COMPARISON PRINCIPLES FOR CONSTRAINED SUBHARMONICS

PH.D. COURSE - SPRING 2019
UNIVERSITÀ DI MILANO

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Abstract. Comparison principles for functions obeying differential inequalities/differential inclusions play a ubiquitous role in mathematical analysis and differential geometry. Elementary examples include subharmonic functions and convex functions, which can be characterized as those upper semi-continuous functions whose second order super-differentials lie in a certain closed convex cone inside the 2-jet bundle. Well known monotonicity properties of the constraint sets is crucial for the use of maximum principle techniques (as laid out in the pioneering work of Crandall-Evans-Ishii-Jensen-Lions [9] on viscosity solutions in the late 80s.) Making use of a general notion of ellipticity due to Krylov [19], Harvey and Lawson [10] have recast the discussion in an elegant way by introducing a notion of Dirichlet duality in the 2-jet bundle and bringing to bear important ideas from convex analysis and intuition from complex geometry. We intend to give a survey of this recent theory and its applications to important classes of fully nonlinear PDE.

1. Meeting times and dates

Lectures to be given in the Aula Dottorato according to the following schedule:

- **10:30 – 12:30** on February 28 and March 5;
- **14:00 – 16.00** on March 11, 15;
- **10:30 – 12:30** on March 19, 21, 26, 28 and April 2, 4, 9,11;
- **10:30 – 11:30** on April 16.

Date: February 28, 2019.

2. Syllabus

Some variation is possibile.

(1) Semi-continuous functions and their differentiability properties.
(2) Monotonicity cones $\mathcal{M}$ and constraint sets $\mathcal{F}$ which are invariant with respect to $\mathcal{M}$.
(3) $\mathcal{F}$-subharmonic functions: definitions, examples and elementary properties.
(4) Dirichlet duality and associated subharmonics.
(5) The *Addition Theorem* and the *Almost Everywhere Theorem* of Harvey and Lawson.
(6) The *Zero Maximum Principle (ZMP)* and reduction of comparison to it.
(7) *Strict approximators*, radial calculus and their use in the verification of the validity of (ZMP).
(8) Analysis of monotonicity cone constraints whose Dirichlet duals satisfy (ZMP).
(9) Applications to fully nonlinear degenerate elliptic and parabolic PDE: comparison principles, principal eigenvalues, Perrons method.

3. Remarks on the references for the course.

Section (1): The main reference is Harvey-Lawson [13], which also makes use of Slodkowski [20] and compares this to the classical approach of Jensen [17]. One might want to consult also the exposition in the classical viscosity solution literature such as in Crandall [6] and [7].

Sections (2) - (4): The main reference is Cirant-Harvey-Lawson-Payne [4]. This approach has its origins in Harvey-Lawson [10], which was also inspired by Krylov [19]. Numerous generalizations have been carried out, such as those in [11, 15] and [5], just to name a few.

Section (5): The main reference is Harvey-Lawson [14], but we will make use of some improvements found in the course of preparing [4].

Sections (6) - (9): Again we will make use of Cirant-Harvey-Lawson-Payne [4], but consulting other Harvey-Lawson papers [10, 11, 15] (and others) might be of use. Depending on which applications we will discuss, [2], [5] may be used. For the applications, one might want to consult the classical viscosity literature such as

Crandall-Ishii-Lions [9], Ishii-Lions [16], Crandall [7], Koike [18] and Barles-Busca [1], for example.

References

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