb{N}\colon \exists p \in \mathcal{P},\ k \in \mathbb{N} \text{ s.t. } n = p + D^k\}$$

(i.e. the of integers which can be represented as a sum of a prime and a D power). Prove that

$$\liminf_{x\to\infty} \frac{1}{x}\#(S_D \cap [0, x]) > 0.$$

Hint: Follow the proof of Romanoff's theorem in the notes.