AN EXPLICIT CHEBOTAREV DENSITY THEOREM UNDER GRH

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Abstract. We prove an explicit version of the Chebotarev theorem for the density of prime ideals with fixed Artin symbol, under the assumption of the validity of the Riemann hypothesis for the Dedekind zeta functions. In appendix we also give some explicit formulas counting non-trivial zeros of Hecke’s $L$-functions, in that case without assuming the truth of the Riemann hypothesis.

DOI: https://doi.org/10.1016/j.jnt.2018.12.005

To Jurek Kaczorowski
for his 60th birthday

1. Introduction

In order to state the results we need to fix some notation. Thus, given a number field $K$ we denote $n_K$ its dimension and $r_1(K), r_2(K)$ the number of its real, respectively imaginary places; the absolute value of its discriminant is denoted as $\Delta_K$, $p$ always denotes a nonzero prime ideal of the integer ring $\mathcal{O}_K$, and $N_p$ its absolute norm; $\Lambda_K$ denotes the analogue of the von Mangoldt function, i.e. the function which is defined on the set of ideals of $\mathcal{O}_K$ and whose value at an ideal $I$ is $\log N_p$ if $I = p^m$ for some $p$ and $m \geq 1$, and zero otherwise. Moreover, let $K \subseteq L$ be a Galois extension of number fields with relative discriminant $\Delta_{L/K}$, and let $\mathfrak{P}$ be a prime ideal of $L$ above a non-ramified $p$ prime ideal of $\mathcal{O}_K$. Then the Artin symbol $[L/K]_p$ denotes the conjugacy class of the Frobenius automorphism corresponding to $\mathfrak{P}/p$, and which is extended multiplicatively on the prime powers in $\mathcal{O}_K$ coprime to $\Delta_{L/K}$.

Let $C$ be any conjugacy class in $G := \text{Gal}(L/K)$ and let $\varepsilon_C$ be its characteristic function. Then the function $\pi_C$ and the Chebyshev function $\psi_C$ are defined as

$$\pi_C(x) := \# \{ p : p \text{ non-ramified in } L/K, N_p \leq x, [L/K]_p = C \}$$

$$= \sum_{p \text{ non-ram.}}\varepsilon_C([L/K]_p),$$

$$\psi_C(x) := \sum_{\mathfrak{J} \subseteq \mathcal{O}_K, \mathfrak{J} \text{ non-ram.}}\varepsilon_C([L/K]_{\mathfrak{J}}) \Lambda_K(\mathfrak{J}).$$

The first function counts the number of non-ramified prime ideals with prescribed Artin symbol, while Chebyshev’s function does the same but with a suitable logarithmic weight supported on prime powers. The celebrated Chebotarev density theorem states that $\pi_C(x) \sim \frac{|C|}{|G|} \log x$ when $x$ diverges, a claim which can be stated equivalently by saying that $\psi_C(x) \sim \frac{|C|}{|G|} x$.

We introduce also two other functions which are closely related to $\pi_C$ and $\psi_C$ but that are easier to deal with. They are built using an arithmetical function which comes from the theory of Artin $L$-functions and extends $\varepsilon_C([L/K]_p)$ to ramifying prime ideals. To wit, for any

2010 Mathematics Subject Classification. Primary 11R42, Secondary 11Y70.
prime ideal \( p \subseteq \mathcal{O}_K \) (possibly ramified) let \( \mathfrak{P} \) be any prime ideal dividing \( p\mathcal{O}_L \), let \( I \) be the inertia group of \( \mathfrak{P} \) and \( \tau \) be one of the \(|I|\) Frobenius automorphisms corresponding to \( \mathfrak{P}/p \). Let

\[
\theta(C; p^m) := \frac{1}{|I|} \sum_{a \in I} \varepsilon_C(\tau^m a).
\]

Notice that \( \theta(C; p^m) \in [0, 1] \), and that for non-ramified primes it is 1 if and only if \( \tau^m \) belongs to \( C \), and 0 otherwise. We define

\[
\pi(C; x) := \sum_{p: Np \leq x} \theta(C; p) \log Np,
\]

\[
\psi(C; x) := \sum_{\mathfrak{P} \subseteq \mathcal{O}_K} \theta(C; \mathfrak{P}) \Lambda_K(\mathfrak{P}).
\]

Observe that \( \psi_C(x) \) and \( \psi(C; x) \) agree except on ramified-prime-powers ideals, being

\[
\psi(C; x) = \psi_C(x) + \mathcal{R}_C(x)
\]

with

\[
\mathcal{R}_C(x) := \sum_{p|\Delta_L/K} \sum_{m \geq 1} \theta(C; p^m) \log Np.
\]

In particular, \( 0 \leq \psi_C(x) \leq \psi(C; x) \) for every \( x \), so that every upper bound for \( \psi(C; x) \) gives also a bound for \( \psi_C(x) \), and a lower bound for \( \psi_C(x) \) produces a lower bound for \( \psi(C; x) \).

Jeffrey Lagarias and Andrew Odlyzko [12] provided versions of Chebotarev’s theorem which are explicit in their dependence on the field \( K \) up to positive universal constants which however are not estimated, and Joseph Oesterlé [15] announced that

\[
|\mathfrak{P}| \psi(C; x) - x| \leq \sqrt{x} \left[ \left( \frac{\log x}{\pi} + 2 \right) \log \Delta_L + \left( \frac{\log^2 x}{2\pi} + 2 \right) n_L \right] \quad \forall x \geq 1
\]

under the assumption of the generalized Riemann hypothesis. On the other hand, Lowell Schoenfeld [22] proved that the Riemann hypothesis implies that

\[
|\psi_q(x) - x| \leq \frac{1}{8\pi} \sqrt{x} \log^2 x \quad \forall x \geq 59.
\]

(He states this result for \( x \geq 73.2 \), but actually it is easy to check that the inequality holds also for \( x \in [59, 73.2) \)). This result shows that it should be possible to improve the constants appearing in Oesterlé’s result. Bruno Winckler [24, Th. 8.1] proved a result similar to (1.4), but with larger coefficients of logs in the \( \log \Delta_L \) and \( n_L \) parts.

In [7] we have proved an analogue of Schoenfeld’s result for the easier case \( K = \mathbb{L} \), where all prime ideals are counted. In this paper we generalize this work to the full set of extensions and classes, as in Oesterlé’s result, but with the improved constants. In fact, the following theorem is our main result.

**Theorem 1.1.** Assume GRH holds. Then \( \forall x \geq 1 \)

\[
|G| \psi(C; x) - x| \leq \sqrt{x} \left[ \left( \frac{\log x}{2\pi} + 2 \right) \log \Delta_L + \left( \frac{\log^2 x}{8\pi} + 2 \right) n_L \right],
\]

\[
|G| \psi_C(x) - x| \leq \sqrt{x} \left[ \left( \frac{\log x}{2\pi} + 2 \right) \log \Delta_L + \left( \frac{\log^2 x}{8\pi} + 2 \right) n_L \right].
\]
From the proof it will be clear that the constants +2 have nothing special and other values are possible. For instance, one can prove that
\[
\left| \frac{G}{C} \psi(C; x) - x \right| \leq \sqrt{x} \left[ \left( \frac{\log x}{2\pi} + 2 \right) \log \Delta + \frac{\log^2 x}{8\pi} n_L \right] + 40,
\]
again for all \( x \geq 1 \). Moreover, the +40 can be removed if \( n_L \geq 7 \), and both +2, +40 can be removed if \( x \) is large enough. One can also prove a result of the form of [7, Corollary 1.3] where \( \log x \) is substituted by \( \log \left( \frac{cx}{\log x} \right) \) for some constant \( c \). All remarks apply also to \( \psi_C(x) \).

By partial summation one deduces the following result.

**Corollary 1.2.** Assume GRH holds. Then \( \forall x \geq 2 \)
\[
\left| \frac{G}{C} \pi(C; x) - \int_2^x \frac{du}{\log u} \right| \leq \sqrt{x} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log x} \right) \log \Delta + \frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} n_L \right],
\]
\[
\left| \frac{G}{C} \pi_C(x) - \int_2^x \frac{du}{\log u} \right| \leq \sqrt{x} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log x} \right) \log \Delta + \frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} n_L \right].
\]

This corollary also could be improved in the secondary terms as in [7, Corollary 1.4] which, unfortunately, was stated incorrectly and should read

**Corollary (7 Corollary 1.4).** Assume GRH holds. Then \( \forall x \geq 2 \)
\[
\left| \pi_\mathbb{K}(x) - \int_2^x \frac{du}{\log u} \right| \leq \sqrt{x} \left[ \left( \frac{1}{2\pi} - \frac{\log \log x}{\pi \log x} + \frac{5.8}{\log x} \right) \log \Delta + \left( \frac{1}{8\pi} - \frac{\log \log x}{2\pi \log x} + \frac{3.6}{\log x} \right) n_\mathbb{K} \log x + 0.3 + \frac{14}{\log x} \right].
\]

The general strategy for the proof is quite similar to the one of [12] and [6]. However, many estimations have to be done with special care, in order to reduce the range of fields \( \mathbb{K} \), extensions \( \mathbb{L}/\mathbb{K} \) and \( x \) where the claims have to be proved directly via explicit computations.

We have made available at the address:
\[\text{http://users.mat.unimi.it/users/molteni/research/chebotarev/chebotarev.gp}\]
the PARI/GP [17] code we have used to compute the constants in this paper.

**Acknowledgements.** We wish to thank Karim Belabas for comments and interesting discussions, and the referee for useful comments and improvements in the text. The authors are members of the INdAM group GNSAGA.

2. Facts

Let
\[
\psi^{(1)}(C; x) := \int_0^x \psi(C; t) \, dt.
\]
As observed by Ingham [10, Ch. 2, Sec. 5], since \( \psi(C; x) \) is non-decreasing as a function of \( x \), one has the double inequality
\[
\psi(C; x) \leq \psi^{(1)}(C; x+h) - \psi^{(1)}(C; x) \quad \text{if } h > 0,
\]
\[
\psi(C; x) \geq \psi^{(1)}(C; x+h) - \psi^{(1)}(C; x) \quad \text{if } -x < h < 0.
\]
We let, for \( s > 1 \),
\[
K(C; s) := \sum_{\mathfrak{p} \in \mathcal{O}_\mathbb{K}} \theta(C; \mathfrak{p}) \Lambda_\mathbb{K}(\mathfrak{p})(N\mathfrak{p})^{-s}.
\]
As in [10, Ch. IV Sec. 4, p. 73] and [12, Sec. 5], we have the integral representation
\[
\psi(1)(C; x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} K(C; s) x^{s+1} \frac{ds}{s(s+1)}
\]
Let \( g \) be any element in \( C \), then the orthogonality of the irreducible characters \( \phi \) of \( G \) allows one to write
\[
\theta(C; p^m) = \frac{|C|}{|G|} \sum_{\phi} \bar{\phi}(g) \phi_K(p^m)
\]
where
\[
\phi_K(p^m) := \frac{1}{|I|} \sum_{a \in I} \phi(\tau^m a).
\]
The definitions of \( \theta(C; \cdot) \) and \( \phi_K \) are modelled on the definition of the Artin \( L \)-functions \( L(s, \phi, L/K) \), giving the equality
\[
K(C; s) = \sum_{\mathfrak{I} \subseteq O_K} \theta(C; \mathfrak{I}) \Lambda_K(\mathfrak{I}) (N \mathfrak{I})^{-s} = -\frac{|C|}{|G|} \sum_{\phi} \bar{\phi}(g) \frac{L'(s, \phi, L/K)}{L(s, \phi, L/K)}
\]
for \( \text{Re } s > 1 \).
Following an argument of Lagarias and Odlyzko (which comes from Deuring [5] and MacCluer [13]) we can modify the identity in order to use only Hecke \( L \)-functions, for which the continuation as holomorphic functions (apart at \( s = 1 \)) in \( C \) is proved: it is [12, Lemma 4.1], but a quick review can be useful.
As above, let \( g \) be any fixed element in \( C \). Let \( H \) be the cyclic group generated by \( g \) and let \( E := \mathbb{L}^H = \mathbb{L}^g \), the subfield of \( \mathbb{L} \) fixed by \( H \). Let \( f_g : H \to \mathbb{C} \) be the characteristic function of \( \{ g \} \). A direct computation shows that it induces on \( G \) the class function \( \text{Ind}^G_H f_g : G \to \mathbb{C} \) whose values are
\[
(\text{Ind}^G_H f_g)(y) = \frac{1}{|H|} \sum_{s \in G} f_g(s^{-1}ys) = \begin{cases} \frac{|G|}{|C||H|} & \text{if } y \in C \\ 0 & \text{otherwise.} \end{cases}
\]
Thus, the characteristic function of \( C \) is \( \frac{|C||H|}{|G|} \text{Ind}^G_H f_g \). By orthogonality of characters \( \chi \) of \( H \) one has
\[
f_g = \frac{1}{|H|} \sum_{\chi} \bar{\chi}(g) \chi,
\]
thus
\[
(\text{Ind}^G_H f_g)(y) = \frac{1}{|H|} \sum_{\chi} \bar{\chi}(g) (\text{Ind}^G_H \chi)(y),
\]
and the characteristic function of \( C \) is now written as \( \frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \text{Ind}^G_H \chi \). Using the definition of \( \theta(C; \cdot) \), we find that
\[
\theta(C; p^m) = \frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \chi_K(p^m)
\]
where
\[
\chi_K(p^m) := \frac{1}{|I|} \sum_{a \in I} (\text{Ind}^G_H \chi)(\tau^m a).
\]
In this way we get
\[
K(C; s) = -\frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \frac{L'}{L} (s, \text{Ind}^G_H \chi, L/K) = -\frac{|C|}{|G|} \sum_{\chi} \bar{\chi}(g) \frac{L'}{L} (s, \chi, L/E),
\]
which means
\[\psi^{(1)}(C; x) = -\frac{|C|}{|G|} \sum_{\chi} \overline{\chi}(g) \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L'(s, \chi, L/E) x^{s+1} \frac{1}{s(s+1)} \; ds,\]
where only abelian (i.e., Hecke, by class field theory) \(L\)-functions appear.

Thus, let \(E \subseteq \mathbb{L}\) be an abelian extension of fields and let \(\chi\) be any irreducible character of \(\text{Gal}(L/E)\). We will use \(L(s, \chi)\) to denote \(L(s, \chi, L/E)\). Also, set \(\delta_{\chi} = 1\) if \(\chi\) is the trivial character, 0 otherwise.

We recall that for each \(\chi\) there exist uniquely determined non-negative integers \(a_{\chi}, b_{\chi}\) such that
\[a_{\chi} + b_{\chi} = n_E\]
and a positive integer \(Q(\chi)\) such that if we define
\[\Gamma_{\chi}(s) := \left[\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\right]^{a_{\chi}} \left[\pi^{-\frac{s+1}{2}} \Gamma\left(\frac{s+1}{2}\right)\right]^{b_{\chi}}\]
and
\[\xi(s, \chi) := [s(s-1)]^{\delta_{\chi}} Q(\chi)^{s/2} \Gamma_{\chi}(s) L(s, \chi),\]
then \(\xi(s, \chi)\) satisfies the functional equation
\[\xi(1-s, \chi) = W(\chi) \xi(s, \chi),\]
where \(W(\chi)\) is a certain constant of absolute value 1. For the trivial character \(\chi\), the Hecke \(L\)-function \(L(s, \chi, L/E)\) coincides with Dedekind’s zeta function \(\zeta(s)\), and in this case \(a_{\chi} = r_1(\mathbb{E}) + r_2(E)\) and \(b_{\chi} = r_2(E)\). Furthermore, \(\xi(s, \chi)\) is an entire function (by class field theory) of order 1 and does not vanish at \(s = 0\), and hence by Hadamard’s product theorem we have
\[\xi(s, \chi) = e^{A_{\chi} + B_{\chi} s} \prod_{\rho \in Z_{\chi}} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}\]
for some constants \(A_{\chi}\) and \(B_{\chi}\), where \(Z_{\chi}\) is the set of zeros (multiplicity included) of \(\xi(s, \chi)\).

Lastly, we introduce a special notation for the type of sum on characters as the one appearing in (2.4), and for any \(f: \text{Gal}(L/E) \to \mathbb{C}\) we set
\[\mathcal{M}_C f := \sum_{\chi} \overline{\chi}(g) f(\chi)\]
where we recall that \(g\) is a fixed element of \(C\).

3. Preliminary inequalities

3.1. Reduction to Dedekind Zeta functions. Differentiating (2.6) and (2.8) logarithmically we obtain the identity
\[\int_{2-i\infty}^{2+i\infty} \frac{L'(s, \chi)}{L(s, \chi)} \; ds = B_{\chi} + \sum_{\rho \in Z_{\chi}} \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) - \frac{1}{2} \log Q(\chi) - \delta_{\chi} \left(\frac{1}{s} + \frac{1}{s-1}\right) - \frac{\Gamma'_{\chi}}{\Gamma_{\chi}}(s),\]
for all complex \(s\). Using (2.5), (2.6) and (3.1) one sees that
\[\frac{L'(s, \chi)}{L(s, \chi)} = \frac{a_{\chi} - \delta_{\chi}}{s} + r_{\chi} + O(s) \quad \text{as } s \to 0,\]
\[\frac{L'(s, \chi)}{L(s, \chi)} = \frac{b_{\chi}}{s+1} + r'_{\chi} + O(s+1) \quad \text{as } s \to -1,\]
Lemma 3.1. Let \( a, b \) parameters such that 

\[
\frac{a}{2} \Gamma \left( \frac{1}{2} \right) - \frac{b}{2} \Gamma \left( \frac{1}{2} \right) - \log \frac{Q(\chi)}{\pi n}.
\]

(3.3)

\[
L'(2, \tilde{\chi}) - \log \frac{Q(\chi)}{\pi n} \frac{\Gamma'(3)}{2} \frac{\Gamma(1)}{2}.
\]

Comparing the previous formula for \( r_\chi \) and (3.1), we get

\[
r_\chi = \frac{L'}{L}(s, \chi) - \sum_{\rho \in \mathbb{Z}_x} \frac{s}{\rho(s-\rho)} + \delta_\chi \left( \frac{1}{s} + \frac{1}{s-1} \right)
\]

\[
+ \frac{a_\chi}{2} \left( \frac{\Gamma'(3)}{2} \frac{\Gamma(1)}{2} \right) + \frac{b_\chi}{2} \left( \frac{\Gamma'(3)}{2} \frac{\Gamma(1)}{2} \right)
\]

for every \( s \in \mathbb{C} \). Setting \( s = 2 \) this formula simplifies to

\[
r_\chi = \frac{L'}{L}(2, \chi) - \sum_{\rho \in \mathbb{Z}} \frac{2}{\rho(2-\rho)} - \frac{5}{2} \delta_\chi + b_\chi.
\]

We come back to the situation where \( g \in C \) and \( \mathbb{E} = \mathbb{L}^2 \), so that \( \mathbb{L}/\mathbb{E} \) is a cyclic extension for which \( g \) is a generator of \( \text{Gal}(\mathbb{L}/\mathbb{E}) \). The following lemma computes the mean values of the parameters \( a_\chi \) and \( b_\chi \) appearing in (2.5). To simplify the formulas, we will write from now on \( r_1 \) and \( r_2 \) for \( r_1(\mathbb{L}) \) and \( r_2(\mathbb{L}) \).

**Lemma 3.1.** Let

\[
S := \begin{cases} 
  r_1 + r_2 & \text{if } g \text{ has order 1}, \\
  r_2 - 2r_2(\mathbb{E}) & \text{if } g \text{ has order 2}, \\
  0 & \text{otherwise},
\end{cases}
\]

and let \( \delta_C \) defined to be 1 if \( C \) is the trivial class and 0 otherwise. Then

\[
\mathcal{M}_C a_\chi = \sum_\chi \tilde{\chi}(g) a_\chi = S,
\]

\[
\mathcal{M}_C b_\chi = \sum_\chi \tilde{\chi}(g) b_\chi = \delta_C n_\mathbb{E} - S = \delta_C n_\mathbb{L} - S.
\]

**Proof.** If \( C \) is the trivial class, i.e. \( g \) has order 1, we have \( \mathcal{M}_C a_\chi = \sum_\chi a_\chi = r_1 + r_2 = S \) because the extension \( \mathbb{L}/\mathbb{E} \) is Galois, hence \( \prod L(s, \chi) = \zeta_L(s) \). We have as well \( \mathcal{M}_C b_\chi = \sum b_\chi = r_2 = n_\mathbb{L} - S \), hence the result is proved. We henceforth assume that \( g \) has order at least 2.

By duality, the set of characters of \( \text{Gal}(\mathbb{L}/\mathbb{E}) \) is cyclic: let \( \varphi \) be a generator. The character \( \chi \) corresponds to a Hecke character \( \tilde{\chi} \) of the idèles of \( \mathbb{E} \). For any real embedding of \( \mathbb{E} \), let \( p_\ell(\chi) \) be 1 if the local component of \( \tilde{\chi} \) at \( \ell \) is the sign character, and 0 otherwise. We furthermore denote \( s_\chi \) the number of \( \ell \)'s for which \( p_\ell(\chi) = 1 \). The construction of Hecke characters and \( L \)-functions shows that \( a_\chi = r_1(\mathbb{E}) + r_2(\mathbb{E}) - s_\chi \), see [9]. In particular,

\[
\sum_\chi \tilde{\chi}(g) a_\chi = - \sum_\chi \tilde{\chi}(g) s_\chi.
\]

For every fixed real embedding \( \ell \) one has \( p_\ell(\chi) = p_\ell(\chi') \) (mod 2), thus \( s_\chi = 0 \) when \( \chi \) is an even power of \( \varphi \), and \( s_\chi = s_\varphi \) otherwise. This shows that if \( |\text{Gal}(\mathbb{L}/\mathbb{E})| \) is odd, then \( s_\chi = 0 \) for every character, while when \( |\text{Gal}(\mathbb{L}/\mathbb{E})| \) is even one gets

\[
\sum_\chi \tilde{\chi}(g) a_\chi = - s_\varphi \varphi(g) \sum_{k=0}^{|\text{Gal}(\mathbb{L}/\mathbb{E})|/2 - 1} (\varphi^2)^k(g).
\]
This is the sum on the subgroup of the square characters, thus it is zero unless \( \varphi^2(g) = 1 \). This happens if and only if \( |\text{Gal}(L/E)| = 2 \), because \( g \) is a generator, and in this case \( \varphi(g) = -1 \). Thus we get:

\[
\sum_{\chi} \tilde{\chi}(g) a_{\chi} = \begin{cases} 
 s_\varphi & \text{if } |\text{Gal}(L/E)| = 2 \\
 0 & \text{otherwise}.
\end{cases}
\]

To conclude, we have \( p_\ell(\varphi) = 1 \) if and only if \( \ell \) ramifies in \( L/E \) hence \( s_\varphi = r_2(L) - 2r_2(E) \). This proves the lemma for the sum of the \( a_\chi \)'s. For the sum of the \( b_\chi \)'s it is sufficient to observe that

\[
\sum_{\chi} \tilde{\chi}(g)(a_\chi + b_\chi) = \sum_{\chi} \tilde{\chi}(g)n_E = 0.
\]

Note that if \( g \) has order 1, then \( S = r_1 + r_2 = \frac{n_1 + r_1}{2} \). In the other cases we have \( 0 \leq S \leq r_2 = \frac{m_2 - r_1}{2} \). Thus in all cases \( 0 \leq S \leq \frac{m_2 - r_1}{2} + \delta_C r_1 \).

**Lemma 3.2.** Let \( L/E \) be a cyclic extension and let \( Z \) be the multiset of non-trivial zeros of the Dedekind zeta function \( \zeta_L \). Let \( f \) be any complex function with \( \sum_{\rho \in Z} |f(\rho)| < \infty \). Then

\[
\mathcal{M}_C \sum_{\rho \in Z} f(\rho) = \sum_{\rho \in Z} \epsilon(\rho) f(\rho)
\]

where, for any \( \rho \in Z \), \( |\epsilon(\rho)| = 1 \) and \( \epsilon(\overline{\rho}) = \overline{\epsilon(\rho)} \).

**Proof.** Since \( \zeta_L = \prod_L L(s, \chi) \), the multiset \( Z \) is the disjoint union of the multisets \( Z_\chi \). Moreover, for each \( \rho \) in \( Z \) there is a well defined character \( \chi \) such that \( \rho \in Z_\chi \); for this \( \rho \) we set \( \epsilon(\rho) := \tilde{\chi}(g) \). This rule respects the formula \( \epsilon(\overline{\rho}) = \epsilon(\rho) \), because \( \rho \) belongs to \( Z_\chi \) if and only if \( \overline{\rho} \) belongs to \( Z_\overline{\chi} \). Thus, we can write

\[
\mathcal{M}_C \sum_{\rho \in Z} f(\rho) = \sum_{\chi} \sum_{\rho \in Z_\chi} \tilde{\chi}(g) f(\rho) = \sum_{\rho \in Z} \epsilon(\rho) f(\rho).
\]

The equality \( |\epsilon(\rho)| = 1 \) is obvious. \( \square \)

**Lemma 3.3.** Let \( Z \) be the multiset of non-trivial zeros of the Dedekind zeta function \( \zeta_L \). Recall that \( L/E \) is a cyclic extension and that \( S \) and \( \epsilon(\rho) \) are defined in Lemmas 3.1 and 3.2 respectively. We have

\[
\mathcal{M}_C r_\chi = 2 \sum_{\rho \in Z} \frac{\epsilon(\rho)}{\rho(2 - \rho)} - \frac{n_L}{n_E |C|} \sum_{\lambda \subseteq \mathcal{O}_E} \theta(C; \lambda) \Lambda_E(\lambda)(N\lambda)^{-2} + n_L \delta_C - S + \frac{5}{2}.
\]

**Proof.** By (3.4) and Lemmas 3.1 and 3.2 we get

\[
(3.5) \quad \mathcal{M}_C r_\chi = 2 \sum_{\rho \in Z} \frac{\epsilon(\rho)}{\rho(2 - \rho)} + \mathcal{M}_C \frac{L'(2, \chi)}{L(2, \chi)} + n_L \delta_C - S + \frac{5}{2}.
\]

Moreover, by (2.3) we have

\[
\mathcal{M}_C \frac{L'(2, \chi)}{L(2, \chi)} = -\frac{|G|}{|C|} K(C; 2)
\]

hence by (2.2) we have

\[
(3.6) \quad \mathcal{M}_C \frac{L'(2, \chi)}{L(2, \chi)} = -\frac{n_L}{n_E |C|} \sum_{\lambda \subseteq \mathcal{O}_E} \theta(C; \lambda) \Lambda_E(\lambda)(N\lambda)^{-2}.
\]

The result follows from (3.5) and (3.6). \( \square \)
Lemma 3.4. We have
\[ |M_C r'_\chi| \leq \log \Delta_L+n \log \left( \frac{\zeta'}{\zeta} \right) (2)+(\log 2\pi+\gamma-1)n_\delta C, \]
where \( \gamma = 0.5772 \ldots \) is the Euler–Mascheroni constant.

Proof. By (3.3) we get
\[ M_C r'_\chi = -M_C \frac{L'}{L}(2, \bar{\chi})-M_C \log Q(\chi)+n_E \left( \log \pi-\frac{1}{2} \left( \frac{3}{2} \right)-\frac{1}{2} \Gamma(1) \right) \]
Replacing \( C \) by \( C_1 = [g^{-1}] \) and \( g \) by \( g^{-1} \) in (2.3) and conjugating, we get
\[ M_C \frac{L'}{L}(2, \bar{\chi}) = -\left| \frac{G}{C} \right| K(C^{-1}; 2) \]
which by (2.2) is estimated by \( \frac{n_E}{|C|} \zeta'(2) \) because \( 0 \leq \theta(C; \cdot) \leq 1 \) by definition. Moreover,
\[ \left| \sum_{\chi} \bar{\chi}(g) \log Q(\chi) \right| \leq \sum_{\chi} \log Q(\chi) = \log \Delta_L, \]
by the product formula for conductors. The result follows because \( \frac{L'(\frac{3}{2})}{L}(1) = 2-4 \log 2-2 \gamma \) and \( n_E M_C 1 = n_\delta C. \)

Lemma 3.5. We define, for any \( x > 1 \) and any character \( \chi \),
\[ f_1(x) := \sum_{r=1}^{\infty} \frac{x^{1-2r}}{2r(2r-1)}, \quad f_2(x) := \sum_{r=2}^{\infty} \frac{x^{2-2r}}{(2r-1)(2r-2)}, \]
\[ R_\chi(x) := -\left( a_\chi-\delta_\chi \right)(x \log x-x)+b_\chi(x \log x+1-a_\chi f_1(x)-b_\chi f_2(x) \]
and
\[ R_C(x) := M_C R_\chi(x). \]
Then for any \( x > 1 \),
\[ R_C(x) = \int_0^x \log u \, du - S \int_1^{x+1} \log u \, du + \delta C \frac{n_L}{2} \left[ \log(x^2-1)+x \log \left( \frac{x+1}{x-1} \right) \right], \]
\[ R'_C(x) = \log x - S \log(x+1) + \delta C \frac{n_L}{2} \log \left( \frac{x+1}{x-1} \right). \]

Proof. We have
\[ f_1(x) = \frac{1}{2} \left[ x \log(1-x^{-2})+\log \left( \frac{1+x^{-1}}{1-x^{-1}} \right) \right], \]
\[ f_2(x) = 1-\frac{1}{2} \left[ \log(1-x^{-2})+x \log \left( \frac{1+x^{-1}}{1-x^{-1}} \right) \right]. \]
Assume first that \( C \) is not the trivial class. By Lemma 3.1
\[ R_C(x) = -(S-1)(x \log x-x)-S(\log x+1)-S f_1(x)+S f_2(x) \]
\[ = x \log x-x+S \left( -(x+1) \log x+x-\frac{x+1}{2} \left( \log(1-x^{-2})+\log \left( \frac{1+x^{-1}}{1-x^{-1}} \right) \right) \right) \]
\[ = x \log x-x+S \left( -(x+1) \log x+x-(x+1) \log(1+x^{-1}) \right) \]
\[ = x \log x-x+S(x-(x+1) \log(x+1)), \]
which produces the formulas for $R_C$ and $R'_C$ stated in the lemma for a non-trivial class. For the trivial class we have to add $n_L$ times
\[ 1 + \log x - f_2(x) = \frac{1}{2} \left[ \log(x^2 - 1) + x \log \left( \frac{x+1}{x-1} \right) \right] \]
to $R_C$ and $\frac{1}{2} \log \left( \frac{x+1}{x-1} \right)$ to its derivative. \hfill $\square$

3.2. Bounds for the ramification term.

**Lemma 3.6.** Let $x \geq 1$. Then
\[ R_C(x) \leq \min \left( \frac{|C|}{p}, 1 \right) n \log x \]
where $p$ is the smallest prime dividing $|G|$, and $n := \sum_{p \mid \Delta_L/K} 1$ is the number of prime ideals of $K$ dividing $\Delta_L/K$.

**Proof.** From its definition \[1.3\] we have
\[
R_C(x) \leq \max_{p \mid \Delta_L/K} \left( \theta(C; p^m) \right) \sum_{p \mid \Delta_L/K} \log N_p \sum_{m \geq 1} 1 = \max_{p \mid \Delta_L/K} \left( \theta(C; p^m) \right) \sum_{p \mid \Delta_L/K} \log N_p \left( \frac{\log x}{\log N_p} \right)
\]
and \[1.1\] immediately shows that $\theta(C; p^m) \leq \min(|C|/|I|, 1)$. The proof concludes because the order of the inertia group is at least $p$ for ramified primes. \hfill $\square$

**Lemma 3.7.** Let $n = \sum_{p \mid \Delta_L/K} 1$ as in Lemma 3.6. We have the following bounds:

i. If $L \neq \mathbb{Q}[\sqrt{3}]$ and $L \neq \mathbb{Q}[\sqrt{15}]$ then $n \leq \frac{\log \Delta_L}{\log 4}$.

ii. If $n_L = 3$, the bound improves to $n \leq \frac{\log \Delta_L}{\log 49}$.

iii. If $n_L/K$ is not prime, the bound improves to $n \leq \frac{\log \Delta_L}{\log 22}$ except for the quartic fields of discriminant in \{144, 225, 400, 441, 3600, 7056, 176400\} (twenty five fields in total).

iv. If $\log \Delta_L > e^{1.1714} n_K$, then
\[
|K| \leq \frac{\log \Delta_L}{\log \Delta_L - \log n_K - 1.1714}.
\]

The proof will make clear that Item iv is valid even when $L/K$ is not Galois. Moreover, the inequality $\log \Delta_L > e^{1.1714} n_K$ holds except for just a few fields when $L \neq K$. Precisely, the only exceptions for $n_K = 1$ are the fields $L$ with $\Delta_L \leq 25$ (i.e., the cubic field of discriminant $-23$ and seventeen quadratic fields), for $n_K = 2$ they are the twenty four quartic fields with $\Delta_L \leq 634$, for $n_K = 3$ the four sextic fields with $\Delta_L \leq 15986$. There are no exceptions with $n_K \geq 4$.

**Proof.** We can assume $|G| \geq 2$ otherwise $n = 0$.

**Item i.$**

Suppose $K \neq \mathbb{Q}$. We split the set of primes dividing $\Delta_L/K$ into three (possibly empty) sets: $\{p_i\}_{i=1}^a$, $\{q_j\}_{j=1}^b$, and $\{s_\ell\}_{\ell=1}^c$, which are the set of primes whose norm is 2, 3 and $\geq 4$, respectively. Note that $a, b, c \leq n_K$. Then
\[
\Delta_L = \Delta_L^{L/K} N(\Delta_L/K) = \Delta_L^{n_L/K}_K N\left( \prod_i p_i \prod_j q_j \prod_\ell s_\ell \right) \geq \Delta_L^{2a} 3^b 4^c.
\]
Moreover by Minkowski’s bound we know that $\Delta_{K}^{1/n_K} \geq \sqrt{3}$, i.e. $\Delta_{K}^{n_K} \geq 3^{n_K}$. Thus we get

$$\Delta_{L} \geq 3^{n_K} \cdot 2^{3\cdot 4^c} = 2^{3\cdot \left(\frac{3}{2}\right)^n_K} 2^{\cdot 3^\cdot 4^c} \geq 2^{4\cdot \left(\frac{3}{2}\right)} 2^{3^\cdot 4^c} = 4^a \cdot 4^c \geq 4^{a+b+c} = 4^n$$

as claimed.

Suppose $K = \mathbb{Q}$. Then $n = \omega(\Delta_{L})$. Let $p_j, j = 2, 3, \ldots$ be the sequence of primes. Note that if $\Delta_{L} \in \prod_{k \leq j} p_k, \prod_{k \leq j+1} p_k)$ then

$$\frac{n}{\log \Delta_{L}} = \frac{\omega(\Delta_{L})}{\log \Delta_{L}} \leq \frac{j}{\log(\prod_{k \leq j} p_k)} = \frac{j}{d(p_j)}.$$ 

The sequence $\vartheta(p_j)/j$ is strictly increasing because it is the sequence of mean values of the increasing sequence $\log p_j$. Since $\frac{j}{d(p_j)} \leq 1/\log 4$ for $j = 4$, and since $\prod_{k \leq 4} p_k = 210$, the previous remark shows that $n \leq \log \Delta_{L}/\log 4$ as soon as $\Delta_{L} \geq 210$. Moreover, $\omega(\Delta_{L}) \leq 3$ when $\Delta_{L} \in [30, 210)$. Thus in this range $n/\log \Delta_{L} \leq 3/\log \Delta_{L}$ so that it is $\leq 1/\log 4$ as soon as $\Delta_{L} \geq 4^2 = 64$. There are only $21 + 19$ (resp. $4 + 1$) quadratic (resp. cubic) fields with $\Delta_{L} < 64$: for all of them the inequality $n \leq \log \Delta_{L}/\log 4$ holds but for $\mathbb{Q}[\sqrt{\pm 3}]$ and for $\mathbb{Q}[\sqrt{\pm 15}]$.

**Item [12]**

Since $L$ has to be a non-trivial Galois extension of $K$, we must have $K = \mathbb{Q}$ and $G$ cyclic of order 3. We thus know that the discriminant of $L$ (hence $\Delta_{L}$) is the square of an integer. By [8] or [3] Th. 6.4.11, p. 341), the only primes that can divide $\Delta_{L}$ are 3 and the primes congruent to 1 modulo 3 and, if $3 | \Delta_{L}$ then $81 | \Delta_{L}$. This proves that $\Delta_{L} \geq 49^3$, as needed.

**Item [12]**

We prove that $p^2 | \Delta_{L/K}$ for each prime ideal $p \subseteq \mathcal{O}_K$ ramifying in $L$. In fact, we are assuming that $|G|$ is not a prime, thus $G$ has a proper subgroup and by Galois duality there is a proper intermediate field $F$, so that $Q \subseteq K \subseteq F \subseteq L$. Thus

$$\Delta_{L/K} = \Delta_{F/K} N_{L/F}(\Delta_{L/F})$$

Let $p \subseteq \mathcal{O}_K$ be a prime ideal ramifying in $L$. If $p$ ramifies in $F$, then $p^{[L:F]} | \Delta_{L/K}$, hence $p^2 | \Delta_{L/K}$.

Suppose now that $p$ does not ramify in $F$. Let $P \subseteq \mathcal{O}_L$ be a prime above $p$. As $L/K$ is Galois, it follows that $q := P \cap F$ ramifies in $L/F$. Thus $q | \Delta_{L/F}$. This proves that $\prod_{q \mid \mathcal{O}_K} q | \Delta_{L/F}$.

Hence $p \mid \Delta_{L/F}$, because $p \mathcal{O}_F = \prod_{q \mid \mathcal{O}_K} q$ (because $p$ does not ramify in $F$, by hypothesis). Therefore $p^{[L:F]} = N_{L/K}(p \mathcal{O}_F) | \Delta_{L/K}$. In particular $p^2 | \Delta_{L/K}$ also in this case.

Suppose $K = \mathbb{Q}$. The previous computation shows that there exist integers $A$ and $B$ such that $\Delta_{L} = A^2 B$ with $B$ squarefree and $B | A$. As a consequence

$$\frac{n}{\log \Delta_{K}} = \frac{\omega(A^2 B)}{\log(A^2 B)} \leq \frac{\omega(A)}{2 \log A}$$

and if $A \in \prod_{k \leq j} p_k, \prod_{k \leq j+1} p_k$) then

$$\frac{n}{\log \Delta_{K}} \leq \frac{j}{2 \vartheta(p_j)}.$$ 

Since $\frac{j}{2 \vartheta(p_j)} \leq 1/\log 22$ for $j = 5$, and since $\prod_{k \leq 5} p_k = 2310$, the previous remark shows that $n \leq \log \Delta_{L}/\log 22$ as soon as $A \geq 2310$. Moreover, $\omega(A) \leq 4$ when $A < 2310$. Thus in this case $n/\log \Delta_{L} \leq 4/\log \Delta_{L}$ which is $\leq 1/\log 22$ as soon as $\Delta_{L} \geq 22^4 = 234256$. Odlyzko’s Table 3 shows that $\Delta_{L} \leq 234256$ is possible only for degrees $n_L \leq 7$, and, given our hypothesis, it remains to test only $n_L = 4$ and $n_L = 6$. All quartic and sextics fields with absolute discriminant up to 234256 appear in negrez table: exploring the table we found that there
are only twenty five quartic fields which are Galois extensions of \( \mathbb{Q} \) and which do not satisfy the bound (they are the fields with discriminant in \( \{144, 225, 400, 441, 3600, 7056, 176400\} \)), and no sextic fields.

Suppose \( K \neq \mathbb{Q} \). We will prove that \( n \leq \frac{\log \Delta_{K}}{\log 24} \). For \( n = 2, 3, 4 \) let \( S_{n} \) be the set of prime ideals dividing \( \Delta_{L/K} \) and whose norm is \( n \) and let \( S_{5} \) be the set of prime ideals dividing \( \Delta_{L/K} \) and whose norm is \( \geq 5 \). For all \( 2 \leq n \leq 5 \), let \( a_{n} \) be the cardinality of \( S_{n} \). Then

\[
\Delta_{L} = \Delta_{K}^{[L:K]} N(\Delta_{L/K}) \geq \Delta_{K}^{n_{L}/n_{K}} (N(\prod_{n=2}^{5} \prod_{p \in S_{n}} p))^{2} \geq \Delta_{K}^{n_{L}/n_{K}} (2^{a_{2}3^{a_{3}}4^{a_{4}}5^{a_{5}}})^{2}.
\]

Hence

\[
\log \Delta_{L} \geq \frac{n_{L}}{n_{K}} \log \Delta_{K} + 2 \sum_{n=2}^{5} a_{n} \log n.
\]

The number appearing on the right-hand side is larger than \( (\log 24) \sum_{n} a_{n} \) as soon as

\[
(3.7) \quad \frac{n_{L}}{n_{K}} \log \Delta_{K} \geq \sum_{n=2}^{5} a_{n} \log(24/n^{2}).
\]

Note that \( a_{3} \leq n_{K} \) and that \( a_{2} + 2a_{4} \leq n_{K} \) (because these primes factorize \( 2\mathcal{O}_{K} \)). As \( n_{L}/n_{K} \geq 4 \), Inequality \((3.7)\) holds for sure when

\[
\log(\Delta_{K}^{1/n_{K}}) \geq \frac{1}{4} \log \left( \frac{24^{2}}{2^{2\cdot 3^{2}}} \right) = \log 2,
\]

i.e. \( \Delta_{K}^{1/n_{K}} \geq 2 \). The root discriminant of \( K \) satisfies this inequality for \( n_{K} \geq 3 \), as one can see from line \( b = 1 \) in Odlyzko’s Table 3. For \( n_{K} = 2 \) this is true for \( \Delta_{K} \geq 4 \), thus \( K = \mathbb{Q}[\sqrt{-3}] \) is the unique exception to this argument. However, in this case \( S_{2} \) is empty and \( a_{3}, a_{4} \leq 1 \), thus the claim is true anyway.

**Item 21**

Set \( p_{0} := 1 \) and let \( A: [0, +\infty) \to \mathbb{R} \) be the function such that

\[
\forall j \geq 0, \forall x \in [\vartheta(p_{j}), \vartheta(p_{j+1})) \quad A(x) := \frac{x - \vartheta(p_{j})}{\log p_{j+1}} + j,
\]

i.e., the continuous and piecewise affine map satisfying \( A(\vartheta(p_{j})) = j \) for every \( j \). It is an increasing and concave map. We also introduce on \( (e^{1.1714}, +\infty) \) the function \( R(x) := \frac{x}{\log x - 1.1714} \). It is increasing for \( x \geq x_{R} := e^{2.1714} \), convex for \( x \leq ex_{R} \) and concave for \( x \geq ex_{R} \). Guy Robin [19] proved that \( \omega(n) \leq R(\log n) \) for all \( n \geq 26 \). As a consequence,

\[
\forall x > e^{1.1714}, \quad A(x) \leq R(x).
\]

Indeed, \( A(\vartheta(p_{j})) = j = \omega(\prod_{k=1}^{j} p_{k}) \leq R(\vartheta(p_{j})) \) when \( j \geq 4 \) by Robin’s result, and \( A(ex_{R}) \leq R(ex_{R}) \), by explicit computation. Thus, \( A(x) \leq R(x) \) for \( x \geq ex_{R} \) because \( A \) is piecewise affine and \( R \) is concave in this range. On \( (e^{1.1714}, ex_{R}) \) the inequality still holds because \( R \) is convex here and the tangent to its graph in \( ex_{R} \) stays above the graph of \( A \).

Let \( j_{0} := \lfloor n/n_{K} \rfloor \) and \( x_{0} := \vartheta(p_{j_{0}}) + (n/n_{K} - j_{0}) \log p_{j_{0} + 1} \), so that \( n = A(x_{0}) n_{K} \).

Let \( p_{j} \), \( j = 1, \ldots, n \) be the primes ramifying in \( L/K \). For each \( j \) let \( p_{k_{j}} \) be the prime integer below \( p_{j} \) and \( f_{j} \) be such that \( N(p_{j}) = p_{k_{j}}^{f_{j}} \). We suppose that the ideals are ordered such that the sequence \( p_{k_{j}} \) is non-decreasing. We have

\[
N(\Delta_{L/K}) = \prod_{j=1}^{n} Np_{j} = \prod_{j=1}^{n} p_{k_{j}}^{f_{j}} \geq \prod_{j=1}^{n} p_{k_{j}}.
\]
For a given $p_k$, there are at most $n_K$ values of $j$ such that $p_k = p_{kj}$, thus we get
\[
\Delta_L \geq N(\Delta_{L/K}) \geq \left( \prod_{k=1}^{j_0} p_k \right)^{n_K} L \prod_{j_0+1}^{n_K},
\]
so that $\log \Delta_L \geq x_0 n_K$. Hence
\[
\frac{n}{n_K} = A(x_0) \leq A\left(\frac{\log \Delta_L}{n_K}\right) \leq R\left(\frac{\log \Delta_L}{n_K}\right) = \frac{1}{n_K \log(\Delta_L/n_K) - 1.1714}
\]
when $\log \Delta_L > e^{1.1714 n_K}$. □

**Lemma 3.8.** For every integer $n$, let $\tilde{\Lambda}_L(n) := \sum_{\mathcal{N} \subseteq \mathbb{L}} \Lambda_L(3)$. Then for any $\ell \geq 1$ and any prime $p$ we have
\[
\sum_{r=1}^{n_L} \tilde{\Lambda}_L(p^{\ell n_L+r}) \geq n_L \log p.
\]

**Proof.** From the definition of $\tilde{\Lambda}_L$, we have
\[
\sum_{r=1}^{n_L} \tilde{\Lambda}_L(p^{\ell n_L+r}) = \sum_{r=1}^{n_L} \sum_{p \mid p} \sum_{m_L^r = p^{\ell n_L+r}} \log(N_p) = \sum_{p \mid m_L^r} f_p \left( \sum_{r=1}^{n_L} \sum_{m_f = \ell n_L+r} 1 \right) \log p,
\]
where $f_p$ is the inertia degree of $p$ in the extension $\mathbb{Q} \subseteq \mathbb{L}$. To conclude, it is sufficient to prove that
\[
\sum_{r=1}^{n_L} \sum_{m_f = \ell n_L+r} 1 \geq e_p,
\]
where $e_p$ is the ramification index of $p$, because $\sum_{p \mid m_f} f_p e_p = n_L$. To prove this inequality, we pick $r \in \{1, ..., f_p\}$ such that $\ell n_L+r = 0 \pmod{f_p}$. We then set $m = (\ell n_L+r)/f_p$, and this contributes by 1 to the inner sum on $m$. We repeat this procedure in the first $e_p$ blocks of length $f_p$: the claim follows since $e_p f_p \leq n_L$. □

### 3.3. Bounds for sums on zeros of Dedekind Zeta functions.

**Lemma 3.9.** Assume GRH. Then we have
\[
\sum_{|\gamma| \leq 2\pi} \frac{1}{|\rho|} + \sum_{|\gamma| > 2\pi} \frac{|1/2+2\pi i|}{|\rho|^2} \leq 1.348 \log \Delta_L - 1.557 n_L + 7.786 - 0.406 r_1 - e_{n_L},
\]
where the sums run over the non-trivial zeros $\rho = \frac{1}{2} + i \gamma$ of $\zeta_L$. Here $e_{n_L}$ is positive, with $e_1 \geq 5.529$, $e_2 \geq 0.751$ and $e_3 \geq 0.313$.

**Proof.** We prove this lemma with the same method of [7 Lemma 3.1]. Thus, let
\[
g(\gamma) := \begin{cases} 
\frac{2}{|1+4\gamma|^2} & \text{if } |\gamma| \leq 2\pi \\
\frac{2|1+4\pi i|}{1+4\pi^2} & \text{otherwise}
\end{cases}
\]
so that
\[
\sum_{|\gamma| \leq 2\pi} \frac{1}{|\rho|} + \sum_{|\gamma| > 2\pi} \frac{|1/2+2\pi i|}{|\rho|^2} = \sum_{\gamma} g(\gamma).
\]
We observe that \( g \) is continuous in \( \mathbb{R} \). Moreover, let \( f(s, \gamma) := 4(2s-1)/((2s-1)^2+4\gamma^2) \) and \( f_\ell(s) := \sum_{\gamma} f(s, \gamma) \). We look for a finite linear combination of \( f(s, \gamma) \) at suitable points \( s_j \) such that

\[
(3.9) \quad g(\gamma) \leq F(\gamma) := \sum_{j} a_j f(s_j, \gamma) \quad \forall \gamma \in \mathbb{R},
\]

so that

\[
(3.10) \quad \sum_{|\gamma| \leq 2\pi} \frac{1}{|\rho|} + \sum_{|\gamma| > 2\pi} \frac{|1/2+2\pi i|}{|\rho|^2} \leq \sum_{j} a_j f_\ell(s_j).
\]

Once (3.10) is proved, we recover a bound for the sum on zeros recalling the identity

\[
(3.11) \quad f_\ell(s) = 2 \text{Re} \frac{\zeta'}{\zeta} + \log \frac{\Delta_{\ell_0}}{\pi n_{\ell_0}} + \text{Re} \left( \frac{2}{s} + \frac{2}{s-1} \right) + (r_1 + r_2) \text{Re} \frac{\Gamma'}{\Gamma} \left( \frac{s}{2} \right) + r_2 \text{Re} \frac{\Gamma'}{\Gamma} \left( \frac{s+1}{2} \right).
\]

To determine a convenient set of constants \( a_j \)'s we set \( s_j = 1+j/2 \) with \( j = 1, \ldots, 23 \), \( \mathcal{Y} := \{0.62, 1, 1.6, 2.1, 2.8, 3.8, 4.6, 5.8, 7.5, 9.3, 12.9, 14, 16, 17, 18, 19, 20, 30, 40, 50, 10^2, 10^3, 10^4 \} \), and we require:

1. \( F(\gamma) = g(\gamma) \) for all \( \gamma \in \mathcal{Y} \cup \{0, 2\pi\} \),
2. \( F'(\gamma) = g'(\gamma) \) for all \( \gamma \in \mathcal{Y} \),
3. \( \lim_{\gamma \to \infty} \gamma^2 F(\gamma) = \lim_{\gamma \to \infty} \gamma^2 g(\gamma) = |1/2+2\pi i| \).

This produces a set of 49 linear equations for the 49 constants \( a_j \)'s ensuring (3.9), at least for \( \gamma \in \mathcal{Y} \). With an abuse of notation we take for \( a_j \)’s the solution of the system, rounded above to \( 10^{-7} \): this produces the numbers in Table 2. Then, using Sturm’s algorithm, we prove that the values found actually give an upper bound for \( g \), so that (3.9) holds with such \( a_j \)'s. These constants verify

\[
(3.12) \quad \sum_{j=1}^{49} a_j = 1.3479 \ldots, \quad \sum_{j=1}^{49} a_j \left( \frac{2}{s_j} + \frac{2}{s_j-1} \right) \leq 7.786,
\]

\[
\sum_{j=1}^{49} a_j \Gamma' \left( \frac{s_j}{2} \right) \leq -0.421, \quad \sum_{j=1}^{49} a_j \Gamma' \left( \frac{s_j+1}{2} \right) \leq 0.392.
\]

This suffices to manage all terms in (3.10) coming from all terms in (3.11) but the first one. However, we observe that \( a_1 > 0, a_2 > 0 \) and the signs of the \( a_j \)'s alternate for \( 2 \leq j \leq 49 \). We write \( \sum_{j=1}^{49} a_j \frac{\zeta'}{\zeta} + \Lambda(n) S(n) \) as

\[
-\sum_n \Lambda(n) S(n) \quad \text{with} \quad S(n) := \sum_{j=1}^{49} \frac{a_j}{n^{s_j}}.
\]

We isolate the first three terms in \( S(n) \), and group the other ones by consecutive pairs

\[
S(n) = \left( \frac{a_1}{n^{3/2}} + \frac{a_2}{n^{2}} + \frac{a_3}{n^{5/2}} \right) + \left( \frac{a_4}{n^{3}} + \frac{a_5}{n^{7/2}} \right) + \left( \frac{a_6}{n^{4}} + \frac{a_7}{n^{9/2}} \right) + \cdots + \left( \frac{a_{48}}{n^{25}} + \frac{a_{49}}{n^{51/2}} \right).
\]

It is easy to verify that each group decreases for \( n \geq 85597 \), and that hence the same holds for \( S(n) \). A direct computation shows that \( S(n+1) < S(n) \) holds also for \( n \leq 85597 \). Thus \( S \) is a decreasing sequence. Since \( a_1 > 0 \) we know that \( S(n) > 0 \) definitively and hence always.

Thus, we can deduce that \( -e_{n_0} := 2 \sum_{j=1}^{49} \frac{a_j}{n^{s_j}}(s_j) = -2 \sum_{n \geq 1} \Lambda(n) S(n) \leq 0 \) which suffices to prove the claim for a generic \( n_L \), via (3.10) (3.12).
With the help of Lemma 3.8 we can produce a better upper bound for \( -e_{n_L} \), at least when \( n_L \) is small. In fact \( S \) is decreasing, so that
\[
\sum_{j=1}^{49} a_j \frac{c_j}{\zeta L}(s_j) = -\sum_p \sum_m \Lambda_L(p^m)S(p^m) = -\sum_p \sum_{\ell=0}^{\infty} \sum_{r=1}^{n_L} \Lambda_L(p^{\ell n_L + r})S(p^{\ell n_L + r}) \leq -\sum_p \sum_{\ell=0}^{\infty} \sum_{r=1}^{n_L} \Lambda_L(p^{\ell n_L + r})S(p^{(\ell+1)n_L}).
\]
From Lemma 3.8 and since \( S \geq 0 \), this is
\[
\leq -n_L \sum_p \sum_{\ell=1}^{\infty} (\log p)S(p^{\ell n_L}) = -n_L \sum_p \sum_{\ell=1}^{\infty} \Lambda(p^\ell)S(p^{\ell n_L}) = n_L \sum_{j=1}^{49} \frac{c_j}{\zeta}(s_j n_L).
\]
Hence
\[
-e_{n_L} = 2 \sum_{j=1}^{49} a_j \frac{c_j}{\zeta L}(s_j) \leq 2n_L \sum_{j=1}^{49} a_j \frac{c_j}{\zeta}(s_j n_L)
\]
whose value for \( n_L = 1 \) is lower than \(-5.529\), for \( n_L = 2 \) is lower than \(-0.751\) and for \( n_L = 3 \) is lower than \(-0.313\) (the gain unfortunately decreases quickly: it is \(-0.149\) for \( n_L = 4 \) and only \(-0.074\) for \( n_L = 5 \)).

**Lemma 3.10.** Assume GRH. Then one has
\[
\sum_{\rho} \frac{1}{|\rho(\rho+1)|} \leq 0.5375\log \Delta_L - 1.0355n_L + 5.3879 - 0.2635r_1,
\]
where the sum runs over the non-trivial zeros \( \rho \) of \( \zeta_L \).

**Proof.** This claim is [6, Lemma 4.1], but now we repeat the computations keeping the extra term which is proportional to \( r_1 \). Since
\[
\sum_j a_j = 0.53747 \ldots, \quad \sum_j a_j \left( \frac{2}{s_j} + \frac{2}{s_j-1} \right) \leq 5.3879,
\]
\[
\sum_j a_j \Gamma' \left( \frac{s_j}{2} \right) \leq -0.6838, \quad \sum_j a_j \Gamma \left( \frac{s_j+1}{2} \right) \leq -0.1567,
\]
the claim follows. \( \square \)

We rewrite Theorem [A.1] for \( E = L \) and trivial character as
\[
(3.13) \quad \left| N_L(T) - \frac{T}{\pi} \log \left( \frac{T}{2\pi e} \right)^{n_L} \Delta_L \right| - 2 + \frac{1}{4} r_1 \leq c_1 W_L(T) + c_2 n_L + c_3
\]
for every \( T \geq T_0 \geq 1 \), where \( W_L(T) := \log \Delta_L + n_L \log(T/2\pi) \), \( c_1 = D_1 \), \( c_2 = D_2 + D_1 \log 2\pi \) and \( c_3 = D_3' \). With \( T_0 = 2\pi \), the last line of Table 1 provides (3.13) with the constants
\[
c_1 = 0.460, \quad c_2 = 2.491, \quad c_3 = 0.593.
\]
Other and smaller values for \( c_1 \) are available in Table 1 but we need also a small value for \( c_2 \) and \( c_3 \); this choice is adequate to our purpose. This proves

**Lemma 3.11.** For all \( T \geq 2\pi \) one has
\[
(3.14) \quad \sum_{|\gamma| \leq T} 1 = N_L(T) \leq T \left( \frac{1}{\pi} + \frac{0.460}{T} \right) W_L(T) - T \left( \frac{1}{\pi} - \frac{2.491}{T} \right) n_L + 2.593 - \frac{r_1}{4}.
\]

As in [7, Second sum], one has
Lemma 3.12. For all $T \geq 2\pi$ one has

\begin{equation}
\sum_{|\gamma| \geq T} \frac{1}{|\rho|^2} \leq \left( \frac{1}{\pi} + \frac{0.920}{T^2} \right) W_L(T) \frac{T}{T} + \left( \frac{1}{\pi} \frac{5.220}{T} \right) n_L \frac{T}{T} + 1.186 \frac{T}{T^2}.
\end{equation}

Proof. The proof remains the same in spite of the difference between the structure of (3.13) and Trudgian’s formula we used in [7] for this purpose, because the term $-1+r_1/4$ disappears in integrations. This provides the upper bound

$$
\sum_{|\gamma| \geq T} \frac{1}{|\rho|^2} \leq \left( \frac{1}{\pi} + \frac{2c_1}{T^2} \right) W_L(T) \frac{T}{T} + \left( \frac{1}{\pi} \frac{\log 2\pi}{12T^2} \right) n_L \frac{T}{T} + \left( \frac{2c_2 + c_1}{2} \right) n_L \frac{T}{T^2} + 2c_3 \frac{T}{T^2},
$$

and the claim follows from the selected values of $c_j$’s. □

Note that the formula improves upon the one in [7] because now $c_1$, $c_2$ and $c_3$ are smaller.

Lemma 3.13. For all $T \geq 2\pi$ one has

$$
\sum_{|\gamma| \leq T} \frac{1}{|\rho|} + \sum_{|\gamma| > T} \frac{|1+2\pi i|}{|\rho|^2} \leq \left( \frac{1}{\pi} \frac{\log \left( \frac{T}{2\pi} \right)}{T} + 1.067 + \frac{2}{T} \right) \log \Delta_L
$$

$$
+ \left( \frac{1}{2\pi} \frac{\log^2 \left( \frac{T}{2\pi} \right)}{T} + \frac{2}{T} \log \left( \frac{\ell T}{2\pi} \right) \right) - 1.633 - \frac{0.460}{T} + \frac{1.446}{T^2} n_L
$$

$$
+ 7.834 - 0.406r_1 - e_{n_L}.
$$

Proof. Let (3.13) be written as $|N_L(T) - A(T)| \leq R(T)$, with $A(T)$ representing the main term and $R(T)$ the bound for the remainder term. To ease notations, we set $\ell := |1/2+2\pi i|$. We write

$$
\sum_{|\gamma| \leq T} \frac{1}{|\rho|} + \sum_{|\gamma| > T} \frac{\ell}{|\rho|^2} = \sum_{|\gamma| \leq 2\pi} \frac{1}{|\rho|} + \sum_{|\gamma| > 2\pi} \frac{\ell}{|\rho|^2} + \sum_{2\pi < |\gamma| \leq T} \left( \frac{1}{|\rho|} - \frac{\ell}{|\rho|^2} \right)
$$

$$
\leq \sum_{|\gamma| \leq 2\pi} \frac{1}{|\rho|} + \sum_{|\gamma| > 2\pi} \frac{\ell}{|\rho|^2} + \sum_{2\pi < |\gamma| \leq T} \left( \frac{1}{|\gamma|} - \frac{2\pi}{\gamma^2} \right),
$$

where the last step follows by the general inequality $\frac{1}{|1/2+i\gamma|} - \frac{\ell}{|1/2+i\gamma|^2} \leq \frac{1}{|\gamma|} - \frac{2\pi}{\gamma^2}$. By partial summation we get

$$
\sum_{2\pi < |\gamma| \leq T} \left( \frac{1}{|\gamma|} - \frac{2\pi}{\gamma^2} \right) \leq \int_{2\pi}^T \frac{1}{\gamma} - \frac{2\pi}{\gamma^2} \, d\gamma + \int_{2\pi}^T \frac{R(4\pi)}{4\pi} \, d\gamma + \int_{2\pi}^T \frac{1}{\gamma} - \frac{2\pi}{\gamma^2} \, d\gamma + \int_{2\pi}^T \left( \frac{1}{\gamma} - \frac{2\pi}{\gamma^2} \right) R'(\gamma) \, d\gamma
$$

because $\frac{1}{\gamma} - \frac{2\pi}{\gamma^2}$ has a maximum at $4\pi$. Since $R'(\gamma) = c_1 n_L / \gamma$ this produces the bound

$$
\sum_{2\pi < |\gamma| \leq T} \left( \frac{1}{|\gamma|} - \frac{2\pi}{\gamma^2} \right) \leq \int_{2\pi}^T \left( \frac{1}{\gamma} - \frac{2\pi}{\gamma^2} \right) \, d\gamma + \frac{R(4\pi)}{4\pi} + c_1 \left( \frac{1}{8\pi} - \frac{1}{T + \pi/2} \right) n_L.
$$

The claim follows from this bound, the equality

$$
\int_{2\pi}^T \frac{1}{\gamma} - \frac{2\pi}{\gamma^2} \, d\gamma = \left( \frac{1}{\pi} \frac{\log \left( \frac{T}{2\pi} \right) - \frac{1}{\pi} + 2}{T} \right) \log \Delta_L + \left( \frac{1}{2\pi} \frac{\log^2 \left( \frac{T}{2\pi} \right)}{T} + \frac{2}{T} \log \left( \frac{\ell T}{2\pi} \right) \right) - \frac{1}{T} n_L.
$$

the result in (3.8) and the chosen values for the $c_j$’s constants. □
4. A parametric result

**Theorem 4.1.** (GRH) For every $x \geq 4$ and $T \geq 2\pi$ we have:

\[
\frac{|G|}{|C|} \psi(C; x) - x \leq L_a(x, T, n_L, \log \Delta_K),
\]

\[
-\left( \frac{|G|}{|C|} \psi_C(x) - x \right) \leq L_a(x, T, n_L, \log \Delta_K) + D(x, T, n_L, \log \Delta_K) + \frac{|G|}{|C|} R_C(x),
\]

with

\[
L_a(x, T, n, \mathcal{L}) := F(x, T)L + G(x, T)n + H(x, T, n),
\]

\[
F(x, T) := \sqrt{x} \left[ \frac{1}{2\pi} \log \left( \frac{T}{2\pi} \right) + 1.704 + \frac{1.858}{T} \right] + 1.075,
\]

\[
G(x, T) := \sqrt{x} \left[ \frac{1}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) + \left( \frac{2}{\pi} + \frac{1.858}{T} \right) \log \left( \frac{T}{2\pi} \right) - 1.633 + \frac{7.729}{T} \right] - 1.501,
\]

\[
H(x, T, n) := H_1(x, T) + H_2(x, T, n),
\]

\[
H_1(x, T) := \frac{x+2}{T} + \sqrt{x} \left( 7.834 + \frac{3.779}{T} \right) + 8.276,
\]

\[
H_2(x, T, n) := -\sqrt{x} \left( (0.406 + \frac{1}{4T}) r_1 + e_n \right) + (1-S) \log x + S - 0.744 n \delta_C - 0.527 r_1,
\]

\[
D(x, T, n, \mathcal{L}) := 2(S-1)(\log x - 1) + 3 - 0.445 n + 2n \delta_C
\]

\[
-\frac{\sqrt{x}}{T} (1.167 + 0.743 \frac{L}{\pi} + 0.743 \log \left( \frac{T}{2\pi} \right)) n.
\]

**Proof.** Following (2.4), we consider for a character $\chi$ of $\text{Gal}(L/\mathbb{E})$ the integral

\[
I_\chi(x) := -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L'(s, \chi) \frac{x^{s+1}}{s(s+1)} \, ds.
\]

Shifting the axis of integration arbitrarily far to the left, one gets for every $x > 1$ the identity

\[
I_\chi(x) = \delta(x) \frac{x^2}{2} - \frac{1}{\rho+1} - x r_\chi + r'_\chi + R_\chi(x)
\]

where $R_\chi(x)$ is defined in Lemma 3.3 and $r_\chi$ and $r'_\chi$ are defined in (3.2). The shift is done in a way similar to [12 § 6], further simplified by the fact that the integral is absolutely convergent on vertical lines. By (2.4), Lemma 3.2 and using $R_C$ as defined in Lemma 3.3, this gives

\[
\frac{|G|}{|C|} \psi^{(1)}(C; x) = M_C I_\chi(x) = \frac{x^2}{2} - \sum_{\rho \not\in Z} \epsilon(\rho) \frac{x^{\rho+1}}{\rho(\rho+1)} - x M_C r_\chi + M_C r'_\chi + R_C(x)
\]

so that for any $h \neq 0$, one has

\[
\frac{|G|}{|C|} \psi^{(1)}(C; x+h) - \psi^{(1)}(C; x) = \frac{x+h}{2} - \sum_{\rho \not\in Z} \epsilon(\rho) \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h(\rho+1)} - M_C r_\chi + R_C(\eta)
\]

for a suitable $\eta$ in the interval between $x$ and $x+h$. By (2.1) we deduce for $h > 0$:

\[
\frac{|G|}{|C|} \psi(C; x) - x \leq \frac{h}{2} + \sum_{\rho \not\in Z} \left| \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h(\rho+1)} \right| - M_C r_\chi + R_C(\eta)
\]
and for $h < 0$

$$\sum_{\rho \in \mathbb{Z}} \frac{(x+h)^{\rho^1} - x^{\rho^1}}{h \rho (\rho + 1)} \leq \frac{\eta}{2} + \sum_{\rho \in \mathbb{Z}} \frac{(x+h)^{\rho^1} - x^{\rho^1}}{h \rho (\rho + 1)} + M_C r_n - R'_C(\eta).$$

To get an upper bound for the sum of zeros we split its contribution into two parts: above and below $T$. Moreover, in the lower range we isolate the contribution of $\sum_{|\gamma| \leq T} x^\rho / \rho$, which will produce the main term. Thus,

$$\sum_{\rho \in \mathbb{Z}} \frac{(x+h)^{\rho^1} - x^{\rho^1}}{h \rho (\rho + 1)} \leq \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + \sum_{|\gamma| \leq T} \frac{(x+h)^{\rho^1} - x^{\rho^1} - h(\rho + 1)x^\rho}{h \rho (\rho + 1)} + \sum_{|\gamma| > T} \frac{(x+h)^{\rho^1} - x^{\rho^1}}{h \rho (\rho + 1)}$$

(4.6)

$$\leq \sum_{|\gamma| \leq T} \frac{\sqrt{x}}{|\gamma|} + \frac{|h|}{\sqrt{x}} \sum_{|\gamma| \leq T} |w_\rho| + \frac{x^{3/2}}{|h|} \left(\sum_{\gamma \leq T} \left(1 + \frac{h}{x}\right)^{3/2} + 1\right) \sum_{|\gamma| > T} \frac{1}{|\rho|^2}$$

with

$$w_\rho := \frac{(1 + \frac{h}{x})^{\rho^1} - 1 - (\rho + 1)\frac{h}{x}}{\rho (\rho + 1) \left(\frac{h}{x}\right)^2}.$$

The technique we apply to bound (4.4) and (4.5) changes in some details. We thus proceed separately for the two cases.

To prove (4.1) we bound the right hand side of (4.4). Let $h > 0$, then $|w_\rho| \leq \frac{1}{2}$ from [7] Lemma 2.1, and (4.6) gives

$$\sum_{\rho \in \mathbb{Z}} \frac{(x+h)^{\rho^1} - x^{\rho^1}}{h \rho (\rho + 1)} \leq \sqrt{x} \sum_{|\gamma| \leq T} \frac{1}{|\rho|} + \frac{h \sqrt{x}}{2} \frac{N_L(T)}{\rho} + \frac{x^{3/2}}{h} \left(\sum_{\gamma \leq T} \left(1 + \frac{h}{x}\right)^{3/2} + 1\right) \sum_{|\gamma| > T} \frac{1}{|\rho|^2}.$$

By (3.14) we know that $N_L(T)$ has order $T W_L(T)$, by (3.15) that $\sum_{|\gamma| > T} \frac{1}{|\rho|^2}$ has order $W_L(T) / T$, and by (3.16) that $\sum_{|\gamma| \leq T} \frac{1}{|\rho|^2}$ has order $(\log T) W_L(T)$. The comparison of the second and the last term, hence, suggests to take $h \approx x / T$. We set $h = 2x / T$. In this way we get:

$$\sum_{\rho \in \mathbb{Z}} \frac{(x+h)^{\rho^1} - x^{\rho^1}}{h \rho (\rho + 1)} \leq \sqrt{x} \sum_{|\gamma| \leq T} \frac{1}{|\rho|} + \frac{\sqrt{x} N_L(T)}{\rho} + \frac{T \sqrt{x}}{2} \left(\sum_{\gamma \leq T} \left(1 + \frac{2}{T}\right)^{3/2} + 1\right) \sum_{|\gamma| > T} \frac{1}{|\rho|^2}.$$

Since $(1 + \frac{2}{T})^{3/2} + 1 \leq 2 + \frac{3}{2} + \frac{3}{2T}$ we conclude

$$\frac{1}{\sqrt{x}} \sum_{\rho \in \mathbb{Z}} \frac{(x+h)^{\rho^1} - x^{\rho^1}}{h \rho (\rho + 1)} \leq \left(\sum_{|\gamma| \leq T} \frac{1}{|\rho|} + \sum_{|\gamma| > T} \frac{2\pi}{|\rho|^2} + \frac{N_L(T)}{\rho} + \frac{3}{4T^2} + \frac{2\pi}{T}\right) \sum_{|\gamma| > T} \frac{T}{|\rho|^2}.$$

Substituting (3.14), (3.15) and (3.16) in this equation, after some rearrangements we get:

$$\frac{1}{\sqrt{x}} \sum_{\rho \in \mathbb{Z}} \frac{(x+h)^{\rho^1} - x^{\rho^1}}{h \rho (\rho + 1)} \leq \left[\frac{1}{\pi} \log \frac{T}{2\pi} + 1.704 + \frac{1.858}{T}\right] \log \Delta_L$$

$$+ \left[\frac{1}{2\pi} \log^2 \frac{T}{2\pi} + \frac{2}{\pi} + \frac{1.858}{T}\right] \log \frac{T}{2\pi} - 1.633 + 7.729 T$$

$$+ 7.834 + 3.877 - \left(0.406 + \frac{1}{4T}\right) r_1 - e_{n_L}.$$

(4.7)

The explicit formula for $R'_C$ in Lemma 3.5 gives

$$R'_C(\eta) \leq \log \eta - S \log (\eta + 1) + 0.256 n_L \delta_C$$
under the assumption that $x \geq 4$. Using that and Lemma 3.3

\begin{equation}
- M_{C'}(x) \leq \sum_{\rho \in \mathcal{D}} \left| \frac{2}{\rho(2-\rho)} \right| - \frac{C(2)}{\xi(2)} n_{L} - n_{L} \delta_{C} + S - \frac{5}{2} + (1-S) \log x + 0.256 n_{L} \delta_{C} + \frac{2}{T}.
\end{equation}

Following (4.4), we sum (4.7) and (4.8), to get:

\begin{equation}
\frac{|G|}{|C|} \psi(C; x) - x \leq \sqrt{x} \left[ \frac{1}{\pi} \log \left( \frac{T}{2\pi} \right) + 1.704 + \frac{1.858}{T} \right] \log \Delta_{L} + \sqrt{x} \left[ \frac{1}{2\pi} \log^{2} \left( \frac{T}{2\pi} \right) + \left( \frac{2}{\pi} + \frac{1.858}{T} \right) \log \left( \frac{T}{2\pi} \right) - 1.633 + \frac{7.729}{T} \right] n_{L}
\end{equation}

\begin{equation}
+ \sqrt{x} \left[ 7.834 + \frac{3.779}{T} - \left( 0.406 + \frac{1}{4T} \right) r_{1} - e_{n_{L}} \right]
\end{equation}

\begin{equation}
+ \sum_{\rho \in \mathcal{D}} \frac{2}{\rho(2-\rho)} \left| \frac{C(2)}{\xi(2)} n_{L} - n_{L} \delta_{C} + S - \frac{5}{2} + (1-S) \log x + 0.256 n_{L} \delta_{C} + \frac{2}{T} \right|
\end{equation}

Moreover, $|2-\rho| = |\rho+1|$ since we are assuming GRH. Thus, by Lemma 3.10

\begin{equation}
\sum_{\rho \in \mathcal{D}} \frac{2}{|\rho(2-\rho)|} \leq 1.075 \log \Delta_{L} - 2.071 n_{L} + 10.776 - 0.527 r_{1}.
\end{equation}

The upper bound in (4.9) thus gives

\begin{equation}
\left| \frac{G}{C} \psi(C; x) - x \right| \leq \sqrt{x} \left[ \frac{1}{\pi} \log \left( \frac{T}{2\pi} \right) + 1.704 + \frac{1.858}{T} \right] \log \Delta_{L}
\end{equation}

\begin{equation}
+ \sqrt{x} \left[ \frac{1}{2\pi} \log^{2} \left( \frac{T}{2\pi} \right) + \left( \frac{2}{\pi} + \frac{1.858}{T} \right) \log \left( \frac{T}{2\pi} \right) - 1.633 + \frac{7.729}{T} \right] n_{L}
\end{equation}

\begin{equation}
+ \sqrt{x} \left[ 7.834 + \frac{3.779}{T} - \left( 0.406 + \frac{1}{4T} \right) r_{1} - e_{n_{L}} \right] + 1.075 \log \Delta_{L} - 2.071 n_{L} + 10.776 - 0.527 r_{1}
\end{equation}

\begin{equation}
+ 0.570 n_{L} - 0.744 n_{L} \delta_{C} + S - \frac{5}{2} + (1-S) \log x + x + \frac{2}{T}.
\end{equation}

This is the bound in (4.11), once the definition of $L_{a}$ is considered.

To prove (4.2) we first bound the right hand side of (4.5). In this case $h < 0$, thus $|w_{\rho}| \leq \frac{1}{2} + \frac{|h|}{6x}$ from \[7, \text{Lemma 2.1}, \] so that (4.6) gives

\begin{equation}
\sum_{\rho \in \mathcal{D}} \left| \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h \rho(h+1)} \right| \leq \sqrt{x} \sum_{|\gamma| \leq T} \frac{1}{|\rho|} + \frac{|h|}{\sqrt{x}} \left( \frac{1}{2} + \frac{|h|}{6x} \right) N_{L}(T) + x^{3/2} \left( \frac{(1+h)^{3/2}}{x} \right) + 1 \sum_{|\gamma| > T} \frac{1}{|\rho|^{2}}.
\end{equation}

Setting $h = -\frac{2\pi}{T}$, and estimating $(1+h)^{3/2} + 1 = (1 - \frac{2\pi}{T})^{3/2} + 1 \leq 2 - \frac{3}{2} + \frac{20}{T^{2}}$ (valid as soon as $T \geq 2$), we get

\begin{equation}
\frac{1}{\sqrt{x}} \sum_{\rho \in \mathcal{D}} \left| \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h \rho(h+1)} \right| \leq \left( \sum_{|\gamma| \leq T} \frac{1}{|\rho|} + \sum_{|\gamma| > T} \frac{2\pi}{|\rho|^{2}} \right) \left( \frac{N_{L}(T)}{T} \right) + \left( \frac{2}{3} \right) \sum_{|\gamma| > T} \frac{T}{|\rho|^{2}}.
\end{equation}

which with (3.14), (3.15) (which can be used because $1 - \frac{3}{2T} + \frac{10}{T^{2}} - \frac{2\pi}{T}$ is positive for $T \geq 2\pi$) and (3.16) produces

\begin{equation}
\frac{1}{\sqrt{x}} \sum_{\rho \in \mathcal{D}} \left| \frac{(x+h)^{\rho+1} - x^{\rho+1}}{h \rho(h+1)} \right| \leq \left( \frac{1}{\pi} \log \left( \frac{T}{2\pi} \right) + 1.704 + \frac{1.115}{T} \right) \log \Delta_{L}
\end{equation}
\[
\frac{1}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) + \frac{2}{\pi} \frac{1.115 - 2.206}{T^2} \log \left( \frac{T}{2\pi} \right) - 1.633 + \frac{6.562}{T} \right] n_L
\]

(4.10)

\[+7.834 + \frac{3.779}{T} - \frac{5.614}{T^2} - \left( 0.406 + \frac{1}{4T} \right) r_1 - e_{n_L} \cdot \]

Then \(-R'_C(\eta) \leq -\log \eta + S \log(\eta+1)\) hence, using Lemma 3.3

(4.11)

\[
\mathcal{M}_C r_C - R'_C(\eta) \leq \sum_{\rho \in \mathcal{Z}} \frac{2}{|\rho(2-\rho)|} + n_L \delta_C - S + \frac{5}{2} + S \log(x+1) - \log \left( x - \frac{2x}{T} \right).
\]

Summing (4.10) and (4.11), we get from (4.5):

\[
G(\psi(C; x) - x) \leq \frac{x}{T} + \sqrt{x} \left[ \frac{1}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) + 1.704 + \frac{1.115}{T} \right] \log \Delta_L
\]

\[
+ \sqrt{x} \left[ \frac{1}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) + \left( \frac{2}{\pi} + \frac{1.115}{T} \right) \log \left( \frac{T}{2\pi} \right) - 1.633 + \frac{6.562}{T} \right] n_L
\]

\[
+ \sqrt{x} \left[ 7.834 + \frac{3.779}{T} - \frac{5.614}{T^2} - \left( 0.406 + \frac{1}{4T} \right) r_1 - e_{n_L} \right]
\]

\[
+ \sum_{\rho \in \mathcal{Z}} \frac{2}{|\rho(2-\rho)|} + n_L \delta_C - S + \frac{5}{2} + (S-1) \log x + \frac{S}{x} \log \left( 1 - \frac{2}{T} \right).
\]

Reorganizing as above we get

(4.12)

\[-\left( \frac{G}{|C|} \right) \psi(C; x) - x \leq L_a(x, T, n_L, \log \Delta_L) + A\]

with

\[A := 2(S-1) \log x + \frac{S}{x} - 2S + 1.744 n_L \delta_C + 5 - 0.570 n_L - \log \left( 1 - \frac{2}{T} \right) - \frac{2}{T} \]

\[+ \frac{1}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) + \frac{2}{\pi} \frac{1.115 - 2.206}{T^2} \log \left( \frac{T}{2\pi} \right) - 1.633 + \frac{6.562}{T} \right] n_L
\]

\[\frac{x}{T} + \sqrt{x} \left[ 7.834 + \frac{3.779}{T} - \frac{5.614}{T^2} - \left( 0.406 + \frac{1}{4T} \right) r_1 - e_{n_L} \right].\]

We observe that, for \(T \geq 2\pi\), we have \(-\log(1-2/T) - 2/T \leq 2.561/T^2 \leq 5.614 \sqrt{x}/T^2\), and that \(S/x \leq 0.256 n_L \delta_C + 0.125 n_L\), under the assumption \(x \geq 4\). We then get

\[A \leq 2(S-1)(\log x - 1) - 0.445 n_L + 3 + 2n_L \delta_C - \frac{\sqrt{x}}{T} (0.743 W_L(T) + 1.167 n_L)\]

(4.13)

\[= D(x, T, n_L, \log \Delta_L).\]

By (1.2), we have (4.2) from (4.12) and (4.13). \(\square\)

5. Proof of Theorem 1.1

For \(L = Q\), the theorem is weaker than Lowell Schoenfeld’s result for \(x \geq 59\), and true in the range \([1, 59]\) by explicit computation. We assume henceforth that \(L \neq Q\), i.e. \(n_L \geq 2\).

Since \(\psi(C; x) \geq \psi_C(x)\), for the proof of the theorem it is sufficient to show that

(5.1)

\[\left( \frac{G}{|C|} \right) \psi(C; x) - x \leq \sqrt{x} \left[ \left( \frac{\log x}{2\pi} + 2 \right) \log \Delta_L + \left( \frac{\log^2 x}{8\pi} + 2 \right) n_L \right],\]

(5.2)

\[-\left( \frac{G}{|C|} \right) \psi_C(x) - x \leq \sqrt{x} \left[ \left( \frac{\log x}{2\pi} + 2 \right) \log \Delta_L + \left( \frac{\log^2 x}{8\pi} + 2 \right) n_L \right],\]

hold \(\forall x \geq 1\). Let then

\[B_a(x, T, n, L) := \frac{L_a(x, T, n, L)}{n \sqrt{x}} - \left( \frac{\log x}{2\pi} + 2 \right) \frac{L}{n} - \left( \frac{\log^2 x}{8\pi} + 2 \right).\]
\[ B_\theta(x, T, n, \mathcal{L}, g) := B_n(x, T, n, \mathcal{L}) + \frac{D(x, T, n, \mathcal{L})}{n^{\sqrt{x}}} + \frac{\mathcal{R}(\mathcal{L}) \log x}{\frac{p}{n} \sqrt{x}}, \]

where \( g \) is an integer, \( p \) is the smallest prime divisor of \( g \) and \( \mathcal{R}(\log \Delta_L) \) is an upper bound for \( n \), as given by Lemma 3.7, that will be made explicit later. To prove (5.1) it is sufficient to show that there is an \( \bar{x}^+ \geq 4 \) such that it is trivial for \( x \in [1, \bar{x}^+] \) and that when \( x \geq \bar{x}^+ \), by (1.1), there exists a value of \( T \geq 2\pi \) such that \( B_\theta(x, T, n_L, \log \Delta_L) \leq 0 \). To prove (5.2) it is sufficient to show that there is an \( \bar{x}^- \geq 4 \) such that it is trivial for \( x \in [1, \bar{x}^-] \) and that when \( x \geq \bar{x}^- \), by (4.2) and Lemma 3.6 there exists a value of \( T \geq 2\pi \) such that \( B_\theta(x, T, n_L, \log \Delta_L) \leq 0 \).

We assume, from now on, that \( T = T(x) := c \sqrt{x}/\log x \) with \( c := 5.2 \). This ensures in particular that \( T \geq 2\pi \) for any \( x > 1 \).

5.1. Upper bound. We first prove (5.1).

**Step 1: Trivial bound.** We notice that \( \psi(C; x) \leq \psi_K(x) \leq \psi_Q(x)n_K \). Hence, given that \( n_L = |G/n_K| \), the bound (5.1) is true if

\[ \sqrt{x} \left( \frac{\log x}{2\pi} + 2 \right) \frac{\log \Delta_L}{n_L} + \left( \frac{\log^2 x}{8\pi} + 2 \right) \geq \psi_Q(x) - \frac{x}{n_L}. \]

We will call this bound the trivial bound. We observe that \( \psi_Q \) is constant on the intervals \([p^m, q^n]\) where \( p^m \) and \( q^n \) are consecutive prime powers, hence if the trivial bound is true in \( p^m \) it is true in the whole interval \([p^m, q^n]\). We check that the bound is true for \( x < 61 \) if \( n_L = 4 \) and for \( x < 71 \) for any other value of \( n_L \in [2, 13] \) using the explicit lower bounds for \( \log \Delta_L \) in [16] and [14, Table 3]. For \( n_L \geq 14 \), \( \log \Delta_L \geq 2.12 \) as follows from entry \( b = 2.1 \) in [14, Table 3]. We this lower bound, we check that the stronger bound without the \( x/n_L \) term is true for \( x < 71 \). This ensures that it is true for \( x < 71 \) and \( n_L \geq 14 \).

Hence (5.1) is a consequence of the trivial bound if either \( n_L = 4 \) and \( x < 61 \) or \( n_L \neq 4 \) and \( x < 71 \).

**Step 2: function \( B_n \) is decreasing in \( \mathcal{L} \).** We have

\[ B_n(x, T(x), n_L, \mathcal{L}) = \left[ \frac{1}{\pi} \log \left( \frac{c/(2\pi)}{\log x} \right) - 0.296 + \frac{1.858}{T} + \frac{1.075}{\sqrt{x}} \right] \frac{\mathcal{L}}{n_L} + \frac{1}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{1}{8\pi} \log^2 x + \left( \frac{2}{\pi} + \frac{1.858}{T} \right) \log \left( \frac{T}{2\pi} \right) - 3.633 + \frac{7.729}{T} - \frac{1.501}{\sqrt{x}} + \frac{1}{n_L} \sqrt{x} \left[ x + 2T \log x + 8.276 - 0.744n_L\delta_C - 0.527r_1 \right] + \frac{1}{n_L} \left[ 7.834 + 3.779 \left( \frac{0.406 + 1}{4T} \right) r_1 - e_{n_L} \right]. \]

Since \( T(x) \) is an increasing function of \( x \geq e^2 \), \( \frac{\partial B_n}{\partial \mathcal{L}} \) is decreasing with \( x \). As \( \frac{\partial B_n}{\partial \mathcal{L}}(61, T(61)) \leq 0 \), we have that \( \frac{\partial B_n}{\partial \mathcal{L}} \leq 0 \) for any \( x \geq 61 \).

**Step 3: function \( B_n \) is decreasing in \( x \).** We have

\[ \frac{\partial B_n}{\partial x}(x, T(x), n_L, \mathcal{L}) \leq \frac{-\log x}{2\pi} \left[ 1 + \frac{1.858T'}{T^2} + \frac{1.075}{2x\sqrt{x}} \right] \frac{\mathcal{L}}{n_L} + \frac{T'}{T} \left( \frac{1}{T} - \frac{1.858}{T} \right) \log \left( \frac{T}{2\pi} \right) - \frac{\log x + 2T'}{4\pi x} \frac{5.621T'}{T^2} + \frac{\log x + 0.772}{2x\sqrt{x}} + \frac{1}{c_nLx} - \frac{4.138}{n_Lx^{3/2}} \]

where we have removed a few terms whose decreasing behaviour is evident, and used the facts that \( \mathcal{L}/n_L \geq \frac{1}{2} \log 3 \), \( \delta_C \leq 1 \), \( S \leq n_L \) and \( r_1 \leq n_L \). Since \( n_L \geq 2 \), we bound the last two terms by \( \max(0, 1/(cx) - 4.138x^{-3/2})/2 \) and the resulting function is an elementary one variable function which is negative for \( x \geq 61 \).
Step 4: estimates for \( n_L \geq 4 \). For \( n_L \geq 4 \), we have \( \log \Delta_L \geq n_L \) (this is true for all number fields except \( \mathbb{Q} \) and the four quadratic fields with \( \Delta_L \leq 7 \)). Given that \( B_a \) is a decreasing function of \( L \) for \( x \geq 61 \), we have

\[
B_a(x, T(x), n_L, \log \Delta_L) \leq B_a(x, T(x), n_L, n_L)
\]
as soon as \( n_L \geq 4 \) and \( x \geq 61 \).

Since \( \delta_C \geq 0 \), \( r_1 \geq 0 \), \( S \geq 0 \) and \( e_{n_L} \geq 0 \), we have

\[
B_a(x, T(x), n_L, n_L) \leq \frac{1}{\pi} \log \left( \frac{c/(2\pi)}{\log x} \right) - 0.296 + \frac{1.858}{T^{\sqrt{x}}} + \frac{1.075}{T^{2/\pi}}
+ \frac{1}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) - \frac{1}{8\pi} \log^2 x + \left( \frac{2}{\pi} + \frac{1.858}{T} \right) \log \left( \frac{T}{2\pi} \right) - 3.633 - \frac{7.729}{T} - \frac{1.501}{\sqrt{x}}
+ \frac{1}{n_L} \left[ x + 2 + \log x + 8.276 \right] + \frac{1}{n_L} \left[ 7.834 + \frac{3.779}{T} \right].
\]

This upper bound is decreasing in \( n_L \), because \( n_L \) only appears as the denominator of a fraction with positive numerator. Since \( B_a(61, T(61), 4, 4) < 0 \), the decreasing behaviour of \( B_a \) in \( x, n \) and \( L \) proves that \( B_a(x, T(x), n_L, \log \Delta_L) < 0 \) if \( n_L \geq 4 \) and \( x \geq 61 \). With the trivial bound in Step 1, we see that \( B_a(x, T(x), n_L, \log \Delta_L) < 0 \) if \( n_L \geq 4 \) and \( x \geq 1 \).

Step 5: estimates for \( n_L = 3 \), \( r_1 = 3 \). In this case \( \Delta_L \geq 49 \) and \( B_a(71, T(71), 3, \log 49) < 0 \) (where we use, as above, that \( \delta_C \geq 0 \) and \( S \geq 0 \)) which, including the trivial bound, concludes the proof.

Step 6: estimates for \( n_L = 3 \), \( r_1 = 1 \). In this case \( \Delta_L \geq 23 \) and we necessarily have \( L = \mathbb{K} \), hence \( \delta_C = 1 \) and \( S = (n_L + r_1)/2 = 2 \). Since \( B_a(71, T(71), 3, \log 23) < 0 \), the proof is complete for \( n_L = 3 \).

Step 7: estimates for \( n_L = 2 \), large \( \Delta_L \) or large \( x \). We observe that the trivial bound extends to \( x < 607 \) when \( \Delta_L \geq 300 \). As above the worst case is for \( \delta_C = 0 \) and \( r_1 = 0 \) and in that case \( S = 1 \). We have \( B_a(607, T(607), 2, \log 300) < 0 \), which means that the case where \( n_L = 2 \), \( \Delta_L \geq 300 \) is proved.

Besides, we observe that also \( B_a(10^5, T(10^5), 2, \log 3) < 0 \), keeping the worst case \( \delta_C = 0 \), \( r_1 = 0 \) and \( S = 1 \), hence \( \square \) for \( n_L = 2 \) is proved also for \( x \geq 10^5 \). Hence \( \square \) is proved for \( n_L = 2 \) if either \( \Delta_L \geq 300 \) or \( x \geq 10^5 \).

Step 8: estimates for \( n_L = 2 \), small \( \Delta_L \) and small \( x \). For the remaining quadratic fields \( L \) the proof will be made together with the lower bound.

5.2. Lower bound. We now turn to \( \square \). Lemma 3.7(iv) shows that \( n \leq \log \Delta_L / (\log \log \Delta_L - \log n_K - 1.1714) \) when \( \log \Delta_L \geq e^{1.1714} n_K \).

To get an easier estimate we use line \( b = 4.1 \) of Table 3 in \( \square \), producing the lower bound

\[
\log \Delta_L - \log n_K - 1.1714 \geq \log(n_L \log 25.585 - 28.36) - \log n_K - 1.1714
\]

which is \( \log \left( |G| \log 25.585 - 28.36 \right) - 1.714 \geq \log(|G| - 8.79). \]

Moreover, Lemma 3.7(iii) implies that \( n \leq 0.4 + \log \Delta_L / \log 22 \) if \( |G| \) is not prime – where the 0.4 has been added to handle the exceptions. We thus define

\[
\mathfrak{N}(L) := \begin{cases} 
0 & \text{if } |G| = 1, \\
\mathcal{L} / \log(|G| - 8.79) & \text{if } |G| \geq 32, \\
\mathcal{L} / \log 4 & \text{if } |G| \text{ is a prime } \leq 31 \text{ and } \neq 3, \\
\mathcal{L} / \log 49 & \text{if } |G| = 3, \\
0.4 + \mathcal{L} / \log 22 & \text{otherwise}.
\end{cases}
\]
We observe that the derivative $\partial B_b / \partial x$ is a constant depending only on $|G|$. Moreover, since $x \geq 16n_L^2 \geq 16|G|^2$, 

$$\frac{\partial}{\partial \mathcal{L}} \left[ \frac{|G|\mathcal{N}(\mathcal{L}) \log x}{p \sqrt{x}} \right] = \frac{|G|\mathcal{N}'(\mathcal{L}) \log x}{p \sqrt{x}} \leq \frac{\mathcal{N}' \log (4|G|)}{2p}.$$ 

By computing the values for $2 \leq |G| \leq 32$, and using the lower bound $x \geq 16|G|^2$, we observe that 

$$\frac{1.075}{\sqrt{x}} + \frac{\mathcal{N}' \log (4|G|)}{2p} \leq 0.51.$$ 

The conclusion holds also for any $|G| > 32$ because 

$$\frac{\mathcal{N}' \log (4|G|)}{2p} \leq \frac{\log (4|G|)}{4 \log (|G| - 8.79)}$$ 

which decreases in $|G|$. We thus get 

$$n_L \frac{\partial B_b}{\partial \mathcal{L}} \leq \frac{1}{\pi} \log \left( \frac{c/(2\pi)}{\log x} \right) - 0.296 + \frac{1.115}{T} + 0.51$$ 

which is negative because $x \geq 64$ hence $T \geq 10$.

**Step 3: function $B_b$ is decreasing in $x$.** We have 

$$\frac{\partial B_b}{\partial x} (x, T(x), n_L, \mathcal{L}, |G|) \leq -\log 3 \left[ \frac{1}{\pi x \log x} + \frac{1.115 T'}{T^2} \right] + \log \left( \frac{T}{2\pi} \right) \leq -\log x + \frac{2T'}{\pi T} \left( 5.197 + 2.473 \right) + \log x - \frac{2}{2n_L} \sqrt{x}$$ 

which is negative as well for $x \geq 64$. 

In this way, from Lemma 3.7 we have $n \leq \mathcal{N}(\log \Delta)$. Before starting the proof, we observe that if $K = L$, then $\mathcal{N}(\mathcal{L}) = 0$. Thus, when we are able to prove that $B_b \leq 0$ for suitable $x, T$ (and a certain value for the parameters $r_1$ and $S$) under the assumption that $K \neq L$, then with the same values for $x$ and $T$, we have $B_b \leq 0$ also for $K = L$ (and the same value for $r_1$ and $S$).

**Step 1: trivial bound.** Bound (5.2) is satisfied if 

$$\left( \frac{\log x}{2\pi} + 2 \right) \log \Delta_{\mathcal{L}} + \left( \frac{\log^2 x}{8\pi} + 2 \right) n_L \geq \sqrt{x}$$ 

because in this case it is weaker than the trivial bound $\psi_C(x) \geq 0$. Since for $n_L \geq 3$ we have $\log \Delta_{\mathcal{L}} \geq n_L$, we see that this is true if $x \leq 16n_L^2$. This extends to $n_L = 2$ by direct computation.

For the end of this subsection, we will assume $x \geq 16n_L^2$ (and hence $x \geq 16|G|^2$ and $x \geq 64$).

**Step 2: function $B_b$ is decreasing in $\mathcal{L}$.** We have 

$$B_b(x, T(x), n_L, \mathcal{L}, |G|) = \frac{1}{\pi} \log \left( \frac{c/(2\pi)}{\log x} \right) - 0.296 + \frac{1.115}{T} + \frac{1.075}{\sqrt{x}} \log x + \frac{|G|}{p} \frac{\mathcal{N}(\mathcal{L}) \log x}{\sqrt{x}}$$

$$+ \frac{1}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) - \frac{3.63}{\pi x 

which decreases in $|G|$. We thus get 

$$n_L \frac{\partial B_b}{\partial \mathcal{L}} \leq \frac{1}{\pi} \log \left( \frac{c/(2\pi)}{\log x} \right) - 0.296 + \frac{1.115}{T} + 0.51$$ 

which is negative because $x \geq 64$ hence $T \geq 10$. 

**Step 3: function $B_b$ is decreasing in $x$.** We have 

$$\frac{\partial B_b}{\partial x} (x, T(x), n_L, \mathcal{L}, |G|) \leq -\log 3 \left[ \frac{1}{\pi x \log x} + \frac{1.115 T'}{T^2} \right] + \log \left( \frac{T}{2\pi} \right) \leq -\log x + \frac{2T'}{\pi T} \left( 5.197 + 2.473 \right) + \log x - \frac{2}{2n_L} \sqrt{x}$$ 

which is negative as well for $x \geq 64$. 

By computing the values for $2 \leq |G| \leq 32$, and using the lower bound $x \geq 16|G|^2$, we observe that 

$$\frac{1.075}{\sqrt{x}} + \frac{\mathcal{N}' \log (4|G|)}{2p} \leq 0.51.$$ 

The conclusion holds also for any $|G| > 32$ because 

$$\frac{\mathcal{N}' \log (4|G|)}{2p} \leq \frac{\log (4|G|)}{4 \log (|G| - 8.79)}$$ 

which decreases in $|G|$. We thus get 

$$n_L \frac{\partial B_b}{\partial \mathcal{L}} \leq \frac{1}{\pi} \log \left( \frac{c/(2\pi)}{\log x} \right) - 0.296 + \frac{1.115}{T} + 0.51$$ 

which is negative because $x \geq 64$ hence $T \geq 10$. 

**Step 3: function $B_b$ is decreasing in $x$.** We have 

$$\frac{\partial B_b}{\partial x} (x, T(x), n_L, \mathcal{L}, |G|) \leq -\log 3 \left[ \frac{1}{\pi x \log x} + \frac{1.115 T'}{T^2} \right] + \log \left( \frac{T}{2\pi} \right) \leq -\log x + \frac{2T'}{\pi T} \left( 5.197 + 2.473 \right) + \log x - \frac{2}{2n_L} \sqrt{x}$$ 

which is negative as well for $x \geq 64$. 


Step 4: estimates for $n_L \geq 4$. We have $\log \Delta_L \geq n_L$. Given that $B_b$ is a decreasing function of $L$ for $x \geq 64$, we have

$$B_b(x, T(x), n_L, \log \Delta_L, |G|) \leq B_b(x, T(x), n_L, n_L, |G|)$$

as soon as $x \geq 64$. We know that $S \leq (n_L + r_1)/2$; introducing this bound in $B_b$, the term depending on $r_1$ in $B_b$ becomes

$$\frac{r_1}{n_L \sqrt{x}} \left( \frac{1}{2} (\log x - 1) - 0.527 \left( \frac{0.406 + \frac{1}{4T}}{\sqrt{x}} \right) \right)$$

which is $\leq 0$ for every $x$. Its larger value is therefore reached for $r_1 = 0$. Once the bound $\delta_C \leq 1$ is also considered, we get the upper bound

$$B_b(x, T, n_L, n_L, |G|) \leq \frac{1}{\pi} \log \left( \frac{c}{2\pi} \right) \log x - 0.296 + \frac{1.115}{\sqrt{x}} + \frac{|G| \Re(n_L) \log x}{p \ n_L \sqrt{x}} + \frac{1}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) - \frac{1}{8\pi} \log^2 x + \left( \frac{2}{\pi} + \frac{1.115}{T} \right) \log \left( \frac{T}{2\pi} \right) - 3.633 + \frac{6.562}{T} \frac{\log x - 2.380}{2\sqrt{x}} + \frac{1}{n_L \sqrt{x}} \left[ \frac{x+2}{T} \log x + 13.276 \right]$$

$$+ \frac{1}{n_L} \left[ 7.834 + 3.779 \right].$$

Once again this is decreasing in $n_L$, as long as $|G|/p$ remains constant and $\Re$ does not change form, since $7.834 \sqrt{x} - \log x > 0$. We check that $B_b$ is negative in the proper range of its arguments by checking that this upper bound is negative, too. Doing this, we can restrict the test to the cases with $|G| \geq 2$: in fact, $\frac{|G| \Re(n_L) \log x}{p \ n_L \sqrt{x}}$ is the unique term depending on $|G|$ appearing there, and it is zero when $|G| = 1$. Moreover, for each $|G|$, we only need to check whether the right hand side with $x = 16n_L^2$, $T = T(16n_L^2)$ is negative when $n_L = |G|$ (if $|G| \geq 4$) or when $n_L = 2|G|$ (if $|G| = 2$ or 3).

If $|G| \geq 32$, then $n_L = |G|$ and

$$\frac{|G| \Re(n_L) \log (16n_L^2)}{p \ n_L \sqrt{16n_L^2}} = \frac{\log (4|G|)}{2p \log (|G| - 8.79)} \leq \frac{\log (4|G|)}{4 \log (|G| - 8.79)}$$

which is decreasing in $|G|$, so, we just need to test the value for $n_L = |G| = 32$.

If $|G| \leq 31$ is not prime, we need to check for $|G|/p \in \{2, ..., 15\}$, but from the decreasing argument (now in $p$ with fixed $|G|/p$) we only need to check the case $p = 2$, i.e. $|G|$ even in $[4, 30]$.

If $|G| \leq 31$ is prime (but different from 3) we have

$$\frac{|G| \Re(n_L) \log (16n_L^2)}{p \ n_L \sqrt{16n_L^2}} = \frac{\log (4n_L)}{2n_L \log 4},$$

which decreases in $n_L$. Thus we just need to check the case $n_L = 4$, and hence $|G| = 2$.

If $|G| = 3$, then $n_L = 6$ and

$$\frac{|G| \Re(n_L) \log (16n_L^2)}{p \ n_L \sqrt{16n_L^2}} = \frac{\log (4n_L)}{2n_L \log 49},$$

which is smaller than what we got previously for the case $|G| = 6$.

In total we have sixteen cases: $n_L = |G| = 32$, $n_L = |G|$ even in $[4, 30]$ and $n_L = 4$ with $|G| = 2$. All sixteen values are negative. We have covered all cases for $|G|/p$ and $\Re$ hence, together with the trivial bound, this proves the lower bound for $n_L \geq 4$. 

Step 5: estimates for \( n_L = 3 \). We have \( \Delta_L \geq 23, \delta_C \leq 1 \). As for the previous case, we estimate \( S \) with \((n_L+r_1)/2\) and the emerging term depending on \( r_1 \) with its largest value, which now corresponds to \( r_1 = 1 \) (because for \( n_L = 3 \) the unique admissible values for \( r_1 \) are 1 and 3). This produces the bound

\[
B_b(x, T(x), 3, \log 23, |G|) \leq \left[ \frac{1}{\pi} \log \left( \frac{c/(2\pi)}{\log x} \right) - 0.296 + \frac{1.115}{T} + \frac{1.075}{\sqrt{x}} \right] \log 23 + \frac{\log 23 \log x}{3 \log 49 \sqrt{x}} + \frac{1}{2\pi} \log^2 \left( \frac{T}{2\pi} \right) - \frac{1}{8\pi} \log^2 x + \frac{\left( \frac{2}{\pi} + \frac{1.115}{T} \right)}{\log \left( \frac{T}{2\pi} \right)} - 3.633 + \frac{6.562}{T} - \frac{1.946}{\sqrt{x}} + \frac{1}{3} \left[ 7.115 + \frac{3.529}{T} \right] + \frac{1}{3} \frac{[x+2+\log x+14.517]}{\sqrt{x}},
\]

which is negative for \( x = 16n_L^2 = 169 \) and \( T = T(169) \). This completes the proof of the claim for \( n_L = 3 \).

Step 6: estimates for \( n_L = 2 \), large \( \Delta_L \) or large \( x \). The worst case happens when \( \delta_C = 1, |G| = 2, S = 1+r_1/2 \) and \( r_1 = 0 \). For \( \Delta_L \geq 300 \), we observe that the trivial bound extends to \( x \leq 598 \) and that \( B_0(598, T(598), 2, \log 300, 2) < 0 \) if \( r_1 = 0 \). This means that the case where \( \Delta_L \geq 300 \) is proved. We observe that \( B_b(10^5, T(10^5), 2, \log 3, 2) < 0 \), hence the claim is proved for \( x \geq 10^5 \).

Step 7: estimates for \( n_L = 2 \), small \( \Delta_L \) and small \( x \). For the remaining fields \( \mathbb{L} \), which are quadratic with \( \Delta_L < 300 \), let \( x_1(\mathbb{L}) \geq 61 \) be such that \( B_B(x_1(\mathbb{L}), T(x_1(\mathbb{L})), n_L, \log \Delta_L) < 0 \) (with \( \delta_C = 0 \)) and \( B_b(x_1(\mathbb{L}), T(x_1(\mathbb{L})), n_L, \log \Delta_L, 2) < 0 \) (with \( \delta_C = 1 \)), where we use the true value of \( n \). As we have seen, for all fields \( x_1(\mathbb{L}) \leq 20 \). To complete the proof of Theorem 1.1 we have built a program that checks for each integer \( x \in [1, x_1(\mathbb{L})] \) that

\[
-B+1 \leq \psi_{r_1}(x) - x \leq B,
-B+1 \leq 2\psi_C(x) - x = 2\psi(C; x) - x \leq B,
\]

where

\[
B : = \sqrt{x} \left[ \left( \frac{\log x}{2\pi} + 2 \right) \log \Delta_L + 2 \left( \frac{\log^2 x}{8\pi} + 2 \right) \right].
\]

6. Proof of Corollary 1.2

The bounds stated in the corollary are certainly true as soon as

\[
\sqrt{x} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log x} \right) \log \Delta_L + \left( \frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) \right] \geq \max \left( \int_2^x \frac{du}{\log u}, \pi(x) - \frac{1}{n_L} \int_2^x \frac{du}{\log u} \right),
\]

because in this case the conclusion is weaker than the elementary bound \( 0 \leq \pi_{\mathbb{C}}(x) \leq \pi(x)n_k \). The first inequality holds when \( x \in [2, 193] \), because \( \frac{1}{n_L} \log \Delta_L \geq \frac{1}{2} \log 3 \), and

\[
\sqrt{x} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log x} \right) \frac{1}{2} \log 3 + \left( \frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) \right] \geq \int_2^x \frac{du}{\log u}
\]

holds in this range. The second inequality

\[
\sqrt{x} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log x} \right) \frac{1}{n_L} \log \Delta_L + \left( \frac{\log x}{8\pi} + \frac{1}{4\pi} + \frac{6}{\log x} \right) \right] \geq \pi(x) - \frac{1}{n_L} \int_2^x \frac{du}{\log u}
\]

is checked for \( x \in [2, 193] \) by testing it for each \( n_L \leq 20 \) (using the lower bound for \( \log \Delta_L \) as follows from Odlyzko’s tables for each degree). The case \( n_L = 20 \) is checked in the stronger version where \( -\frac{1}{n_L} \int_2^x \frac{du}{\log u} \) is removed, so that its validity implies the validity also for all
\[ n_L \geq 20. \] In this way the corollary is fully proved up to 193.

Let
\[ \vartheta(C; x) := \sum_{p \mid Np} \theta(C; p) \log Np. \]

Then by partial summation
\[
\frac{|G|}{|C|} \pi(C; x) - \int_2^x \frac{du}{\log u} = \frac{|G|}{|C|} \pi(C; 73) - \frac{|G|}{|C|} \vartheta(C; 73) \log 73 + \int_2^x \frac{du}{\log u} \left( \vartheta(C; x) - x \right) \log x + \int_{73}^x \frac{|G|}{|C|} \left( \vartheta(C; u) - u \right) \log u \, du.
\]

Assuming \( x \geq 193 \), we have
\[
0 \leq \pi(C; 73) - \frac{\vartheta(C; 73)}{\log 73} \leq \sum_{p \leq 73} \left( 1 - \frac{\log Np}{\log 73} \right) \leq \sum_{p \leq 73} \left( 1 - \frac{\log p}{\log 73} \right) n_K \leq 5.65 n_K \leq 2.15 \sqrt{x} n_K,
\]

and
\[
0 \leq \int_2^x \frac{du}{\log u} - \frac{73}{\log 73} \leq 6.1 \leq 1.16 \sqrt{x} n_L,
\]

by [20, Th. 13]. We deduce that
\[
\frac{|G|}{|C|} \pi(C; x) - \int_2^x \frac{du}{\log u} \leq \left( \frac{1}{2\pi} + \frac{2}{\log x} \right) \log \Delta_L + \frac{4.59}{\log x} n_L + \int_2^x \frac{|G|}{|C|} \psi(C; u) - u \right) \frac{\log u}{u \log^2 u} \, du
\]
\[
\leq \sqrt{x} \left[ \left( \frac{1}{2\pi} + \frac{2}{\log x} \right) \log \Delta_L + \frac{4.59}{\log x} n_L \right]
\]
\[
+ \int_{73}^x \frac{\log u + 2}{\sqrt{\log x} \log^2 u} \, du.
\]

Since \( \int_{73}^x \frac{\log u + 2}{\sqrt{\log x} \log^2 u} \, du \leq \sqrt{x} \) and \( \int_{73}^x \frac{du}{\sqrt{\log x} \log^2 u} \leq 0.33 \sqrt{x} \) (for \( x \geq 193 \)), we get
\[
\leq \sqrt{x} \left[ \left( \frac{1}{2\pi} + \frac{3}{\log x} \right) \log \Delta_L + \frac{1}{4\pi} + \frac{6}{\log x} \right] n_L,
\]

which concludes the proof of the claim for \( \pi(C; x) \). For \( \pi_C(x) \) the argument is the same.

**Appendix A. Number of zeros**

Trudgian [23] showed how to take advantage of both Backlund’s and Rosser’s approaches to produce good explicit bounds for the function \( N(T) \) counting non-trivial zeros \( \rho \) with \( |\text{Im} \rho| \leq T \) for Dirichlet and Dedekind \( L \)-functions. Note that, contrary to the rest of this paper, Trudgian’s approach does not require to assume any form of the Riemann Hypothesis. Studying his paper we have found some possible improvements in the way some terms are bounded. We have also noted that the original paper does not isolate the role of a special constant (the analogue of the constant \(-7/8\) appearing for Riemann’s zeta in [4, Ch. 15, (1)])

However, isolating this term allows to formulate the bound with smaller constants, and this is very useful when sums on zeros of type \( \sum_{|\text{Im} \rho| > a} f(\rho) \) with \( a > 0 \) are estimated via partial summation, because in this case that term does not contribute and only the smaller constants appear. This is very important for our application, since we need to take advantage of every possible method to improve the constants, in order to reduce the set of explicit computations.
which are needed to prove Theorem 1.1.

Moreover, we have also noticed that essentially the same strategy can be applied to study the zeros of all Hecke’s \( L \)-functions of finite order Größencharakter, thus we have formulated the results for this more general set, for possible future reference.

We stress once again that the main strategy for this computation has to be credited to Trudgian, our contribution being limited to the points cited above.

Let \( E \) be a number field. Let \( \chi \) be a Hecke Größencharakter of \( E \) which is primitive and of finite order. Let \( f(\chi) \) denote the conductor of \( \chi \) and set \( Q(\chi) = \Delta_E N_{E/Q}(f(\chi)) \). Let \( \delta_\chi \) be 1 if \( \chi \) is trivial and 0 otherwise. Let \( N(T,\chi) \) be the number (multiplicity included) of non-trivial zeros \( \rho \) (i.e. with \( \text{Re} \rho \in (0,1) \)) with \( |\text{Im} \rho| \leq T \) for \( L(s,\chi) \).

**Theorem A.1.** Unconditionally,

\[
\left| N(T,\chi) - \frac{T}{\pi} \log \left( \frac{Q(\chi)}{2\pi e} \right)^{n_E} - 2\delta_\chi \frac{a_\chi-b_\chi}{4} \right| \leq D_1 (\log Q(\chi) + n_E \log T + D_2 n_E + \delta_\chi D'_3
\]

when \( T \geq T_0 \), for \( T_0, D_1, D'_2 \) and \( D'_3 \) as in Table 1.

If \( \chi \) is the trivial character, then \( E = L \) and \( N(T,\chi) = N_L(T) \) is the number of non-trivial zeros of \( \zeta_L \) with imaginary part in \([-T,T]\). In that case \( Q(\chi) = \Delta_L \) and \( a_\chi-b_\chi = r_1 \). If one want to compare this result with the analogue contained in [23, Theorem 2] one has to take note of the extra term \(-2+\frac{1}{4} r_1\) that we have put in evidence (as for Riemann’s zeta in [4, Ch. 15, (1)])

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
D_1 & T_0 = 1 & D'_2 & D'_3 & T_0 = 2\pi & D'_2 & D'_3 & T_0 = 10 \\
\hline
0.230 & 16.577 & 1.330 & 16.032 & 0.033 & 16.004 & 0.014 \\
0.247 & 8.180 & 1.435 & 7.614 & 0.083 & 7.585 & 0.062 \\
0.265 & 6.416 & 1.515 & 5.834 & 0.150 & 5.805 & 0.129 \\
0.282 & 5.409 & 1.598 & 4.812 & 0.213 & 4.783 & 0.192 \\
0.299 & 4.696 & 1.699 & 4.083 & 0.275 & 4.053 & 0.254 \\
0.316 & 4.158 & 1.814 & 3.526 & 0.335 & 3.495 & 0.313 \\
0.333 & 3.735 & 1.961 & 3.082 & 0.400 & 3.050 & 0.371 \\
0.350 & 3.425 & 2.185 & 2.731 & 0.429 & 2.698 & 0.402 \\
0.367 & 3.206 & 2.426 & 2.467 & 0.453 & 2.432 & 0.423 \\
0.384 & 3.043 & 2.687 & 2.257 & 0.478 & 2.221 & 0.444 \\
0.401 & 2.918 & 2.966 & 2.083 & 0.503 & 2.044 & 0.465 \\
0.460 & 2.666 & 4.082 & 1.645 & 0.593 & 1.598 & 0.540 \\
\hline
\end{array}
\]

**Proof.** We first suppose that \( \chi \) is non-trivial. Let \( \sigma_1 \in (1,2) \) and let \( \mathcal{R} \) be the rectangle with vertices \( \sigma_1 \pm iT \) and \( 1-\sigma_1 \pm iT \), positively oriented. We furthermore assume that \( T \) is not the imaginary part of any zero of \( L(s,\chi) \). The conclusion for the missing \( T \)'s follows because \( N(T,\chi) \) is upper-continuous and all other functions are continuous. Cauchy’s argument principle shows that

\[
2\pi N(T,\chi) = \Delta_R \arg \xi(s,\chi),
\]

where \( \Delta_R \arg \xi(s,\chi) \) is the variation of the argument of \( \xi(s,\chi) \) along \( \mathcal{R} \). The functional equation shows that the variation of the argument we have in the left half-rectangle equals the variation in the right half-rectangle. Hence

\[
\pi N(T,\chi) = \Delta_C \arg \xi(s,\chi)
\]
where $C$ is the path $1/2-iT \to \sigma_1-iT \to \sigma_1+iT \to 1/2+iT$ and $\Delta_c$ is the variation along $C$. Hence
\[
\pi N(T, \chi) = \Delta_c \arg (Q(\chi)^{1/2}) + \Delta_c \arg \Gamma(s) + \Delta_c \arg L(s, \chi)
\]
\[
= \Delta_c \arg (Q(\chi)^{1/2} + a_c \Delta_c \arg (\pi^{-2} \Gamma(1/2)) + b_c \Delta_c \arg (\pi^{-2} \Gamma(s+1/2)) + \Delta_c \arg L(s, \chi).
\]
Letting $q(\chi) := Q(\chi)^{1/\pi}$ it becomes:
\[
= \Delta_c \arg \left(\left(\frac{q(\chi)}{\pi}\right)^{sn_{e/2}}\right) + a_c \Delta_c \arg \left(\frac{s}{2}\right) + b_c \Delta_c \arg \left(\frac{s+1}{2}\right) + \Delta_c \arg L(s, \chi).
\]
We define the function $g(\alpha, T)$ by
\[
(A.1) \quad \text{Im} \log \Gamma\left(\frac{1+2\alpha}{4} + \frac{iT}{2}\right) = -\frac{T}{2} \log \frac{T}{2e} + (2\alpha-1) \frac{\pi}{8} + g(\alpha, T)
\]
for $T > 0$, and by Stirling's formula we know that $g(\alpha, T) = O(1/T)$ as $T \to +\infty$. Thus, in terms of $g(\alpha, T)$ we get
\[
\pi N(T, \chi) = n_{2E} \log \left(\frac{q(\chi)T}{2\pi e}\right) + \frac{\pi}{4} (b_\chi - a_\chi) + 2 \alpha_\chi g(0, T) + 2 \beta_\chi g(1, T) + \Delta_c \arg L(s, \chi).
\]
We first show that $g(1, T) \leq g(0, T)$ for every $T \geq 0$. In fact, setting $z := \frac{1}{4} + iT$, by Euler's reflection formula
\[
\frac{\Gamma(\frac{1}{4} + iT)}{\Gamma(\frac{3}{4} + iT)} = \frac{|\Gamma(z)|^2}{\pi \sqrt{2}} \left(\cosh \left(\frac{\pi T}{2}\right) - i \sinh \left(\frac{\pi T}{2}\right)\right).
\]
Since this fraction is in the fourth quadrant, this equality implies that
\[
g(0, T) - g(1, T) = \frac{\pi}{4} \arg \left(\frac{\Gamma(\frac{1}{4} + iT)}{\Gamma(\frac{3}{4} + iT)}\right) = \frac{\pi}{4} \arg \left(\tan \left(\frac{\pi T}{2}\right)\right) > 0.
\]
For $g(\alpha, T)$ we have the equalities:
\[
(A.2) \quad g(\alpha, T) = -\frac{2\alpha - 1}{4} \\text{atan} \left(\frac{2\alpha + 1}{2T}\right) + \frac{T}{4} \log \left(1 + \left(\frac{2\alpha + 1}{2T}\right)^2\right) - \frac{T}{64 + \alpha + iT} \left(\frac{3 T \theta}{40 + \alpha + iT}\right)
\]
for some $\theta \in [-1, 1]$ (see \cite{1} Th. 1.4.2], with $m = 2$), and
\[
g(\alpha, T) = -\frac{2\alpha - 1}{4} \\text{atan} \left(\frac{2\alpha + 1}{2T}\right) + \frac{T}{4} \log \left(1 + \left(\frac{2\alpha + 1}{2T}\right)^2\right) + \int_0^{+\infty} \left(\frac{1}{2} \frac{1}{t^2 + 1} \frac{e^{-2(\alpha + 1)/4}}{t} \sin \left(\frac{T t}{2}\right)\right) dt
\]
(see \cite{1} Th. 1.6.3 (i)) when $2\alpha + 1 > 0$. The first formula is strong enough to prove that $g(1, T) > 0$ for $T \geq 1.5$ (but an explicit computation shows that this holds also for $T \in [1, 1.5]$). The second one (with some tedious but elementary work) shows that $g(0, T)$ decreases for $T \geq 1$. Therefore
\[
(A.3) \quad \left|N(T, \chi) - \frac{n_{2E} T}{\pi} \log \left(\frac{q(\chi)T}{2\pi e}\right) + \frac{a_\chi - b_\chi}{4}\right| \leq \frac{2n_{2E}}{\pi} g(0, T_0) + \frac{1}{\pi} |\Delta_c \arg L(s, \chi)|
\]
for every $T \geq T_0 \geq 1$.
To bound $\Delta_c \arg L(s, \chi)$ we split $C$ in three segments $C_1$, $C_2$ and $C_3$ where $C_2$ is the vertical one. We have
\[
(A.4) \quad |\Delta_c \arg L(s, \chi)| \leq 2 |\log \zeta(E(\sigma_1))| \leq 2n_{2E} \log \zeta(\sigma_1).
\]
To bound $\Delta c_1 \arg L(s, \chi)$ and $\Delta c_3 \arg L(s, \chi)$ we apply Backlund’s argument \cite{2}, in the version given by Trudgian \cite{23}. Let
\begin{equation}
(A.5) \quad f(s) := \frac{1}{2} \left( L(s+iT, \chi)^N + L(s-iT, \bar{\chi})^N \right)
\end{equation}
for some positive integer $N$. Suppose that there are $n$ distinct zeros of $f(\sigma)=\text{Re}(L(\sigma+iT, \chi)^N)$ for $\sigma \in \left[\frac{1}{2}, \sigma_1\right]$. These zeros partition the segment into $n+1$ intervals. On each interval $\arg(L(\sigma+iT, \chi)^N)$ can vary by at most $\pi$. Thus
\begin{align*}
|\Delta c_3 \arg L(s, \chi)| &= \frac{1}{N} |\Delta c_3 \arg L(s, \chi)^N| \leq \frac{(n+1)\pi}{N}.
\end{align*}
By symmetry the same bound applies on $C_1$, thus (A.3) becomes
\begin{equation}
(A.6) \quad \left| N(T, \chi) - \frac{n E}{\pi} \log \left( \frac{q(\chi)T}{2\pi e} + \frac{a\chi - b_\chi}{4} \right) \right| \leq \frac{2n E}{\pi} \left( g(0,T) + \log \zeta(\sigma_1) \right) + \frac{2(n+1)}{N}.
\end{equation}
In order to bound $n$ we apply Jensen’s formula, see \cite{11} (8) or \cite{21} Th. 15.18 p. 307,
\begin{align*}
\log \frac{R^m}{|a_1 a_2 \cdots a_m|} &= \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(a + R e^{i\phi})| \, d\phi - \log |f(a)|
\end{align*}
where $f$ is any function which is holomorphic in the disc centred in $a$ and radius $R$, $f(a)$ is assumed to be not zero, and $a_j$ for $j = 1, \ldots, m$ is the list of all zeros of $f$ in the disc (further assuming that there are no zeros on the boundary). We set $a = 1+\eta$ with $\eta \in (0,1]$, $R = r(\frac{1}{2} + \eta)$, $r > 0$ and apply Jensen’s formula to the function in (A.5). Assuming for the moment that $f(1+\eta) \neq 0$, \cite{23} Lemma 2 \cite{23} (a special realization of Backlund’s trick) shows that if $\sigma_1 = \frac{1}{2} + \sqrt{2(\frac{1}{2} + \eta)}$ and $1 - \sigma_1 > a - R$ (which corresponds to $r > 1 + \sqrt{2}$) there are $n' \geq n - 2 - \frac{NE}{r}$ real zeros in the circle and smaller than $1/2$ which coupled with the $n$ zeros allow one to prove that
\begin{align*}
\log \frac{R^m}{|a_1 a_2 \cdots a_m|} &\geq \log \frac{R^{n+n'}}{|a_1 a_2 \cdots a_{n+n'}|} \geq (n+n') \log r \geq 2 \left( n - 1 - \frac{NE}{2\pi} \right) \log r,
\end{align*}
where $E$ is any upper bound for
\begin{equation}
(A.7) \quad |\Delta_+ \arg L(s, \chi) + \Delta_- \arg L(s, \chi)|,
\end{equation}
where $\Delta_+ \arg$ denotes the change of the argument between the points $\frac{1}{2} \pm \delta + iT$, with $\delta := \sigma_1 - \frac{1}{2}$, and the point $\frac{1}{2} + iT$, proviso that
\begin{equation}
(A.8) \quad |\Delta c_1 \arg L(s, \chi)^N| \geq 3\pi + NE.
\end{equation}
An argument of Heath-Brown \cite{23} Subsection 3.1 shows that the same conclusion holds also if $\sigma_1 < a - R$ but without the assumption $1 - \sigma_1 > a - R$. As a consequence, for $n$ (the number of zeros of $f(\sigma)$ in $[\frac{1}{2}, \sigma_1]$) we have the bound
\begin{equation}
(A.9) \quad n \leq 1 + \frac{NE}{2\pi} + \frac{1}{4\pi \log r} \int_{0}^{2\pi} \log |f(a + Re^{i\phi})| \, d\phi - \frac{1}{2\log r} \log |f(a)|,
\end{equation}
when (A.8) holds. To bound the integral, we first use the inequality $|f(s)| \leq |L(s, \chi)^N|$. For $\phi \in [-\pi/2, \pi/2]$, we bound $L(s, \chi)$ with what we get from its representation as Dirichlet series on the half-circle $a + Re^{i\phi}$. Thus,
\begin{align*}
\frac{1}{N} \int_{-\pi/2}^{\pi/2} \log |f(a + Re^{i\phi})| \, d\phi &\leq \frac{1}{N} \int_{-\pi/2}^{\pi/2} \log |L(a+iT+Re^{i\phi}, \chi)^N| \, d\phi \\
&\leq \int_{-\pi/2}^{\pi/2} \log(\zeta(a+R \cos \phi)) \, d\phi \leq n E \int_{-\pi/2}^{\pi/2} \log(\zeta(a+R \cos \phi)) \, d\phi.
\end{align*}
For the remaining part of the domain, following [23] Subsection 4.1, we use Lindelöf’s convexity bound [18] on the strip \( p \leq \sigma \leq a \), where the negative parameter \( p \) has to satisfy both \( p \geq -1/2 \) to use [18], and \( p \leq a - R \) so that the left half-circle is included in the strip. In fact, by (2.5), (2.6), (2.7) and [18] Lemmas 1, 2 we get

\[
|L(s, \chi)| = \left( \frac{Q(\chi)}{\pi^{n_\chi}} \right)^{\frac{1}{2} - \sigma} \Gamma\left( \frac{1+\frac{a}{2}}{2} \right) \Gamma\left( \frac{\sigma + \frac{a}{2}}{2} \right) \left| \chi \Gamma\left( \frac{\sigma + \frac{a}{2}}{2} \right) L(1-s, \chi) \right| 
\]

\[
\leq \left( \frac{Q(\chi)}{(2\pi)^{n_\chi}} \right)^{\frac{1}{2} - \sigma} \left| 1 + s \right|^{\frac{1}{2} - \sigma} n_\chi |L(1-s, \chi)|
\]

for \( \sigma \in [-\frac{1}{2}, \frac{1}{2}] \). In particular, for \( p \in [-\frac{1}{2}, 0] \)

\[
|L(p+it, \chi)| \leq \left( \frac{q(\chi) |1+p+it|}{2\pi} \right)^{\frac{1}{2} - p} n_\chi (1-p)^{n_\chi}
\]

and by [18] Th. 2 we conclude

\[
|L(s, \chi)| \leq \left\{ \left( \frac{q(\chi) |1+s|}{2\pi} \right)^{(1/2-p)(1+\eta-\sigma)} \zeta(1-p)^{1+\eta-\sigma} \zeta(1+\eta)^{\sigma-p} \right\}^{n_\chi/(1+\eta-p)},
\]

valid for \( p \leq \sigma \leq 1+\eta \) where \(-\frac{1}{2} \leq p < 0 < \eta \leq \frac{1}{2} \). We thus have

\[
\frac{1}{N} \int_{\pi/2}^{3\pi/2} \log |f(a+Re^{i\phi})| d\phi \leq \frac{1}{N} \int_{\pi/2}^{3\pi/2} \log |L(a+it+Re^{i\phi}, \chi)| d\phi
\]

\[
\leq \frac{1-2p}{1+\eta-p} R n_\chi \log \left( \frac{\zeta(T)}{2\pi} \right) + \pi n_\chi \log \zeta(1+\eta) + \frac{2R n_\chi}{1+\eta-p} \log \left( \frac{(1-p)}{\zeta(1+\eta)} \right)
\]

(A.11)

\[
+ \frac{1/2-p}{1+\eta-p} R n_\chi \int_{\pi/2}^{3\pi/2} (-\cos \phi) \log \left( w(T, \phi, \eta, R) \right) d\phi
\]

where, as in [23] (4.8) but using \( R \) instead of \( r \) as the last argument of \( w \)

\[
w(T, \phi, \eta, R)^2 = 1 + \frac{2R \sin \phi}{T} + \frac{R^2 + (2+\eta)^2 + 2R(2+\eta) \cos \phi}{T^2}
\]

To bound this integral we use the elementary inequality \( \log x \leq \frac{x^2-1}{2} \), which applied to \( w \) produces a function which can be explicitly integrated. The resulting function is decreasing in \( T \), so that it can be bounded with its value at \( T_0 \). With this method from (A.11) we get

\[
\frac{1}{N} \int_{\pi/2}^{3\pi/2} \log |f(a+Re^{i\phi})| d\phi \leq \frac{1-2p}{1+\eta-p} R n_\chi \log \left( \frac{\zeta(T)}{2\pi} \right) + \pi n_\chi \log \zeta(1+\eta)
\]

(A.12)

\[
+ \frac{2R}{1+\eta-p} n_\chi \log \left( \frac{(1-p)}{\zeta(1+\eta)} \right) + \frac{1/2-p}{1+\eta-p} R n_\chi \frac{2R^2 + (2+\eta)^2 - \pi R(2+\eta)}{2T_0^2}
\]

valid for all \( T \geq T_0 \geq 1 \), as long as \(-1/2 \leq p < 0 < \eta \leq 1/2 \), \( p \leq a - R \) and \( \sigma_1 < a + R \).

We still have to bound \(- \log |f(a)| \) and for that we let \( N \) diverge along a sequence such that \( N \arg L(a+iT, \chi) \) tends to 0 modulo 2\( \pi \). In the limit we get \( \lim_{a \rightarrow T_0} \frac{1}{N} \log |f(a)| = \log |L(a+it, \chi)| \). We use

\[
\log |L(a+iT, \chi)| = \left| \prod_p \left( 1 - \chi(p)Np^{-a-iT} \right)^{-1} \right| \geq \prod_p (1+Np^{-a})^{-1}
\]

(A.13)

\[
= \prod_p \prod_{j=1}^{g_p} (1+p^{-a}f_j)^{-1} \geq \prod_p (1+p^{-a})^{-n_\chi} = \left( \frac{\zeta(2a)}{\zeta(a)} \right)^{n_\chi}.
\]
In order to compute a convenient bound for $E$ in (A.7), we notice that the functional equation (2.7) shows that $\Delta_+ \arg \xi(s, \chi) = -\Delta_+ \arg \xi(s, \chi)$, and that $\Delta_+ \arg (Q(\chi) \pi^{-n\chi})^{s/2} = 0$, thus (A.7) equals

$$|\Delta_+ \arg \Gamma_+(s) + \Delta_- \arg \Gamma_+(s)|.$$  

Recalling the definition of $\Gamma_+$ and the bound in (A.1)–(A.2), this may be estimated by

$$a_\chi G(0, \delta, T) + b_\chi G(1, \delta, T) \leq n_G G(0, \delta, T)$$

where

$$G(\alpha, \delta, T) := \frac{1}{2} \left( \alpha \frac{1}{2} + \delta \right) \tan \left( \frac{\alpha + \frac{1}{2} + \delta}{T} \right) + \frac{1}{2} \left( \alpha - \frac{1}{2} - \delta \right) \tan \left( \frac{\alpha - \frac{1}{2} - \delta}{T} \right) - \left( \alpha - \frac{1}{2} \right) \tan \left( \frac{\alpha + \frac{1}{2}}{T} \right) - \frac{T}{4} \log \left( 1 + \frac{2\delta^2 (T^2 - (\frac{1}{2} + \alpha)^2)^2}{(T^2 + (\frac{1}{2} + \alpha)^2)^2} \right) + \frac{1}{4} \left( \frac{1}{\frac{1}{2} + \alpha + iT} - \frac{1}{\frac{1}{2} - \alpha + iT} + \frac{2}{\frac{1}{2} + \alpha + iT} \right)$$

and we have used the inequalities $0 < G(1, \delta, T) \leq G(0, \delta, T)$. Observing that $G(0, \delta, T)$ is decreasing in $T$ for $T \geq 1$, we have

(A.14) \[ |\Delta_+ \arg L(s, \chi) + \Delta_- \arg L(s, \chi)| \leq n_G G(0, \delta, T_0) \]

for $T \geq T_0 \geq 1$. We thus let $E := n_G G(0, \delta, T_0)$.

In the final inequality (A.15) the coefficient of $\log(q(\chi)T)$ is $\frac{(1/2-p)R}{2(1+\eta-p)\log r}$. It is minimal for $r = \frac{1+\eta-p}{1/2+\eta}$, hence this is the choice we make. We then have $R = 1+\eta-p$, hence $a-R = p$ and $a+R = 2+2\eta-p > \frac{1}{2} + \sqrt{2}(\frac{1}{2} + \eta) = \sigma_1$. From (A.6), (A.9), (A.10), (A.12), (A.13) and (A.14) we have, recalling that $r = \frac{1+\eta-p}{1/2+\eta}$,

(A.15) \[ \left| \frac{N(T, \chi)}{n_G} - \frac{T}{\pi} \log \left( \frac{q(\chi)T}{2\pi e} \right) + \frac{a_\chi - b_\chi}{4n_G} \right| \leq C_1 \log(q(\chi)T) + C'_2 \]

with

(A.16) \[ C_1 := \frac{1/2-p}{\pi \log r} \]

and

$$C'_2 := \frac{2}{\pi} \left( g(0, T_0) + \log \zeta \left( \frac{1}{2} + \sqrt{2} \left( \frac{1}{2} + \eta \right) \right) + \frac{1}{2} G \left( 0, \sqrt{2} \left( \frac{1}{2} + \eta \right), T_0 \right) \right)$$

\[ + \frac{1}{\pi \log r} \int_{-\pi/2}^{\pi/2} \log(\zeta(a+(1+\eta-p) \cos \phi)) d\phi \]

\[ + \frac{1/2-p}{4\pi T_0^2 \log r} \left[ 2(1+\eta-p)^2 + 2(2+\eta)^2 - \pi(1+\eta-p)(2+\eta) \right] \]

\[ - \frac{1/2-p}{\pi \log r} \log(2\pi) + \frac{\log(\zeta(1+\eta))}{2 \log r} + \frac{\log \left( \frac{\zeta(1-p)}{\zeta(1+p)} \right)}{\pi \log r} + \frac{1}{\pi \log r} \log \left( \frac{\zeta(1+\eta)}{\zeta(2(1+\eta))} \right) \]

valid for $-1/2 \leq p < 0 < \eta \leq 1/2$ and $T \geq T_0 \geq 1$, and proviso that (A.8) holds. In case (A.8) is false, by (A.3), (A.4) and (the opposite of) (A.8) we still get (A.15) but with

(A.18) \[ C_1 := 0, \]

(A.19) \[ C'_2 := \frac{2}{\pi} \left( g(0, T_0) + \log \zeta \left( \frac{1}{2} + \sqrt{2} \left( \frac{1}{2} + \eta \right) \right) + G \left( 0, \sqrt{2} \left( \frac{1}{2} + \eta \right), T_0 \right) \right). \]
To obtain the values in Table 1, we observe that by \(\text{[A.16]}\) we have
\[
\eta = \frac{1/2-p}{\exp((1/2-p)/(\pi C_1))-1} - \frac{1}{2}
\]
for every given \(C_1\) and \(p \in [-\frac{1}{2}, 0)\).

Coming to the case where \(\chi\) is trivial, we follow the proof of \([23, \text{Theorem 2]}\) with the modifications we have made above, and we observe that \(\Delta_C s(s-1) = 2\pi\), which accounts for the \(-2\delta_\chi\) in the main term of \(N(T, \chi) = N_{L_\chi}(T)\).

For the remaining terms, we observe that \(g(T) := \text{Im } \Gamma(1/2+iT)-T \log(T/e)\) and \(g(0, T)\) both decrease to 0 as \(T \to \infty\), and that \(g(T) \leq g(0, T)\), hence we can use \(D_1 := C_1\) and \(D_2' := C_2'\).

Moreover using that \(\log x \leq (x^2-1)/2\) to bound the integrals in the expression of \(D_3\) of \([23, (5.12)]\), we can use
\[
D_3' := \frac{1}{\pi \log r} \log \left(\frac{1-p}{1+p}\right) + \frac{1}{\pi} F\left(\sqrt{2}(\frac{1}{2} + \eta), T_0\right) + \frac{\pi r^2(\frac{1}{2} + \eta)^2 - 4r(\frac{1}{2} + \eta) + \pi \eta^2 + 2\pi \eta + 2\pi}{2\pi T_0^2 \log r}
\]
where \(F(\delta, T) := 2 \arctan \frac{1}{2T} - \arctan \frac{1/2+\delta}{T} - \arctan \frac{1/2-\delta}{T}\).

We use the formula given above for \(\eta\) in terms of \(C_1 = D_1\) and \(p\), we compute the values of \(D_3' = C_3'\) for a suitable choice of \(p\) as given by \(\text{[A.17]}\) and we test that it is greater than the value produced by \(\text{[A.19]}\); an upper bound for \(D_2'\), a rounding of the computed value of \(\eta\) and the chosen value of \(p\) are indicated in the table below (the sequences of values of \(D_1\) are the same in the three subtables and are those indicated in \([23, \text{Table 2]}\), plus the two extremal values 0.230 and 0.460).
\[ T_0 = 1 \]

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<th>( p )</th>
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\[ T_0 = 2\pi \]

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<th>( D_3' )</th>
<th>( \eta )</th>
<th>( p )</th>
<th>( D_2' )</th>
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<th>( p )</th>
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Table 2: Constants for Lemma [x9]
References


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