NUMBER THEORY, EXAM OF 1/02/2012

Name: ....

Student ID number: ....

e-mail: ....

1) Describe the structure of the class group of $K = \mathbb{Q}[\sqrt{-57}]$.

2) Let K be the field generated by a root α of the polynomial $f(x) = x^3 - x^2 + 2x - 3$. Compute $\mathrm{disc}(f(x))$ and deduce that $d_K \in \{-7, -5^2 \cdot 7\}$. Use Minkowski's bound to prove that $d_K = -5^2 \cdot 7$, and that therefore the ring of integers R_K of K is equal to $\mathbb{Z}[\alpha]$.

3) Let $K = \mathbb{Q}[\sqrt{D}]$ be a real quadratic field, with ring of integers R, group of units $R^\times$ and set of embeddings $\sigma_1, \sigma_2 : K \longrightarrow \mathbb{R}$.
(i) For any non-zero ideal $I \subset R$, show that

$$\{(\sigma_1(a), \sigma_2(a)) \in \mathbb{R}^2 : a \in I\}$$

is a full lattice in $\mathbb{R}^2$.
(ii) Use (i) to show that

$$\{(\log|\sigma_1(u)|, \log|\sigma_2(u)|) \in \mathbb{R}^2 : u \in R^\times\}$$

is a lattice in $\mathbb{R}^2$ of rank $t \leq 1$. Deduce that

$$R^\times \simeq \{\pm 1\} \times \mathbb{Z}^t, \qquad t \leq 1.$$

(iii) (**Optional**) Show that $t = 1$.

4)
(i) For p an odd prime, let $K_n = \mathbb{Q}[e^{2\pi i/p^n}]$, and let R_n be the ring of integers of K_n. Let $q\mathbb{Z}$ be a maximal ideal of $\mathbb{Z}$, with q a prime $\neq p$. Show that the number of prime divisors Q_n of qR_n is bounded as $n \longrightarrow \infty$.
(ii) (**Optional**) Assume that there exists a sequence of Galois extensions $K_n/\mathbb{Q}$ with $\mathrm{Gal}(K_n/\mathbb{Q})$ dihedral of order $2p^n$, $n \geq 0$. Write R_n for the ring of integers of K_n. Let $\mathcal{S}$ be the set of primes $q\mathbb{Z}$ which are inert in the quadratic field K_0 (i.e., for which the ideal qR_0 is prime). For any $q \in \mathcal{S}$, show that the number of prime divisors of qR_n has unbounded cardinality as $n \longrightarrow \infty$. Is $\mathcal{S}$ infinite?