Upper and lower bounds at $s = 1$ for certain Dirichlet series with Euler product

Giuseppe Molteni  
Dipartimento di Matematica  
Università di Milano  
Via Saldini 50  
20133 Milano  
ITALY  
e-mail: giuseppe.molteni@mat.unimi.it


Abstract
Estimates of the form $L^{(j)}(s, A) \ll_{c,j,A} R_A$ in the range $|s - 1| \ll 1/\log R_A$ for general $L$-functions, where $R_A$ is a parameter related to the functional equation of $L(s, A)$, can be quite easily obtained if the Ramanujan hypothesis is assumed. We prove the same estimates when the $L$-functions have Euler product of polynomial type and the Ramanujan hypothesis is replaced by a much weaker assumption about the growth of certain elementary symmetrical functions. As a consequence, we obtain an upper bound of this type for every $L(s, \pi)$, where $\pi$ is an automorphic cusp form on $\text{GL}(d, A)$. We employ these results to obtain Siegel-type lower bounds for twists by Dirichlet characters of the third symmetric power of a Maass form.

Mathematics Subject Classification (2000): 11M41

1 Definitions and results
We consider the class of functions satisfying the following axioms:

(A1) (Euler product) Let $A = \{A_p\}_p$, $p$ prime, be a sequence of complex square matrices of order $d$, with monic characteristic polynomial $P_p(x) = P^A(x) \in \mathbb{C}[x]$ and roots $\alpha_j(p) = \alpha^A_j(p)$. We define the general $L$-function $L(s, A)$ as

$$L(s, A) = \prod_p \prod_{j=1}^d (1 - \alpha_j(p)p^{-s})^{-1} = \sum_{n=1}^{\infty} a_n n^{-s}$$

and we suppose the series absolutely convergent for $\sigma > 1$.

(A2) (Continuation) There exists $m = m(A) \in \mathbb{N}$ such that $(s - 1)^m L(s, A)$ has entire continuation over $\mathbb{C}$ as a function of finite order.
(A3) (Growth) There exists $0 < \delta < 1/2$ such that $L(s, \mathcal{A}) \ll \exp(\epsilon |t|) \forall \epsilon > 0$ as $|t| \to \infty$, uniformly in $-\delta < \sigma < 1 + \delta$.

(A4) (Functional equation) There exists a second sequence of complex square matrices $\mathcal{A}^* = (\mathcal{A}_p^*)_p$ satisfying axioms (A1), (A2), (A3), and there exist $Q, \lambda_1, \ldots, \lambda_r \in \mathbb{R}^+, \mu_1, \ldots, \mu_r \in \mathbb{C}$ with $\Re \mu_j \geq 0 \forall j$ and $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that the functions

$$
\Phi(s, \mathcal{A}) = Q \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) L(s, \mathcal{A}), \quad \Phi(s, \mathcal{A}^*) = Q^* \prod_{j=1}^{r} \Gamma(\lambda_j s + \bar{\mu}_j) L(s, \mathcal{A}^*)
$$

satisfy the functional equation

$$
\Phi(1 - s, \mathcal{A}) = \omega \Phi(s, \mathcal{A}^*).
$$

By definition, the main parameter of $L(s, \mathcal{A})$ is the quantity

$$
\mathcal{R}_\mathcal{A} = (1 + Q) \prod_{j=1}^{r} (1 + |\mu_j|).
$$

(A5) (Tensor product) There exists a finite, possibly empty, set of primes $\mathcal{P}_\mathcal{A}$, the exceptional set, and complex numbers $\gamma_i(p), \delta_i(p) \in \mathbb{C}$ for $i = 1, \ldots, d^2, p \in \mathcal{P}_\mathcal{A}$ with

$$
|\gamma_i(p)|, |\delta_i(p)| \leq p^{1-\rho} \quad \text{for some } \rho > 0, \quad \forall i = 1, \ldots, d^2, \quad p \in \mathcal{P}_\mathcal{A},
$$

such that, if we define

$$
L(s, \mathcal{A} \otimes \bar{\mathcal{A}}) = \prod_{p} \prod_{i,j=1}^{d} (1 - \alpha_i(p)\bar{\alpha}_j(p)p^{-s})^{-1},
$$

$$
P(s, \mathcal{A} \otimes \bar{\mathcal{A}}) = \prod_{p \in \mathcal{P}_\mathcal{A}} \prod_{i=1}^{d^2} (1 - \gamma_i(p)p^{-s})(1 - \delta_i(p)p^{-s})^{-1},
$$

$$
L(s, \mathcal{A} \otimes \bar{\mathcal{A}}) = P(s, \mathcal{A} \otimes \bar{\mathcal{A}}) L(s, \mathcal{A} \otimes \bar{\mathcal{A}}),
$$

then $L(s, \mathcal{A} \otimes \bar{\mathcal{A}})$ satisfies axioms (A1)-(A4). By abuse of notation we denote by $\mathcal{R}_{\mathcal{A} \otimes \bar{\mathcal{A}}}$ the main parameter of $L(s, \mathcal{A} \otimes \bar{\mathcal{A}})$ and we assume that

$$
\sum_{p \in \mathcal{P}_\mathcal{A}} 1 \ll \log \mathcal{R}_{\mathcal{A} \otimes \bar{\mathcal{A}}}, \quad (1)
$$

$$
\log \mathcal{R}_{\mathcal{A} \otimes \bar{\mathcal{A}}} \asymp \log \mathcal{R}_{\mathcal{A}}. \quad (2)
$$

It is important to remark that we do not assume anything about the size of the eigenvalues $\alpha_j$; in particular, the Ramanujan hypothesis is not assumed. This is the only difference with a class of functions introduced and widely studied in Carletti-Monti Bragadin-Perelli [3], so we refer to that paper for the basic properties of the functions $L(s, \mathcal{A})$ and for comments about the axioms; here we only recall that the exceptional set $\mathcal{P}_\mathcal{A}$ of axiom (A5) is related to the existence of ramified primes.
Remark. For well known classes of $L$-functions the main parameter $\mathcal{R}_A$ captures, although in a quite rough form, most of the algebraic informations about $L(s, A)$: for examples, for the Dirichlet $L(s, \chi)$-functions, $R \asymp q$, i.e., the modulus of $\chi$; for Hecke $L$-functions related to holomorphic cuspidal forms, $R \asymp kN$, i.e., the product of the level $N$ and the weight $k$; for Maass $L$-functions, $R \asymp \lambda N$, i.e., the product of the level $N$ and the eigenvalue $\lambda$; and in general, for the $L$-functions associated with cuspidal automorphic representations of the groups $GL(d)$, $R$ is of the order of the \textit{analytic conductor} introduced by Iwaniec and Sarnak in [13]. For this reason our results will be uniform in the $\mathcal{R}$-aspect but will depend on all other parameters appearing in axioms. It is useful, therefore, to introduce the following notation: let $D_A$ be the set

$$D_A = \{m_A, m_A \otimes A, r_A, r_A \otimes A, d_A, \lambda_j(A), \lambda_j(A \otimes \overline{A})\}$$

i.e., $D_A$ contains the order of pole $m_A$, the number of gamma factors $r_A$, the degree $d_A$ and the $\lambda_j$ coefficients, both of $L(s, A)$ and of $L(s, A \otimes \overline{A})$. In this way we will say our results are $\mathcal{R}_A$-independent and $D_A$-dependent.

The aim of this paper is proving an upper bound for $L(1, A)$, and more generally for $L^{(j)}(s, A)$, of type

$$L^{(j)}(s, A) \ll_{\epsilon,j, D_A} \mathcal{R}_A \quad \text{for} \quad |s - 1| \ll 1/\log \mathcal{R}_A. \quad (3)$$

Such estimates are an essential tool for a general Siegel-type estimate, as we will see in Section 2. Obtaining these estimates is quite easy when the Ramanujan hypothesis is assumed (see [3]). However, this hypothesis has been proved only for a limited class of functions (the Hecke $L$-functions, the Artin $L$-functions and the $L$-functions coming from the cuspidal holomorphic forms for congruence groups, see Deligne [4]), although it is generally believed that all the $L$-functions appearing in number theory should satisfy the Ramanujan hypothesis; for example, it is conjectured to hold for the $L$-functions associated with cuspidal automorphic representations.

In general, only partial and rather poor estimates for the coefficients are at our disposal, hence it is interesting to consider the possibility to obtain (3) without the Ramanujan hypothesis.

For Maass forms ($d = 2$) this has been done firstly by Iwaniec [12] (in the $Q$-aspect, in that paper). He remarks that a preliminary estimate for $S(x) = \sum_{n \leq x} |a_n|/n$ of the form $S(x) \ll_{\epsilon} Q^c x^\epsilon$ for some constant $c$ and for every $\epsilon > 0$ can be proved in a standard way. Then, the multiplicative properties of the coefficients of these functions can be employed to obtain an estimate for $S(x)^2$ in terms of $S(x^2)$. By iterating this relation the fundamental estimate $S(x) \ll_{\epsilon} (Qx)^\epsilon$ is deduced, and the claim easily follows. This method has been also used by Hoffstein-Lockhart [10] to get an analogous estimate in the case of the symmetric square $L$-functions ($d = 3$) associated with Maass forms.

We modify Iwaniec’s idea in such a way to obtain the required estimate for functions of any degree $d$; in this sense the most original part of our work is Section 2, where the basic Proposition 1 below is proved.

We introduce the following notation: $s_j(p)$ denotes the $j$-th elementary symmetric function of the roots $\alpha_1(p), \ldots, \alpha_d(p)$, i.e.,

$$s_j(p) = \sum_{1 \leq i_1 < \cdots < i_j \leq d} \alpha_{i_1}(p) \cdots \alpha_{i_j}(p);$$
moreover, let $R(p) = \sqrt{2} \sum_{j=2}^{d} |s_j(p)|^{1/j}$ and let $R(d)$ be the completely multiplicative function generated by $R(p)$.

**Proposition 1.** We have

$$|a_n a_m| \leq \sum_{(d,nm) \mathcal{d}}^* R(d)|a_{nm}| \quad \forall n, m \geq 1,$$

where $\sum^*$ denotes that the summation is restricted to square-full divisors $d$, i.e., either $d = 1$ or $d > 1$ and $p^2 \mid d$ for every $p \mid d$.

**Remark.** When $d = 1$ the estimate (4) is reduced to the trivial $|a_n a_m| \leq |a_{nm}|$.

Obviously, some hypothesis about the size of the coefficients has to be assumed in order to prove (3). We assume

**Hypothesis (R).** There exists $\rho > 0$ such that $|\alpha_j(p)| \leq p^{1-\rho}$ for every prime $p$, and the estimate

$$s_j(p) \ll p^{j/2} \quad \forall 2 \leq j \leq d,$$

uniformly over $\mathcal{R}$.

**Remark.** The first part of the hypothesis means that the local components are convergent when $\sigma > 1 - \rho$. Moreover, the estimate on the elementary symmetric functions is satisfied in an obvious way when the estimate $a_n \ll n^{1/2}$ holds, uniformly on $\mathcal{R}$. Nevertheless, hypothesis (R) does not assume anything about the first symmetric function $s_1(p)$, in this way hypothesis (R) is satisfied as well in some cases where the global estimate $a_n \ll n^{1/2}$ is not known.

At last, it is also interesting to remark that the quality of the estimate assumed in (R) does not depend on the degree $d$, as instead a consideration of similar situations could suggest.

Our main result is

**Theorem 1.** Let $L(s, \mathcal{A})$ satisfy axioms (A1)-(A5) and hypothesis (R) and define $f(s) = (s - 1)^{m_{\mathcal{A}}} L(s, \mathcal{A})$. Then

$$f^{(j)}(s) \ll_{s, \mathcal{D}} (j + 1)c^j R^j \log^j R + \frac{j^j}{2^j R^j} \text{ for } |s - 1| \ll 1/\log R,$$

uniformly on $j$, for a suitable positive constant $c = c(\mathcal{D})$, independent of $\mathcal{R}$.

**Remark.** Under the same hypotheses and with some minor change to the proof of this theorem we caught obtain upper bounds of similar type for the values of $L(s, \mathcal{A})$ when $s$ is centered at a different point of the line $\sigma = 1$; for example, for $\theta \neq 0$ we have

$$L(s + i\theta, \mathcal{A}) \ll_{s, \mathcal{D}} (R(1 + |\theta|))^s \text{ for } |s - 1| \ll 1/\log(R(1 + |\theta|)).$$

It is interesting to remark that this more general result is already included in Theorem 1 when $L(s, \mathcal{A})$ is entire. In fact, in this case the shifted $L$-function $L(s, \mathcal{A}(\theta)) = L(s + i\theta, \mathcal{A})$ satisfies axioms (A1)-(A5) and (R) with $\log R_{\mathcal{A}(\theta)} \asymp_{\mathcal{D}, \mathcal{A}} \log R_{\mathcal{A}} + \log(1 + |\theta|)$ so that (5) for $L(s, \mathcal{A}(\theta))$ gives (6) for $L(s, \mathcal{A})$. 

4
For the applications it is important to know estimate (3) for the generic tensor product function

\[ L(s, A \otimes B) = \prod_{\ell} \prod_{i=1}^{d_{A}} \prod_{j=1}^{d_{B}} (1 - \alpha_{i}(p)\beta_{j}(p)p^{-s})^{-1} \]

as well; to deal with this function, axiom \((A5)\) has to be modified in the following way.

\((A5')\) There exists a finite, possibly empty, set of primes \(P\) and, assume that both

\[ L(s, A \otimes B) \]

such that, if we define

\[ P(s, A \otimes B) = \prod_{p \in P, \ell} \prod_{i=1}^{d_{A}} \prod_{j=1}^{d_{B}} (1 - \gamma_{i}(p)p^{-s})(1 - \delta_{j}(p)p^{-s})^{-1} \]

then \(L(s, \widehat{A} \otimes \widehat{B})\) satisfies axioms \((A1)-(A4)\). As before, by abuse of notation we denote by \(R_{A \otimes B}\) the main parameter of \(L(s, A \otimes B)\) and we assume that

\[ \sum_{p \in P, \ell} \ll 1 \ll \log R_{A \otimes B}, \]

\[ \log R_{A \otimes B} \asymp \log R_{A} + \log R_{B}. \]  

Remark. Under hypotheses (7) and (8), the estimate

\[ (R_{A}R_{B})^{-e} \ll e \ll P(1, A \otimes B) \ll (R_{A}R_{B})^{e} \]

holds, so that any upper bound of type (3) for \(L(s, \widehat{A} \otimes \widehat{B})\) gives a similar upper bound for \(L(s, A \otimes B)\) and vice versa. Moreover, in the context of \(L\)-functions from representations of \(GL(d, \mathbb{C})\), \(L(s, \widehat{A} \otimes \widehat{B})\) is the Rankin-Selberg convolution and estimates (1), (2), (7) and (8) about the ramified primes are satisfied.

An upper bound for \(L(s, \widehat{A} \otimes \widehat{B})\) can be deduced by Theorem 1, but in this case we have to assume axiom \((A5)\) for \(L(s, A \otimes B)\), i.e., the validity of \((A1)-(A4)\) for \(L(s, A \otimes \hat{A} \otimes B \otimes \hat{B})\). This fact agrees with well known conjectures, but it is not proved in general, hence it is important for applications to obtain an upper bound for \(L(s, A \otimes B)\) under a different set of hypotheses. The following Theorems 2 and 3 achieve this purpose.

**Theorem 2.** Assume that both \(L(s, A)\) and \(L(s, B)\) satisfy axioms \((A1)-(A5)\) and \((A5')\). Moreover, assume that both \(L(s, A \otimes \hat{A})\) and \(L(s, B \otimes \hat{B})\) satisfy hypothesis \((R)\) and that \(\left| \prod_{i=1}^{d_{A}} \alpha_{i}(p) \right|, \left| \prod_{j=1}^{d_{B}} \beta_{i}(p) \right| \leq 1 \) for every \(p\). Define \(f(s) = (s - 1)^{m_{A} \otimes B} L(s, A \otimes B)\). Then

\[ f(j)(s) \ll_{c, D_{A}, D_{B}} (j + 1)^{c j} (R_{A}R_{B})^{j} \log^{j}(R_{A}R_{B}) + \frac{j!}{2^{j} R_{A}R_{B}} \]  

for \(|s - 1| \ll 1 / \log(R_{A}R_{B})\), uniformly on \(j\), for a suitable positive constant \(c = c(D_{A}, D_{B})\), independent of \(R_{A}, R_{B}\).
Nevertheless, there are cases where (R) is not proved for \( L(s, A \otimes \bar{A}) \), but the following stronger hypothesis about \( L(s, A) \) holds.

**Hypothesis (R’).** There exists \( \rho > 0 \) such that \( |\alpha_j(p)| \leq p^{1-\rho} \) for every prime \( p \) and the estimate

\[
s_j(p) \ll p^{j/4} \quad \forall \ 2 \leq j \leq d , \text{ uniformly over } \mathcal{R}
\]

holds.

We can prove upper bounds for \( L(s, A \otimes B) \) under this hypothesis as well.

**Theorem 3.** Assume that both \( L(s, A) \) and \( L(s, B) \) satisfy axioms (A1)-(A5) and (A5’). Moreover, assume that both \( L(s, A) \) and \( L(s, B) \) satisfy hypothesis (R’) and that \( \prod_{j=1}^{dk} \beta_j(p) \leq 1 \) for every \( p \). Then, defining \( f(s) = (s-1)^m A \otimes B L(s, A \otimes B) \) we have

\[
f^{(j)}(s) \ll_{\epsilon, D_A, D_B} (j+1)c^j(R_A R_B)^j \log^j(R_A R_B) + \frac{j!}{2^{j} R_A R_B} \quad \text{for } |s-1| \ll 1/ \log(R_A R_B),
\]

uniformly on \( j \), for a suitable positive constant \( c = c(D_A, D_B) \), independent of \( R_A, R_B \).

According to the original Iwaniec’s idea, Theorems 1-3 are deduced in a standard way from suitable upper bound for the coefficients of \( L(s, A) \) and \( L(s, A \otimes B) \); we state here explicitly the upper bound giving Theorem 1 since it has an interest of its own; the similar upper bounds for the coefficients of \( L(s, A \otimes B) \) under the hypotheses of Theorems 2 and 3 are contained in their proofs.

**Theorem 4.** Let \( L(s, A) \) satisfy axioms (A1)-(A5) and hypothesis (R). Then

\[
\sum_{n \leq x} \frac{|a_n|}{n} \ll_{\epsilon, D} (Rx)^{\epsilon} \quad \forall \epsilon > 0.
\]

**Examples.** Let \( \alpha(p) \) and \( \alpha^{-1}(p) \) be the coefficients of the Euler product of the \( L \)-function \( L(s, f) \) associated with a Maass form \( f \). Then \( d = 2 \) and \( s_2(p) = 1 \) so (R) is satisfied for \( L(s, f) \), hence for this function estimate (5) holds, as already proved by Iwaniec [12].

Moreover, Kim and Shahidi [15] proved that \( p^{-5/34} < |\alpha(p)| < p^{5/34} \). The square-symmetric \( L \)-function \( L(s, \text{sym}^2 f) \) associated with \( f \) satisfies axioms (A1)-(A5) by the works of Shimura [20], Gelbart-Jacquet [5] and Moeglin-Waldspurger [17]. It has an Euler product of degree 3 and \( \alpha^2(p), 1, \alpha^{-2}(p) \) as coefficients, thus \( s_3(p) = 1 \) and \( |s_2(p)| = |\alpha^2(p) + 1 + \alpha^{-2}(p)| \leq 3p^{5/17} \), hence (R) is satisfied and the estimate \( L(1, \text{sym}^2 f) \ll \mathcal{R}^{\epsilon} \) holds by Theorem 1. This has been proved already by Hoffstein-Lockhart [10].

It is easy to verify that the estimate of Kim and Shahidi for the coefficients of a Maass form is sufficiently strong to prove that \( L(s, \text{sym}^2 f \otimes \text{sym}^2 f) \) satisfies (R), so that by Theorem 2 it follows that \( \text{res}_{s=1} L(s, \text{sym}^2 f \otimes \text{sym}^2 f) \ll \mathcal{R}^{\epsilon} \), again a result already proved in [10] (in that paper the upper bound for \( \text{res}_{s=1} L(s, \text{sym}^2 f \otimes \text{sym}^2 f) \) was not deduced by Theorem 2 but by a specific version of Theorem 3 because only the weaker estimate \( p^{-1/5} < \alpha(p) < p^{1/5} \) was known at that time).

At last, Maass functions are examples of Langlands \( L \)-functions, a general and extremely important class of functions. This theory associates an \( L \)-function, \( L(s, \pi) \), to every automorphic
cuspidal representation $\pi$ of $\text{GL}(d, A_K)$, where $A_K$ is the Adèle ring of a global field $K$ (see [6]). These functions satisfy axioms (A1)-(A5) by the work of many authors (see [7], [19] and [17]); by Luo-Rudnick-Sarnak [16] the local coefficients of these functions satisfy
\[ |\alpha_j(p)| < p^{1/2-1/(d^2+1)} \quad \text{for any } j = 1, \ldots, d, \]
hence all the hypotheses of Theorem 1 are satisfied (but it is interesting to remark that the old result of Jacquet and Shalika [14] asserting the bound $|\alpha_j(p)| < p^{1/2}$ is sufficient for this purpose) so that Corollary below immediately follows.

**Corollary.** Let $\pi$ be an automorphic cuspidal representation of $\text{GL}(d, A_K)$ and $L(s, \pi)$ be the associated $L$-function. Let $f(s) = (s - 1)^m L(s, \pi)$, where $m$ is the order of pole of $L(s, \pi)$ at $s = 1$, then
\[ |f^{(j)}(s)| \leq c R^\epsilon \quad \text{for} \quad |s - 1| \ll 1 / \log R, \]
for a suitable $c = c(\epsilon, j, m, d) > 0$.

(For the sake of simplicity, we ignore here uniformity on the order $j$.) An unconditional and general statement corresponding to Theorem 2 or 3 is not possible for the Rankin-Selberg convolution $L(s, \pi \otimes \pi')$ since it is not known if this function satisfies axiom (A5) or if $L(s, \pi)$ satisfies Hypothesis (R').

**Siegel-type lower bounds**

Upper bounds of Theorems 1-3 are necessary ingredients for the proof of Siegel-type theorems, i.e., for the proof of lower bounds of the form
\[ L(1, A) \gg_{\epsilon, D} R^{-\epsilon} \quad \forall \epsilon > 0; \quad (11) \]
the importance of such a type of results justifies the previous researches about lower bounds for $L$-functions whose coefficients are not known to satisfy Ramanujan hypothesis. A careful analysis about Siegel-type lower bounds is carried out in [18], where the results of Golubeva-Fomenko [8, 9] about a lower bound for holomorphic cusp forms, the well known estimate (11) for symmetric square power of a Maass form proved by Hoffstein-Lockhart [10] and similar results are deduced as consequences of a coherent and axiomatic approach. In [18] new results are deduced as well; in the following we show how some of these results, i.e., estimates (12)-(14) below, can be proved.

Let $f \in S_0(\Gamma_0(N), \psi)$, i.e., let $f$ be a Maass cusp form for the Hecke congruence subgroup of level $N$ with the real character $\psi$ modulus $N$ as multiplier. As we have shown in previous examples, both $L(s, f)$ and $L(s, \text{sym}^2 f)$ satisfy (R'). Moreover, let $\chi$ and $\chi'$ be different real primitive and non-principal characters modulo $q$ and $q'$ respectively. It is known that $L_\chi(s, f) = L(s, \chi \otimes f)$, $L_\chi(s, \text{sym}^2 f) = L(s, \chi \otimes \text{sym}^2 f)$ and $L_\chi(s, f \otimes \text{sym}^2 f) = L(s, f_\chi \otimes \text{sym}^2 f)$ are entire functions satisfying axioms (A1)-(A5) (by [17]) and that
\[ \log R_{\chi \otimes f} \asymp \log R_f + \log q \]
and the same is true for $R_{\chi \otimes \text{sym}^2 f}$ and $R_{f_k \otimes \text{sym}^2 f}$. We consider
\[
F(s) = L(s, (1 + \chi) \otimes (1 + \chi') \otimes (f + \text{sym}^2 f) \otimes (f + \text{sym}^2 f))
= \zeta(s) L(s, \text{sym}^2 f) L(s, \text{sym}^2 f \otimes \text{sym}^2 f) L^2(s, f \otimes \text{sym}^2 f) \\
\times L(s, \chi) L_\chi(s, \text{sym}^2 f) L_\chi(s, \text{sym}^2 f \otimes \text{sym}^2 f) L^2(s, f \otimes \text{sym}^2 f) \\
\times L(s, \chi') L'_\chi(s, \text{sym}^2 f) L'_\chi(s, \text{sym}^2 f \otimes \text{sym}^2 f) L^2(s, f \otimes \text{sym}^2 f) \\
\times L(s, \chi\chi') L_{\chi\chi'}(s, \text{sym}^2 f) L_{\chi\chi'}(s, \text{sym}^2 f \otimes \text{sym}^2 f) L^2(s, f \otimes \text{sym}^2 f);
\]
this function has a representation as a Dirichlet series with positive coefficients, as it can be verified by considering its logarithmic derivative. It has a double pole at $s = 1$ and is divisible by $L(s, f \otimes \text{sym}^2 f) L_\chi(s, f \otimes \text{sym}^2 f)$ with multiplicity two. By Theorems 1-3 upper bounds of the form $\ll_{f, \epsilon} (q q')^\epsilon$ follow for the derivatives of every order of any function appearing into $F(s)$. Therefore, by the standard approach to Siegel-type theorems (see [8], for example), we prove that
\[
L_\chi(1, f \otimes \text{sym}^2 f) \gg_{f, \epsilon} q^{-\epsilon} \quad \forall \epsilon > 0.
\]
Since $L_\chi(s, f \otimes \text{sym}^2 f) = L(s, f) L_\chi(s, \text{sym}^3 f)$, where $L(s, \text{sym}^3 f)$ is the $L$-function investigated by Kim and Shahidi [15], we deduce that
\[
L_\chi(1, \text{sym}^3 f) \gg_{f, \epsilon} q^{-\epsilon} \quad \forall \epsilon > 0.
\]
In a similar way we can prove that
\[
L_\chi(1, f \otimes \text{sym}^2 g) \gg_{f, g, \epsilon} q^{-\epsilon},
\]
\[
L_\chi(1, \text{sym}^2 f \otimes \text{sym}^2 g) \gg_{f, g, \epsilon} q^{-\epsilon},
\]
for $f$ and $g$ both Maass forms, $f \neq g$.

We recall that $\beta_\mathcal{A}$, the largest real zero of $L(s, \mathcal{A})$, is called Siegel zero of $L(s, \mathcal{A})$ if $1 - \beta_\mathcal{A} \ll 1/\log R_\mathcal{A}$, and that (11) is equivalent to
\[
1 - \beta_\mathcal{A} \gg_{\epsilon, \mathcal{A}} R_\mathcal{A}^{-\epsilon}.
\]
Banks [1] and Hoffstein-Ramakrishnan [11] have proved that $L$-functions related to automorphic cuspidal representations $\pi$ of $GL(2)$ and $GL(3)$ have not Siegel-zero, so that for these functions the stronger estimate
\[
L(1, \pi) \gg \frac{1}{\log R}
\]
holds. However, estimates (12)-(14), involving functions conjecturally related to automorphic representations of $GL(4)$, $GL(6)$ and $GL(9)$, are non-trivial.

2 Algebraic relations

2.1 Local relations

Let $\{b_h\}_{h=0}^\infty$ be a sequence with $b_0 = 1$, extended to $h < 0$ by setting $b_h = 0$ in this range. We assume that the sequence $b_h$ verifies the relation
\[
b_1 b_h = b_{h+1} + \sum_{u=2}^d x_u^u b_{h+1-u},
\]
(15)
where $d \geq 1$ is a fixed integer, the degree of the sequence, and \( \{x_u\}_{u=2}^d \) are non-negative real parameters.

**Remark.** When $d = 1$ the relation is reduced to $b_1b_h = b_{h+1}$.

With these assumptions we prove

**Proposition 2.** There exist polynomials $P_{u,l} \in \mathbb{N}[x_2, \cdots, x_d]$ such that

\[
b_1b_h = \sum_{u=0}^{d} P_{u,l}(x) b_{h+l-u} \quad \forall l, h \geq 0.
\]  

Moreover, if we assume $P_{0,0} = 1$ and $P_{u,l} = 0$ in the following cases: $u < 0$ or $l < 0$ or $u \geq dl$ or $u = 1$, such polynomials satisfy the recursion

\[
P_{u,l+1}(x) = P_{u,l}(x) + \sum_{g=2}^{d} x_g^g (P_{u-g,l}(x) - P_{u-g,l+1-g}(x)).
\]

**Proof.** We introduce the variables $x_0$ and $x_1$, since in this way we can write $b_1b_h = \sum_{u=0}^{d} x_u b_{h+1-u}$ and recover relation (15) by setting $x_1 = 0$. We prove the claim recursively on $l$. By (15), it is true when $l = 1$, for every $h$. Assume now the claim for the values $l \leq l$ and for every $h$, we will verify it for $l+1$, for every $h$ again. We have

\[
b_1(b_1b_h) = \sum_{u=0}^{d} P_{u,l}(x) b_1 b_{h+l-u} = \sum_{u=0}^{d} P_{u,l}(x) \sum_{g=0}^{d} x_g^g b_{h+l+1-u-g}
\]

\[
= \sum_{u=0}^{d} \sum_{g=0}^{d} x_g^g P_{u,l}(x) b_{h+l+1-u-g} = \sum_{\gamma=0}^{d(l+1)} \left( \sum_{g+u\gamma}^{d} x_g^g P_{u,l}(x) \right) b_{h+l+1-\gamma}
\]

\[
= b_{h+l+1} + \sum_{\gamma=1}^{d(l+1)} \left( \sum_{\gamma \leq g \leq d} x_g^g P_{u,l}(x) \right) b_{h+l+1-\gamma},
\]

moreover,

\[
(b_1b_1b_h) = \sum_{u=0}^{d} x_u b_{1+1-u} b_h = b_{l+1} b_h + \sum_{u=1}^{d} x_u b_{l+1-u} b_h
\]

\[
= b_{l+1} b_h + \sum_{\rho=0}^{d(l+1-u)} \sum_{u=1}^{d} x_u P_{\rho,l+1-u}(x) b_{h+l+1-\rho-u}
\]

\[
= b_{l+1} b_h + \sum_{\gamma=1}^{d(l+1)} \left( \sum_{\gamma \leq u \leq d} x_u P_{\rho,l+1-u}(x) \right) b_{h+l+1-\gamma},
\]

Comparing these identities we get

\[
b_{l+1} b_h = b_{h+l+1} + \sum_{\gamma=1}^{d(l+1)} \left( \sum_{\gamma \leq u \leq d} x_g^g P_{u,l+1-g}(x) \delta_{\gamma \leq d} \right) b_{h+l+1-\gamma},
\]
where \( \delta_{\gamma} \) is 1 or 0 according to \( \gamma \leq d \) or otherwise. The existence of the polynomials \( P_{u,l+1} \) follows from (18), completing in this way the proof of (16). Moreover,

\[
P_{\gamma,l+1}(x) = \sum_{g+u=\gamma \atop 0 \leq g \leq d} x^g P_{u,l}(x) - \left[ \sum_{g+u=\gamma \atop 1 \leq g \leq d \atop 0 \leq u \leq d(l+1-g)} x^g P_{u,l+1-g}(x) \right] \delta_{\gamma \leq d}l, \tag{19}
\]

where the assumption \( x_1 = 0 \) has been introduced. Since \( P_{u,l} = 0 \) for \( u < 0 \), or \( d < 0 \) or \( u > d l \), (19) implies (17).

The particular form of (17) suggests the analysis of the polynomials \( D_{l,h} = P_{u,l} - P_{u,l-1} \); we get

**Proposition 3.** The polynomial sequence \( D_{u,l} = P_{u,l} - P_{u,l-1} \) verifies the recursion

\[
\begin{cases}
D_{u,l+1}(x) = \sum_{g=2}^{d} x^g \sum_{\rho=1}^{g-1} D_{u-g,l+1-\rho}(x) \\
D_{0,0} = 1 \\
D_{u,l} = 0 \text{ when } u < 0, \text{ or } u > d l, \text{ or } l < 0.
\end{cases}
\tag{20}
\]

Moreover, \( D_{u,l} \in \mathbb{N}[x_2, \ldots, x_d] \) and it is homogeneous of degree \( u \), i.e., \( D_{u,l}(\lambda x) = \lambda^u D_{u,l}(x) \) for every \( \lambda \) and \( x \).

**Proof.** The recursive relation (20) is deduced from (17). The other claim is an easy consequence of (20).

We need another property of the sequence \( D_{u,l} \).

**Proposition 4.** If \( l \geq u + d \), then \( D_{u,l} = 0 \).

**Proof.** We prove the claim recursively on \( u \); it is true when \( u = 0 \) and \( u = 1 \) for every \( l > 0 \). We suppose the claim to hold up to \( u \), and we prove it for \( u + 1 \). Let \( l \geq u + 1 + d \). By the identity

\[
D_{u+1,l} = \sum_{g=2}^{d} x^g \sum_{\rho=1}^{g-1} D_{u+1-g,l-\rho},
\]

the claim is the same as proving that every \( D_{u+1-g,l-\rho} \) is zero. The inductive hypothesis assures it if \( l - \rho \geq u + 1 + g + d \), and this inequality holds since we assumed \( l \geq u + 1 + d \) and \( \rho < g \).

**Remark.** Let \( l_0(u) \) be the smallest value such that \( D_{u,l} = 0 \) when \( l \geq l_0(u) \). Proposition 4 states that \( l_0(u) \leq u + d \), but it is easy verify that the inequality is not sharp. Nevertheless, the exact value of \( l_0(u) \) is not necessary here.
Every $D_{u,l}(x) \in \mathbb{N}[x]$, therefore $P_{u,l_1}(x) \leq P_{u,l_2}(x)$ when $l_1 \leq l_2$ (recall we assume $x_i \in \mathbb{R}^+)$ since $P_{u,l} = \sum_{g=0}^{l} D_{u,g}$; nevertheless, the sequence $P_{u,l}$ is bounded on $l$ since by Proposition 4 the following series is finite

$$P_u(x) = \sum_{g=0}^{\infty} D_{u,g}(x)$$

so that

$$P_{u,l}(x) \leq P_u(x) \quad \forall u, l, \quad \forall x_i \in \mathbb{R}^+.$$

The existence of the polynomials $P_u$ giving an upper bound for $P_{u,l}$, uniform on $l$, is an essential fact; we summarize their principal properties in the following proposition.

**Proposition 5.** The sequence $P_u(x)$ satisfies the recurrence

$$\begin{cases}
P_u(x) = \sum_{g=2}^{d} (g-1)x_g^2 P_{u-g}(x) & \text{if } u \geq 2 \\
P_u = 0 & \text{if } u < 0 \\
P_0 = 1, P_1 = 0.
\end{cases}$$

Moreover, $P_u$ belongs to $\mathbb{N}[x_2, \ldots, x_d]$, is homogeneous with degree $u$ and satisfies the estimate

$$P_u(x) \leq [\sqrt{2}(x_2 + \cdots + x_d)]^u.$$  \hfill (23)

**Proof.** By definition $P_u = \sum_{i=0}^{\infty} D_{u,l}$, therefore $P_u$ belongs to $\mathbb{N}[x]$ and is homogeneous with degree $u$ because the same properties hold for every $D_{u,l}$. The recursive relation (22) is recovered from (20). In order to prove (23) we first remark that if set

$$f_M(y) = \sum_{i=1}^{M} \left[ \frac{y_i}{\sum_{j=1}^{M} y_j} \right]^i,$$

then $f_M(y) \leq 1$ when $y_i \in \mathbb{R}^+$ for every $i$; we prove this fact inductively over $M$. It is true for $M = 1$. By the inductive hypothesis $f_{M-1}(y_1, \ldots, y_{M-1}) \leq 1$, we get

$$1 \geq f_{M-1}(y_1 + y_M, y_2, \ldots, y_{M-1}) = \frac{y_1}{\sum_{j=1}^{M} y_j} + \frac{y_M}{\sum_{j=1}^{M} y_j} + \sum_{i=2}^{M-1} \left[ \frac{y_i}{\sum_{j=1}^{M} y_j} \right]^i$$

$$\geq \frac{y_1}{\sum_{j=1}^{M} y_j} + \left[ \frac{y_M}{\sum_{j=1}^{M} y_j} \right]^M + \sum_{i=2}^{M-1} \left[ \frac{y_i}{\sum_{j=1}^{M} y_j} \right]^i = f_M(y_1, y_2, \ldots, y_M).$$

Now we can prove (23) once again by induction. It is true when $u = 0$ and it is trivial when $u < 0$. By the recursive law (22), we have

$$P_u(x) = \sum_{g=2}^{d} (g-1)x_g^2 P_{u-g}(x) \leq \sum_{g=2}^{d} \frac{(g-1)x_g^2}{(\sqrt{2}\sum_{j=2}^{d} x_j)^g} \left( \sqrt{2} \sum_{j=2}^{d} x_j \right)^u$$

$$\leq \sum_{g=2}^{d} \frac{x_g^2}{(\sqrt{2}\sum_{j=2}^{d} x_j)^g} \left( \sqrt{2} \sum_{j=2}^{d} x_j \right)^u \leq \left( \sqrt{2} \sum_{j=2}^{d} x_j \right)^u,$$

where we have used the inequality $g - 1 \leq 2g/2$ and the previous estimate $f_M(y) \leq 1$. \hfill $\square$

**Remark.** Since every $D_{u,l}$ is homogeneous, $P_{u,l}$ is homogeneous too (as directly verified by (17)). This immediately implies that there exists a constant $c = c(l)$ such that $P_{u,l}(x) \leq (c(l)(x_2 + \cdots + x_d))^u$, for every $u$. Obtaining (23) is important since it shows that $c(l)$ is actually independent of $l$. 

11
2.2 Euler product: proof of Proposition 1

Now we use the results of the previous section for the study of the coefficients of an Euler product. Let \( \{a_j\}_{j=1}^{d} \) be complex numbers, and let \( \{b_h\}_{h=0}^{\infty} \) be the sequence generated by the relation

\[
\sum_{h=0}^{\infty} b_h x^h = \prod_{j=1}^{d} (1 - a_j x)^{-1}.
\]

Let \( \{s_j\}_{j=1}^{d} \) be the elementary symmetric polynomials of the variables \( a_j \), then the sequence \( b_h \) satisfies the recursion

\[
\begin{aligned}
&b_h - s_1 b_{h-1} + s_2 b_{h-2} + \cdots + (-1)^d s_d b_{h-d} = 0 \quad \forall h > 0 \\
&b_0 = 1.
\end{aligned}
\]

But \( s_1 = b_1 \), so the recursive relation can be formulated as

\[
b_1 b_{h-1} = b_h + s_2 b_{h-2} + \cdots + (-1)^d s_d b_{h-d}.
\]

Comparing this relation with (15), by (16), (21) and (23) we get

\[
|b_h b_h| \leq |b_{h+1}| + \sum_{u=2}^{d \min (l,h)} R^u |b_{l+h-u}|, \quad \text{with} \quad R = \sqrt{2} \sum_{j=2}^{d} |s_j|^{1/j}.
\]

Proposition 1 follows from this estimate by multiplicativity.

2.3 A useful proposition

For the proof of the theorems we will need the following lemma establishing an inequality between the coefficients of the series \( L(s, \mathcal{A}) \) and those of the convolution series \( \mathcal{A} \otimes \mathcal{A} \).

**Proposition 6.** Let \( L(s, \mathcal{A}) = \sum_{n=1}^{\infty} a_n n^{-s} \) and \( L(s, \mathcal{A} \otimes \mathcal{A}) = \sum_{n=1}^{\infty} A_n n^{-s} \). Then \( A_n \geq |a_n|^2 \) for every \( n \).

The identity \( L(s, \mathcal{A} \otimes \mathcal{A}) = \zeta(2s) \sum_{n=1}^{\infty} |a_n|^2 n^{-s} \) holds when the degree is 2 by a formula of Ramanujan, and in this case the statement easily follows by this fact because \( \zeta(2s) \) is a Dirichlet series with non-negative coefficients. For larger degrees the relation \( L(s, \mathcal{A} \otimes \mathcal{A}) = F(s) \sum_{n=1}^{\infty} |a_n|^2 n^{-s} \) again holds for a suitable Dirichlet series \( F(s) \) but in this case the coefficients of \( F(s) \) can be negative so that the general inequality of Proposition 6 cannot be proved in this way.

**Proof.** Both \( L(s, \mathcal{A}) \) and \( L(s, \mathcal{A} \otimes \mathcal{A}) \) have a representation as an Euler product, thus it is sufficient prove the claim for the local components, i.e., verify that if \( f(x) = \prod_{j=1}^{d} (1 - \alpha_j x)^{-1} = \sum_{h=0}^{\infty} a_h x^h \) and \( F(x) = \prod_{i,j=1}^{d} (1 - \alpha_i \alpha_j x)^{-1} = \sum_{h=0}^{\infty} A_h x^h \), then \( A_h \geq |a_h|^2 \). We remark that

\[
\sum_{h=0}^{\infty} h a_h x^h = x \frac{d}{dx} f(x) = f(x) \sum_{j=1}^{d} \frac{\alpha_j x}{1 - \alpha_j x} = f(x) \sum_{j=1}^{d} \sum_{h=1}^{\infty} \alpha_j^h x^h = \sum_{h=0}^{\infty} a_h x^h \cdot \sum_{h=1}^{\infty} \tau_h x^h,
\]
where we have set $\tau_h = \sum_{j=1}^d \alpha_j^h$; in a similar way, for $F(x)$ we have
\[
\sum_{h=0}^\infty h A_h x^h = \frac{d}{dx} F(x) = F(x) \sum_{i,j=1}^d \frac{\alpha_i \alpha_j x}{1 - \alpha_i \alpha_j x} = F(x) \sum_{i,j=1}^d \sum_{h=1}^\infty \alpha_i^h \alpha_j^h x^h = \sum_{h=0}^\infty A_h x^h \cdot \sum_{h=1}^\infty |\tau_h|^2 x^h.
\]
Therefore, the following recursions hold
\[
\begin{aligned}
{h a_h} &= \sum_{l=1}^h \tau_l a_{h-l} \quad \forall h > 0 \\
{a_0} &= 1,
\end{aligned}
\quad
\begin{aligned}
{h A_h} &= \sum_{l=1}^h |\tau_l|^2 A_{h-l} \quad \forall h > 0 \\
{A_0} &= 1.
\end{aligned}
\tag{24}
\]
These relations prove that for every $h \geq 0$ there exists a polynomial $P_h \in \mathbb{N}[x_1, \ldots, x_h]$ such that
\[
{h! a_h} = P_h(\tau_1, \ldots, \tau_h)
\]
\[
{h! A_h} = P_h(|\tau_1|^2, \ldots, |\tau_h|^2),
\]
therefore, proving $A_h \geq |a_h|^2$ is the same as showing that
\[
{h! P_h(|\tau_1|^2, \ldots, |\tau_h|^2)} \geq |P_h(\tau_1, \ldots, \tau_h)|^2.
\tag{25}
\]
At last, we observe that
\[
{P_h(1, \ldots, 1)} = h!,
\tag{26}
\]
since if we set $\tau_l = 1$ for every $l$, then the recursion (24) is solved by $A_h = 1$ for every $h$.
We know that $P_h \in \mathbb{N}[x_1, \ldots, x_h]$, hence by (26) we can consider $P_h$ as a sum of $h!$ monomials, every one with unitary coefficient, so that (25) follows from the Cauchy-Schwarz inequality $h! \sum_{l=1}^h |y_l|^2 \geq |\sum_{l=1}^h y_l|^2$.

\textbf{Remark.} Proposition 6 also follows from Littlewood’s lemma quoted in Subsection 3.3, but we believe our proof is conceptually easier.

\section{Proof of Theorems}

\subsection{Theorem 4}
Let $\mathcal{S}$ be a finite set of primes, independent of $\mathcal{R}$ and which we will choose later on. We follow the notation of the axioms and we consider the quantity
\[
S_\mathcal{S}(x) = \sum_{n \leq x \atop (n,\mathcal{S}) = 1} |a_n|/n.
\]
The first step is proving that there exists a constant $c_1 = c_1(\mathcal{D}) > 0$, independent of $\mathcal{R}$, such that
\[
S_\mathcal{S}(x) \ll \epsilon, \mathcal{D}^{c_1} \mathcal{R}^\epsilon x^\epsilon, \quad \forall \epsilon > 0.
\tag{27}
\]
Proof. By the Stirling asymptotic formula, when \( \lambda > 0, \Re \mu \geq 0, \) and \( s = \sigma + it \) with \( \sigma < 0, \) we get
\[
\frac{\Gamma(\lambda(1-s) + \mu)}{\Gamma(\lambda s + \mu)} \ll_{s, \lambda} |(1 + |\mu|)(1 + |t|)|^{\lambda(1-2\sigma)},
\]
uniformly on \( t \) and \( \mu. \) This upper bound, axioms (A1)-(A4) and the Lindelöf principle imply that the function \( L(s, A) \) has a polynomial growth in the \( \mathcal{R}t \)-aspect on every vertical strip, i.e., that the estimate
\[
(s-1)^{m,A}L(s, A) \ll (\mathcal{R}A(1 + |t|))^{c_1}
\]
holds when \( a < \sigma < b, \) for some constant \( c_1 = c_1(a, b, D) > 0. \) By axiom (A5) a similar estimate holds for \( L(s, A \otimes \tilde{A}), \) and by assumptions (1) and (2) about the exceptional set \( \mathcal{P}_A, \) the same estimate holds for \( L(s, A \otimes \tilde{A}), \) i.e.,
\[
(s-1)^{m,A \otimes A}L(s, A \otimes \tilde{A}) \ll (\mathcal{R}(A(1 + |t|))^{c_1}.
\]
Therefore,
\[
\sum_{n \leq x} A_n \ll \sum_{n \leq 2x} \frac{A_n}{n} \left(1 - \frac{n}{2x}\right)^r \ll c_1 \mathcal{R}^{c_1} x^r \quad \forall \epsilon > 0, \text{ for some } c_1 = c_1(D) > 0.
\]
By Proposition 6 we have \( |a_n| \leq 1 + |a_n|^2 \leq 1 + A_n, \) so that \( S_{\mathbb{Z}}(x) \leq \sum_{n \leq x} \frac{1+A_n}{n}, \) and (27) immediately follows from (30).

Iwaniec’s work suggests the next step, that is, showing that
\[
S_{\mathbb{Z}}^2(x) \leq c_2 x^s S_{\mathbb{Z}}(x^2) \quad \text{for some } c_2 = c_2(\epsilon, D) > 0.
\]

Proof. By Proposition 1 we have
\[
S_{\mathbb{Z}}^2(x) = \sum_{n,m \leq x, (nm,\mathbb{Z})=1} \frac{|a_n a_m|}{nm}
\leq \sum_{n,m \leq x, (nm,\mathbb{Z})=1} \frac{1}{mn} \sum_{d|(n,m)^A} \frac{R(d)|a_{nm}|}{d} = \sum_{n,m \leq x, (nm,\mathbb{Z})=1} \sum_{d|(n,m)^A} \frac{R(d)|a_{nm}|}{d}
\leq \left( \sum_{d \leq x^2, (d,\mathbb{Z})=1} \frac{R(d)}{d} \right) \left( \sum_{A \leq x^2, (A,\mathbb{Z})=1} \frac{|a_A|}{A} \right) \max_{A,D \leq x^2} \# \{(n,m) : n,m < x, nm = AD\},
\]
so that
\[
S_{\mathbb{Z}}^2(x) \ll \log x \left( \sum_{d \leq x^2, (d,\mathbb{Z})=1} \frac{R(d)}{d} \right) S_{\mathbb{Z}}(x^2).
\]
Now we estimate the sum in (32). Recalling that $R$ is a completely multiplicative function and that $\sum^*$ denotes a sum over the square-full divisors only, we get
\[ \sum_{\substack{d \leq x^2 \\ (d,S)=1}} \frac{R(d)}{d} \leq \prod_{p \leq x^2} \left( 1 + \frac{R^2(p)}{p^2} + \frac{R^3(p)}{p^3} + \cdots \right) \leq \prod_{p \leq x^2} \left( 1 + \frac{R^2(p) \sum_{j=0}^{\infty} \left( \frac{R(p)}{p} \right)^j}{p^2} \right). \]

By hypothesis (R) and the definition of $R(p)$ there exists $c_3$ dependent on $D$ but independent of $R$ such that $R(p) \leq c_3 p^{1/2}$ for every prime. Hence, the series in the infinite product is always convergent if a convenient set $S$ is assumed, for example $S = \{ p : p \leq c_3^2 + 1 \}$. With this choice of $S$ the estimate becomes
\[ \sum_{\substack{d \leq x^2 \\ (d,S)=1}} \frac{R(d)}{d} \ll_D \prod_{p \leq x^2} \left( 1 + \frac{c_3 R^2(p)}{p^2} \right) \leq \prod_{p \leq x^2} \left( 1 + \frac{c_6}{p} \right) \ll_D \log^{c_6} x, \]

and (31) is proved in the stronger form $S^{2^M}_S(x) \ll_D \log^{c_6+1} x \ S_S(x^2)$. \hfill \Box

By iterating (31) and using (27) we obtain, for every $M > 0$,
\[ S_{S}^{2^M}(x) \leq c_M x^{2^M} S_{S}(x^{2^M}) \leq c R x^{2^M} c^\epsilon \quad \text{for some } c = c(\epsilon, D, M) > 0, \]
hence, taking the $2^M$-th root with a suitable $M = M(c_1, \epsilon) = M(\epsilon, D)$, we get
\[ S_{S}(x) \ll_{\epsilon, D} (R x)^{\epsilon}. \quad (33) \]

Since the local factors are convergent for $\sigma > 1 - \rho$ by (R), the series $\sum_{n=0}^{\infty} |a_p|/p^h$ converges for every $p$ and is bounded by a constant independent of $R$. Hence, from (33) we have
\[ \sum_{n \leq x} \frac{|a_n|}{n} \leq \left( \sum_{n \leq x} \frac{|a_n|}{n} \right) \prod_{p \leq S} \sum_{h=0}^{\infty} |a_p|/p^h \ll S_S(x) \ll_{\epsilon, D} (R x)^{\epsilon}. \]

### 3.2 Theorem 1

Theorem 1 is deduced by Theorem 4 in the following way. From (33) we get
\[ \sum_{n \leq x} \log^j n \frac{|a_n|}{n} \leq \log^j x \sum_{n \leq x} \frac{|a_n|}{n} \ll_{\epsilon, D} (R x)^{\epsilon} \log^j x, \]
uniformly on $j$. Let $I^{(j)}(x)$ be defined by
\[ I^{(j)}(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{f^{(j)}(s+1)x^s}{s(s+1)\cdots(s+r)} ds \quad (34) \]
with $r = r(D)$, independent of $j$ and sufficiently large to assure the convergence of the integral. We remark that
\[ \frac{d^j}{ds^j} \left( \frac{s^m}{n^s} \right) = \frac{P_{j,m}(s, \log n)}{n^s} \]
with \( P_{j,m}(x, y) \) a polynomial bounded by \( m^{j+1}(j+1)x^my^j \) uniformly on \( m \) and \( j \). Moreover,

\[
\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{s^m}{s(s+1)\cdots(s+r)} \left( \frac{x}{n} \right)^s ds = \begin{cases} 
0 & \text{if } n > x \\
\ll \frac{\log n}{\sqrt{n}} & \text{if } n \leq x,
\end{cases}
\]

hence,

\[ I^{(j)}(x) \ll m^{j+1}(j+1)\frac{x^{m+2r}}{r!} \sum_{n \leq x} \log^j n \frac{|a_n|}{n} \ll_{c,\mathcal{D}} (j+1)c^j (\mathcal{R}x)^c \log^j x \]

uniformly on \( j \), for some \( c = c(\mathcal{D}) > 0 \). Moreover, by (28) and the Cauchy theorem about the derivative of an holomorphic function, we get

\[ f^{(j)}(s) = ((s-1)^{m_A}L(s,A))^{(j)} \ll \mathcal{D} j! 2^{-j} (\mathcal{R}(1+|t|))^c \]

uniformly on \( j \), with \( a < \sigma < b \).

We move the integration line in (34) to \( \sigma = -1/2 \) so that by the residue theorem we obtain

\[ I^{(j)}(x) = f^{(j)}(1)/r! + O(\frac{\mathcal{R}c^j}{2^{j/\sqrt{x}}}) \]

choosing \( x = \mathcal{R}^{2c_1+2} \) we get the estimate

\[ f^{(j)}(1) \ll_{c,\mathcal{D}} (j+1)c^j \mathcal{R}^c \log^j \mathcal{R} + \frac{j!^c \mathcal{R}}{2^{j/\sqrt{x}}} \]

uniformly on \( j \),

with some \( c = c(\mathcal{D}) > 0 \). At last, the power series expansion \( f^{(j)}(s) = \sum_{n=0}^{\infty} \frac{f^{(j+n)}(1)}{n!} (s-1)^n \) gives (5) in the range \( |s-1| \ll 1/\log \mathcal{R} \).

### 3.3 Theorems 2 and 3

The following algebraic lemma (see Bump [2], Section 2.2) is necessary for the proofs.

**Lemma (E. D. Littlewood).** Let \( \{\alpha_i\}_{i=1}^{d} \) and \( \{\beta_j\}_{j=1}^{d'} \) be non-zero complex numbers, and define \( A = \prod_{i=1}^{d} \alpha_i \), \( B = \prod_{j=1}^{d'} \beta_j \). Then

\[
\prod_{i=1}^{d} \prod_{j=1}^{d'} (1 - \alpha_i \beta_j x)^{-1} = (1 - ABx^d)^{-1} \sum_{k_1,\ldots,k_{d-1}=0}^{\infty} S_{k_1,\ldots,k_{d-1}}(\alpha) S_{k_1,\ldots,k_{d-1}}(\beta) x^{k_1+2k_2+3k_3+\cdots+(d-1)k_{d-1}}
\]

if \( d = d' \), and

\[
\prod_{i=1}^{d} \prod_{j=1}^{d'} (1 - \alpha_i \beta_j x)^{-1} = \sum_{k_1,\ldots,k_{d-1}=0}^{\infty} S_{k_1,\ldots,k_{d-1}}(\alpha) S_{k_1,\ldots,k_{d-1}}(\beta) A^{k_d} x^{k_1+2k_2+3k_3+\cdots+d k_d}
\]

if \( d < d' \), with

\[
S_{k_1,\ldots,k_{d-1}}(\alpha) = \begin{vmatrix} 
\alpha_1^{d-k_d-1} & \cdots & \alpha_{d-1+k_d-1} & \cdots & \alpha_1 \alpha_2 \cdots \alpha_{d} \\
0 & \alpha_1 \cdots \alpha_{d-1+k_d-1} & \cdots & \alpha_{d-2+k_d-1} & \cdots \\
\vdots & \vdots & \alpha_1 & \alpha_2 & \cdots \\
0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{d} \\
1 & \cdots & \cdots & \cdots & 1
\end{vmatrix}
\]
The polynomials $S_{k_1,\ldots,k_{d-1}}(\alpha)$ are called Schur’s polynomials.

Let $A(n)$ and $B(n)$ be the totally multiplicative functions defined by $A(p) = \prod_{i=1}^{d_A} \alpha_i(p)$ and $B(p) = \prod_{j=1}^{d_B} \beta_j(p)$. By Littlewood’s lemma we have

$$
\prod_p \left(1 - \frac{|A(p)|^2}{p^{d_A s}}\right) \sum_{n=1}^{\infty} A_n n^s = \prod_p \left(1 - \frac{|A(p)|^2}{p^{d_A s}}\right) L(s, A \otimes \bar{A}) = \sum_{n_1,\ldots,n_{d_A-1}=1}^{\infty} \frac{|a(n_1,\ldots,n_{d_A-1})|^2}{(n_1n_2^2\cdots n_{d_A-1})^s}, \quad (35)
$$

$$
\prod_p \left(1 - \frac{|B(p)|^2}{p^{d_B s}}\right) \sum_{n=1}^{\infty} B_n n^s = \prod_p \left(1 - \frac{|B(p)|^2}{p^{d_B s}}\right) L(s, B \otimes \bar{B}) = \sum_{n_1,\ldots,n_{d_B-1}=1}^{\infty} \frac{|b(n_1,\ldots,n_{d_B-1})|^2}{(n_1n_2^2\cdots n_{d_B-1})^s}, \quad (36)
$$

and

$$
\prod_p \left(1 - \frac{A(p)B(p)}{p^{ds}}\right) L(s, A \otimes B) = \sum_{n_1,\ldots,n_{d_A-1}=1}^{\infty} \frac{a(n_1,\ldots,n_{d_A-1})b(n_1,\ldots,n_{d_A-1})}{(n_1n_2^2\cdots n_{d_A-1})^s} \quad (37)
$$

if $d_A = d_B = d$: we have

$$
L(s, A \otimes B) = \sum_{n_1,\ldots,n_{d_A-1}=1}^{\infty} \frac{a(n_1,\ldots,n_{d_A-1})b(n_1,\ldots,n_{d_A-1},1,\ldots,1)A(n_{d_A})}{(n_1n_2^2\cdots n_{d_A})^s}. \quad (38)
$$

if $d_A < d_B$. Where the coefficients $a(\cdot)$ and $b(\cdot)$ are multiplicative in every entry and are defined as

$$a(p^{k_1},\ldots,p^{k_{d-1}}) = S_{k_1,\ldots,k_{d-1}}(\alpha(p)) \quad b(p^{k_1},\ldots,p^{k_{d-1}}) = S_{k_1,\ldots,k_{d-1}}(\beta(p)).$$

**Theorem 2.** In this case we assume that $L(s, A \otimes \bar{A})$ and $L(s, B \otimes \bar{B})$ satisfy (R), hence we can prove

$$
\sum_{n \leq x} A_n \ll \epsilon, d_A (R_A x)^\epsilon, \quad \sum_{n \leq x} B_n \ll \epsilon, d_B (R_B x)^\epsilon
$$

by (30) and using the Iwaniec’s idea. We can assume $d_A \leq d_B$. By (35), (36) and by the assumptions of Theorem 2 giving $|A(n)|, |B(n)| \leq 1$ for every $n$, we have

$$
\sum_{n_1n_2^2\cdots n_{d_A-1}=1}^{x} \frac{|a(n_1,\ldots,n_{d_A-1})|^2}{n_1n_2^2\cdots n_{d_A-1}} \leq \sum_{n \leq x} \frac{A_n d_A}{n} \ll \log x \sum_{n \leq x} \frac{A_n}{n} \ll \epsilon, d_A (R_A x)^\epsilon \quad (39)
$$

and

$$
\sum_{n_1n_2^2\cdots n_{d_A}^2=1}^{x} \frac{|b(n_1,\ldots,n_{d_A},1,\ldots,1)|^2}{n_1n_2^2\cdots n_{d_A}} \leq \sum_{n_1n_2^2\cdots n_{d_B-1}=1}^{x} \frac{|b(n_1,\ldots,n_{d_B-1})|^2}{n_1n_2^2\cdots n_{d_B-1}} \leq \sum_{n \leq x} \frac{B_n d_B}{n} \ll \log x \sum_{n \leq x} \frac{B_n}{n} \ll \epsilon, d_B (R_B x)^\epsilon. \quad (40)
$$
Suppose now that \( \mathbf{d}_A = \mathbf{d}_B = \mathbf{d} \). By (37), (39) and (40) it follows that

\[
\left| \sum_{n_1 n_2^2 \cdots n_{d-1}^{d-1} \leq x} \log^j (n_1 n_2^{d-1}) a(n_1, \ldots, n_{d-1}) b(n_1, \ldots, n_{d-1}) \right|
\leq \log^j x \left( \sum_{n_1 n_2^{d-1} \leq x} \frac{|a(n_1, \ldots, n_{d-1})|^2}{n_1 n_2^{d-1}} \right)^{1/2} \left( \sum_{n_1 n_2^{d-1} \leq x} \frac{|b(n_1, \ldots, n_{d-1})|^2}{n_1 n_2^{d-1}} \right)^{1/2}
\ll_{\epsilon, D_A, D_B} (\mathcal{R}_A \mathcal{R}_B x)^\epsilon \log^j x
\]  
uniformly on \( j \). We define

\[
L^*(s, \mathcal{A} \otimes \mathcal{B}) = \prod_p \left( 1 - \frac{A(p) B(p)}{p^{d s}} \right) L(s, \mathcal{A} \otimes \mathcal{B}).
\]

The function \( L(s, \mathcal{A} \otimes \mathcal{B}) \) has a polynomial behavior on the strip \( 1 - \frac{1}{2d} < \sigma < 2 \), and the same holds for \( \prod p^{-A(p) B(p) / p^{d s}} \) since \( |A(p)|, |B(p)| \leq 1 \) and \( \mathbf{d} > 1 \) by hypothesis. Therefore, also \( L^*(s, \mathcal{A} \otimes \mathcal{B}) \) has a polynomial behavior in that strip. We define

\[
I^{(j)}(x) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left( s^{m_A \otimes B} L^*(1 + s, \mathcal{A} \otimes \mathcal{B}) \right)^{(j)} x^s ds
\]
with \( r = r(D_A, D_B) \) independent of \( j \) and sufficiently large to assure the convergence of the integral. As for Theorem 1, from (37) and (41) we get

\[
I^{(j)}(x) \ll_{\epsilon, D_A, D_B} (j + 1) c^j (\mathcal{R}_A \mathcal{R}_B x)^\epsilon \log^j x
\]  
uniformly on \( j \). Moving the integration line to \( \sigma = -1/2d \) we have

\[
I^{(j)}(x) = \left. \left( s^{m_A \otimes B} L^*(1 + s, \mathcal{A} \otimes \mathcal{B}) \right)^{(j)} \right|_{s=0} + O \left( \frac{j! (\mathcal{R}_A \mathcal{R}_B)^{c_1}}{2^j x^{1/2d}} \right),
\]  
for some positive constant \( c_1 = c_1(D_A, D_B) \). Choosing \( x = (\mathcal{R}_A \mathcal{R}_B)^{2d(c_1+1)} \), the claim follows by (42), (43) and some easy algebraic manipulations.

Suppose now that \( \mathbf{d}_A < \mathbf{d}_B \). Then, by (38) and (40), we get

\[
\left| \sum_{n_1 n_2^{d-1} \cdots n_{d-1}^{d-1} \leq x} \log^j (n_1 n_2^{d-1}) a(n_1, \ldots, n_{d-1}) b(n_1, \ldots, n_{d-1} A(n_{d-1}) \right|
\leq \log^j x \left( \sum_{n_1 n_2^{d-1} \cdots n_{d-1}^{d-1} \leq x} \frac{|a(n_1, \ldots, n_{d-1})|^2}{n_1 n_2^{d-1} \cdots n_{d-1}^{d-1}} \right)^{1/2} \left( \sum_{n_1 n_2^{d-1} \cdots n_{d-1}^{d-1} \leq x} \frac{|b(n_1, \ldots, n_{d-1})|^2}{n_1 n_2^{d-1} \cdots n_{d-1}^{d-1}} \right)^{1/2}
\ll_{\epsilon, D_A, D_B} (\mathcal{R}_A \mathcal{R}_B x)^\epsilon \log^j x
\]  
uniformly on \( j \). The claim of Theorem 2 again follows from this estimate.

**Theorem 3.** Following the notations of the axioms, let

\[
S_S(x) = \sum_{n \leq x} \frac{|a_n|^2}{n},
\]
and consider

\[18\]
as we done for the proof of Theorem 1. By Proposition 6 we have $|a_n|^2 \leq A_n$, so that

$$S_2(x) \ll_{\varepsilon, D, A} (\mathcal{R}_A)^c x^\varepsilon$$

holds from (30) for every $\varepsilon > 0$, for some $c = c(\mathcal{D}) > 0$. Similarly to (32), in this case we obtain

$$S_2^2(x) \ll \log^3 x \left( \sum_{d \leq x^2 \atop (d, S) = 1} \frac{1}{d} \right) S_2(x^2),$$

where $\sum^*$ is convergent by $(R')$; from this recursive upper bound and with a suitable choice of $S$ we get

$$\sum_{n \leq x} \frac{|a_n|^2}{n} \ll_{\varepsilon, D, A} (\mathcal{R}_A x)^\varepsilon; \quad (44)$$

therefore, under the hypotheses of Theorem 3, estimates (39) and (40) are easy consequences of (44), and upper bounds (9) follows from these ones as in Theorem 2.

Acknowledgments: This paper is a part of my Ph.D. thesis; I would like to thank Prof. A. Perelli, my Ph.D. advisor. Moreover, I thank the referee for his careful reading and for his suggestions which have considerably improved the previous version of this paper.

References


