Introduction to Domain Decomposition Methods for Finite Element Discretizations of PDEs

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- very important in contemporary large-scale problems arising in applied math., engineering, computer science, computational sciences (www.ddm.org)

Main motivations

- Parallel and distributed computing
- Complex geometries
- Different models and/or numerical methods on different parts of the domain
- Adaptive meshes and local refinement

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Main references

- 19+1 conference Proceedings (1988–2011)
- B. Smith and P. Bjørstad and W. Gropp, *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge Univ. Press, 1996
- A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations*, Oxford Univ. Press, 1999
- A. Toselli and O. Widlund, *Domain Decomposition Methods: Theory and Algorithms*, Springer, 2005
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$$\Downarrow$$

iterative method

For simplicity, consider simplest elliptic PDE: laplacian, 0 b.c.:

$$\left\{\begin{array}{rcll} -\Delta u = -\mathrm{div}(\mathrm{grad} u) & = & f & \text{in} \quad \Omega, \\ u & = & 0 & \text{su} \quad \partial\Omega \end{array}\right.$$

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Galerkin method: variational formulation in a finite dimensional subspace V of $H_0^1(\Omega)$

$$u \in V: \quad a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx \quad \forall v \in V$$

Finite Element Method (FEM)

- Triangulation of Ω with elements (2D: triangles, quadrilaterals; 3D: tetrahedra, hexahedra, prisms)

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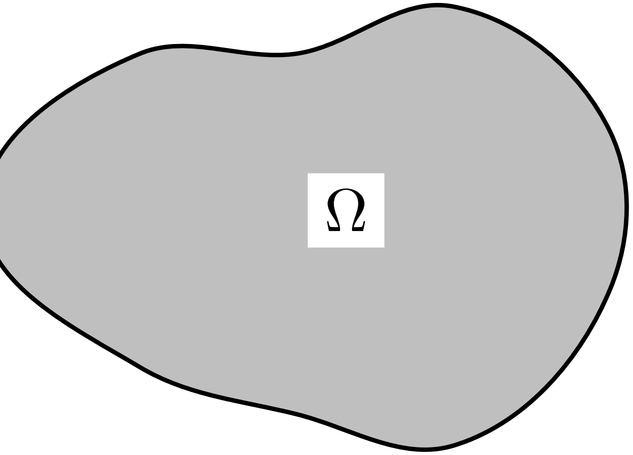
- V = space of piecewise polynomial functions on each element
- Chosen a basis $\{\phi_i\}_{i=1}^{dim(V)}$ for V, the variational problem becomes a linear system

$$Au = f$$

con $A_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dx, \qquad f_j = \int_\Omega f \cdot \phi_j \, dx$

Decomposition of Ω

Let us decompose Ω in two subdomains:



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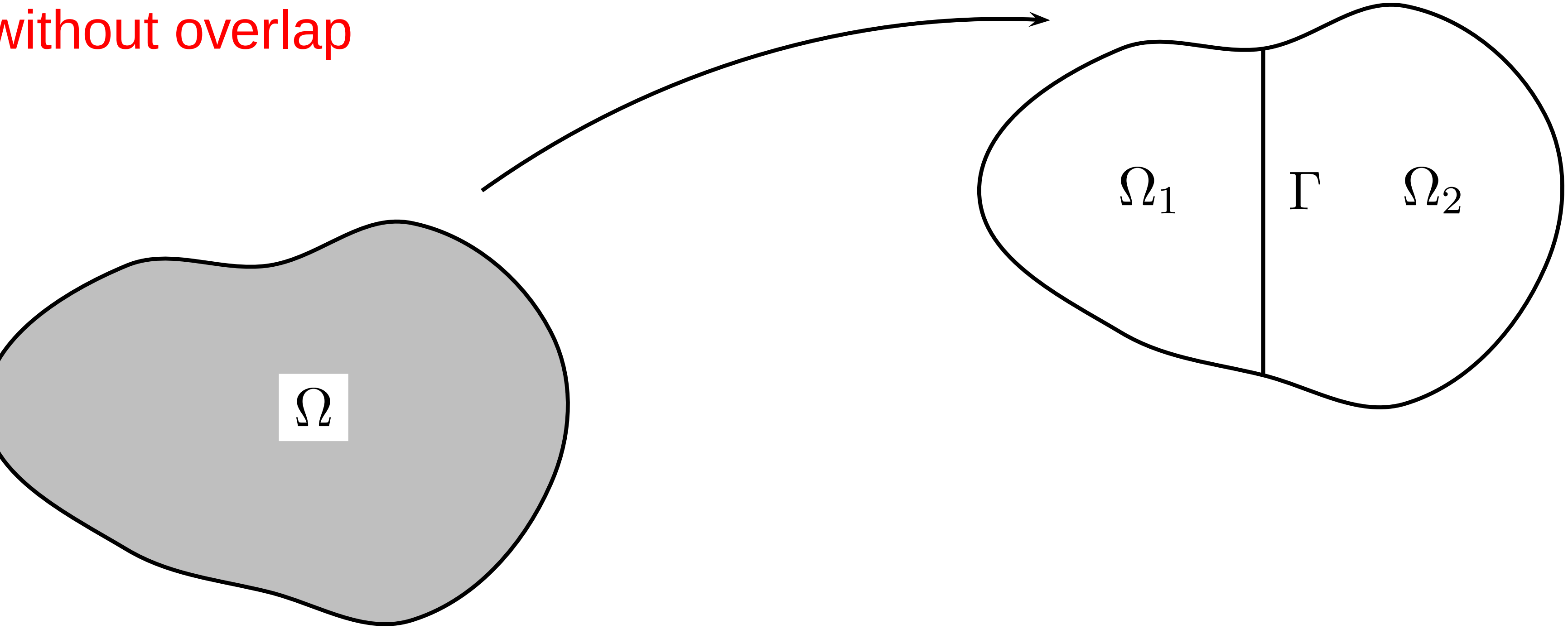

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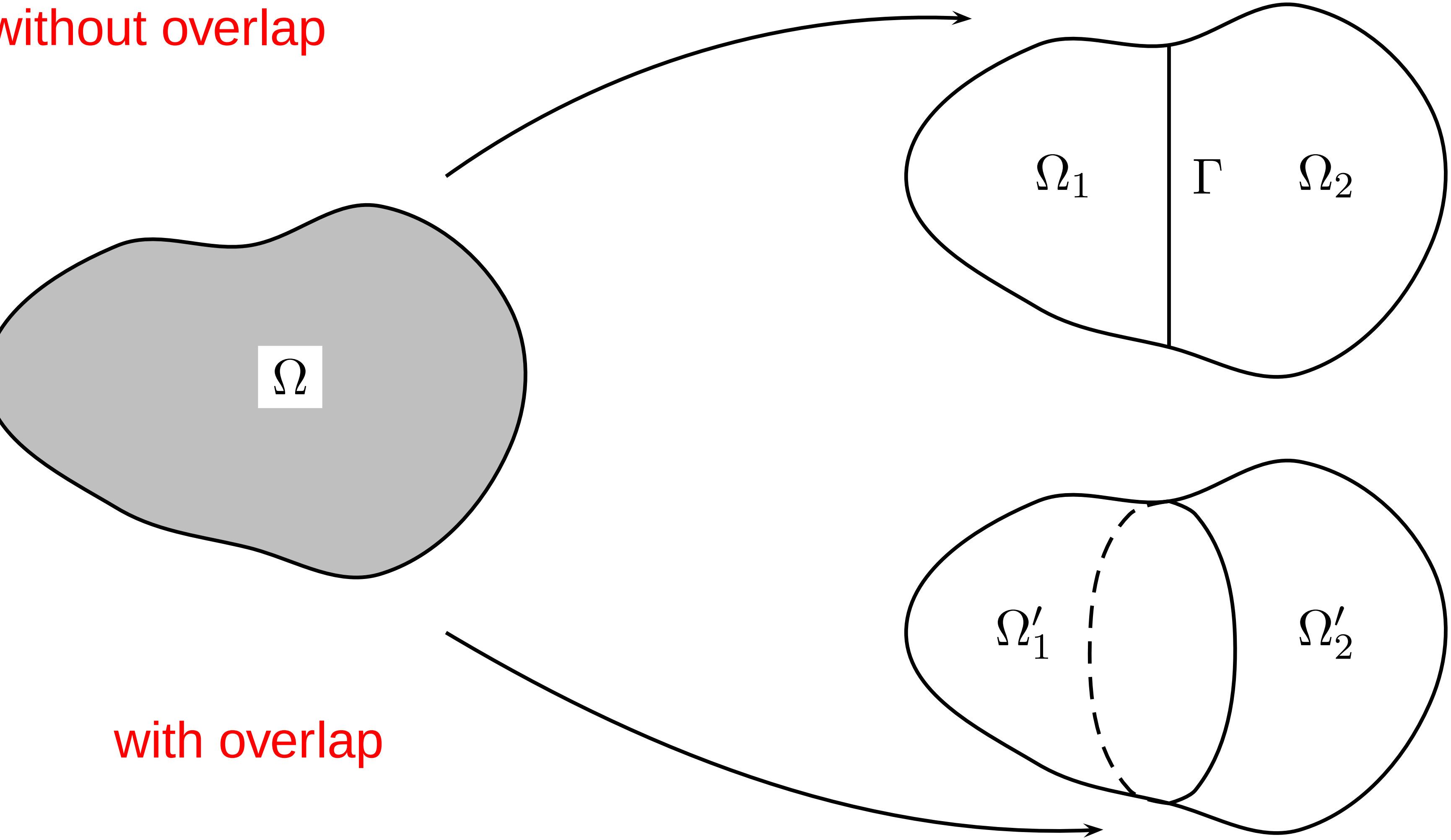

with overlap

Nonoverlapping DD methods (iterative substructuring)

he original problem is equivalent to the coupled problems

$$\left\{\begin{array}{rcllr} -\Delta u_i & = & f & \text{in } \; \Omega_i, & i = 1,2 \\ u_i & = & 0 & \text{on } \; \partial\Omega_i \setminus \Gamma & \\ & & & & \\ u_1 & = & u_2 & \text{on } \; \Gamma, & \text{transmission} \\ \dfrac{\partial u_1}{\partial n_1} & = & -\dfrac{\partial u_2}{\partial n_2} & \text{on } \; \Gamma & \text{conditions} \end{array}\right.$$

$= \partial\Omega_1 \cap \partial\Omega_2$ = interface

$_i$ outer normal on $\partial\Omega_i$

Decomposition of discrete problem

Let us partition the unknowns

$$u = \begin{pmatrix} u_I^1 \\ \\ \\ \end{pmatrix}, \quad f = \begin{pmatrix} f_I^1 \\ \\ \\ \end{pmatrix} \quad interior\ \ to\ \ \Omega_1$$

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$$A = \begin{pmatrix} A_{II}^1 & 0 & A_{I\Gamma}^1 \\ 0 & A_{II}^2 & A_{I\Gamma}^2 \\ A_{\Gamma I}^1 & A_{\Gamma I}^2 & A_{\Gamma\Gamma} \end{pmatrix}$$

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The stiffness matrix A and rhs f are obtained by subassembly ($\int_\Omega = \int_{\Omega_1} + \int_{\Omega_2}$) from their local components

$$A_{\Gamma\Gamma} = A_{\Gamma\Gamma}^1 + A_{\Gamma\Gamma}^2, \quad f_\Gamma = f_\Gamma^1 + f_\Gamma^2$$

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$$\int_\Gamma \frac{\partial u_i}{\partial n_i}\phi_j ds = \int_{\Omega_i}(\Delta u_i\phi_j + \nabla u_i \cdot \nabla \phi_j)dx = \int_{\Omega_i}(-f\phi_j + \nabla u_i \cdot \nabla \phi_j)dx$$

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By approximating u_i with finite elements, we find the discrete flux con λ^i

$$\lambda^i = A^i_{\Gamma I} u^i_I + A^i_{\Gamma\Gamma} u^i_\Gamma - f^i_\Gamma$$

Decomposition of the discrete problem - cont.

$Au = f$

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s equivalent to the problem

$$\begin{cases} A^i_{II} u^i_I + A^i_{I\Gamma} u^i_\Gamma = f^i_I \quad i = 1, 2 & (i) \\ u^1_\Gamma = u^2_\Gamma = u_\Gamma & (D) \\ A^1_{\Gamma I} u^1_I + A^1_{\Gamma\Gamma} u^1_\Gamma - f^1_\Gamma = -(A^2_{\Gamma I} u^2_I + A^2_{\Gamma\Gamma} u^2_\Gamma - f^2_\Gamma) = \lambda_\Gamma & (N) \end{cases}$$

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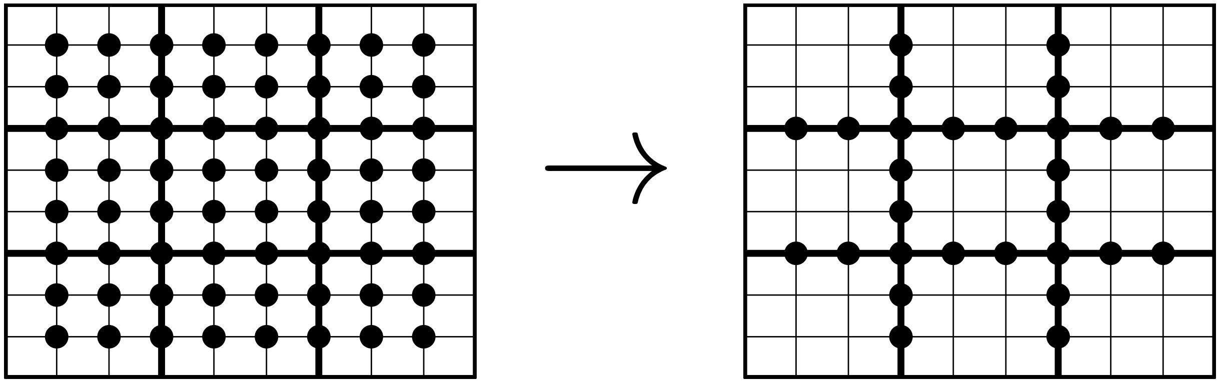
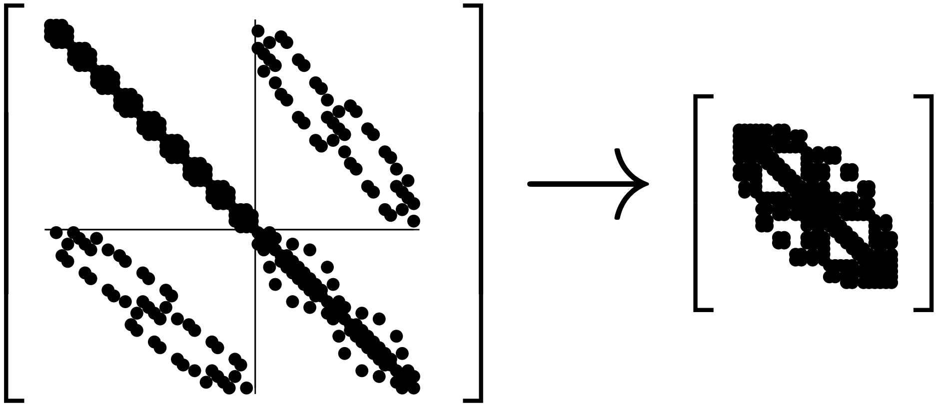
- 4 main possibilities: D-N, N-D, N-N, D-D (other possibilities with mixed conditions of Robin type)
- Richardson iterative method is used here only for simplicity; in he practice of scientific computing we must use a Krylov method (usually PCG or GMRES)!

Schur complement for u_Γ

Let us eliminate the interior unknowns u_I^i in each subdomain Ω_i



$$\left[\begin{array}{c|c} A_{II} & A_{I\Gamma} \\ \hline A_{\Gamma I} & A_{\Gamma\Gamma} \end{array}\right] =$$



$$= S$$

$$S = A_{\Gamma\Gamma} - A_{\Gamma I} A_{II}^{-1} A_{I\Gamma}$$

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Reduced system for u_Γ:

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$$g_\Gamma = g^1_\Gamma + g^2_\Gamma, \qquad g^i_\Gamma = f^i_\Gamma - A^i_{\Gamma I} {A^i_{II}}^{-1} f^i_I$$

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$$g_\Gamma = g^1_\Gamma + g^2_\Gamma, \qquad g^i_\Gamma = f^i_\Gamma - A^i_{\Gamma I} A^{i^{-1}}_{II} f^i_I$$

Once u_Γ in known, the interior unknowns can be found by solving he local Dirichlet problems

$$u^i_I = A^{i^{-1}}_{II} (f^i_I - A^i_{I\Gamma} u_\Gamma)$$

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Eliminating the interior unknowns u_I^i we find $u_\Gamma^i = S^{i^{-1}}(g_\Gamma^i + \lambda_\Gamma^i)$ and since $u_\Gamma^1 = u_\Gamma^2$

$$(S^{1^{-1}} + S^{2^{-1}})\lambda_\Gamma = d_\Gamma$$

with $d_\Gamma = -S^{1^{-1}} g_\Gamma^1 + S^{2^{-1}} g_\Gamma^2$

Dirichlet-Neumann (D-N) method

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- update

$$u_\Gamma^1 = \theta u_2^1 + (1-\theta) u_\Gamma^0 \text{ on } \Gamma,$$

with $\theta \in (0, \theta_{MAX})$

D-N: differential form

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$$(D)\begin{cases} -\Delta u_1^{n+1/2} = f & \text{in} \quad \Omega_1, \\ u_1^{n+1/2} = 0 & \text{on} \quad \partial\Omega_1 \setminus \Gamma \\ u_1^{n+1/2} = u_\Gamma^n & \text{on} \quad \Gamma \end{cases}$$

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$$(N)\left\{\begin{array}{rclll} -\Delta u_2^{n+1} & = & f & \text{in} & \Omega_2, \\ u_2^{n+1} & = & 0 & \text{on} & \partial\Omega_2 \setminus \Gamma \\ \dfrac{\partial u_2^{n+1}}{\partial n_2} & = & -\dfrac{\partial u_1^{n+1/2}}{\partial n_1} & \text{on} & \Gamma \end{array}\right.$$

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update:

$$u_\Gamma^{n+1} = \theta u_2^{n+1} + (1-\theta) u_\Gamma^n \text{ on } \Gamma,$$

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$$u^{n+1}_\Gamma = \theta \tilde{u}^{n+1}_\Gamma + (1-\theta) u^n_\Gamma$$

Eliminating the interior unknowns $u_I^{1,n+1/2}$ e $u_I^{2,n+1}$:

$$S^2(u_\Gamma^{n+1} - u_\Gamma^n) = \theta(g_\Gamma - Su_\Gamma^n)$$

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- substitute the "old" Richardson method with a "modern" Krylov method (e.g. conjugate gradient or GMRES)

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$$u^{n+1}_\Gamma = u^n_\Gamma - \theta(u_\Gamma^{1,n+1} + u_\Gamma^{2,n+1})$$

where

$$r_\Gamma = (A^1_{\Gamma I} u_I^{1,n+1/2} + A^1_{\Gamma\Gamma} u^n_\Gamma - f^1_\Gamma) + (A^2_{\Gamma I} u_I^{2,n+1/2} + A^2_{\Gamma\Gamma} u^n_\Gamma - f^2_\Gamma)$$

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Hence

$$u_\Gamma^{n+1} - u_\Gamma^n = \theta(S^{1^{-1}} + S^{2^{-1}})(g_\Gamma - S u_\Gamma^n)$$

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yielding the preconditioner

$$D^1 {S^1}^{-1} D^1 + D^2 {S^2}^{-1} D^2$$

Dirichlet-Dirichlet (D-D) method

(FETI, dual of N-N)

Given an initial flux λ_Γ^0 on Γ

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2. Overlapping Schwarz method

Given $u^0 = 0$ on $\partial\Omega$

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Multiplicative Schwarz

Using the variational formulation of this algorithm, we find

$$u^{n+1} - u = (I - P_2)(I - P_1)(u^n - u)$$

$P_i : H_0^1(\Omega) \longrightarrow H_0^1(\Omega_i')$ orthogonal projection defined by

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with g a proper rhs

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- Must use a Krylov method (e.g. conjugate gradient if the problem is SDP)

N subdomains

These methods can be extended to the case of $N \geq 2$ subdomains

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For increasing N these 1 level methods are not scalable for elliptic problems: it is proven that their iteration counts to have $err < toll$ is proportional to N

$\Rightarrow$ we need to use more levels (multilevel methods)

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n order to have scalability (independence of N), which is crucial in elliptic problems, we must add at least a second level $\Rightarrow$ solve a coarse (global) problem with few unknowns per subdomain (usually a problem associated with the subdomain mesh)

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3. Abstract Schwarz Theory

(P.L. Lions, Bramble, Dryja, Widlund,...)

V Hilbert space with $dim(V) < \infty$ (e.g. fem or spectral elements).
Decompose V in subspaces

$$V = V_0 + V_1 + \cdots + V_N$$

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- projection-like operators $P_i : V \rightarrow V_i$ defined by

$$b_i(P_i u, v) = a(u,v) \qquad \forall v \in V_i.$$

with $b_i(\cdot,\cdot) : V_i \times V_i \rightarrow R$ local SPD bilinear forms on each subspace V_i

Three assumptions

- (Stable Decomposition) $\exists\ C_0$ such that every $u \in V$ admits a decomposition $u = \sum u_i, u_i \in V_i$, with

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- (Local Stability) $\exists\, \omega \ge 1$ such that for $i = 0, 1, \dots, N$

$$a(u_i, u_i) \le \omega b_i(u_i, u_i), \qquad u_i \in Range(P_i) \subset V_i$$

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vhere$\rho(E) =$ spectral radius of the matrix $E = (\epsilon_{ij})$

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where$\rho(E)$ = spectral radius of the matrix $E = (\epsilon_{ij})$

(multiplicative version)

$$\boxed{\|I - P_{ms}\|_a^2 \leq 1 - \frac{2-\omega}{C_0^2(2\hat{\omega}^2\rho(E)^2+1)} \leq 1}$$

with $\hat{\omega} = \max(1,\omega)$

Convergence rate estimates (FEM)

- With a proper coarse space V_0 (complex in 3D) the best nonoverlapping DD methods satisfy

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- With a proper coarse space V_0 (simple even in 3D), Overlapping Schwarz methods with overlap δ satisfy

$$\boxed{cond(P_{as}) \leq C \left(1 + \frac{H}{\delta}\right)}$$