COMPARISON PRINCIPLES FOR CONSTRAINED SUBHARMONICS

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1. Introduction

Constant coefficient differential inequalities and inclusions, constraint sets, monotonicity cones, duality, jet addition, the zero maximum principle, classical subharmonics, convexity and suffaffine functions, the Monge-Ampére operator. Goals and prerequisites.

2. Constant coefficient constraint sets and their subharmonics.

Subequation constraint sets $\mathcal{F} \subset \mathcal{J}^2$ (the space of 2-jets $(r, p, A) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n)$), directional cones $\mathcal{D} \subset \mathbb{R}^N$. Classical notion of u being $\mathcal{F}$-subharmonic on domains $\Omega \subseteq \mathbb{R}^n$ for constraints sets $\mathcal{F}$ and the viscosity definition of $\mathcal{F}$-subharmonics for upper semi-continuous functions in terms of upper test jets $J^{2,+}_{x_0} u \subset \mathcal{F}$ for each $x_0 \in \Omega$. The space $\mathcal{F}(\Omega)$ of USC($\overline{\Omega}$) functions which are $\mathcal{F}$-subharomic on Ω. Examples: convex, subaffine and Laplace subharmonics. Various equivalent formulations of upper test jets. The "bad test jet" lemma. Illustration of some of the implications of properties (P), (N) and (T) for subequations. The coherence lemma and local existence of smooth subharmonics.

3. Dirichlet duality

Definition of the Dirichlet dual $\widetilde{\mathcal{F}}$ of a subequation $\mathcal{F}$ and simple examples. Elementary properties of the dual. Definition of $\mathcal{F}$-harmonics and $\mathcal{F}$-superharmonics using duality. Equivalent formulation for $\mathcal{F}$-superharmonics in terms of lower contact jets $J^{2,-}_{x_0} u$. The Definitional Comparison Lemma.

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4. Monotonicity cones for constant coefficient subequations.

Monotonicity sets $\mathcal{M} \subset \mathcal{J}^2$ and the minimal monotonicity set $\mathcal{M}_0$. The maximal monotonicity cone $\mathcal{M}_{\mathcal{F}}$ for a subequation $\mathcal{F}$ and its properties. Definition of a monotonicity cone subequation $\mathcal{M}$. $\mathcal{M}$-monotonicity of $\mathcal{F}$ implies property (T) (as well as (P) and (N)) for $\mathcal{F}$. Constructions of monotonicity cone subequations as products and by intersections of elementary examples. A fundamental family of monotonicity cone subequations $\mathcal{M}(\gamma, \mathcal{D}, R)$ with $\gamma \in [0, +\infty), R \in (0, +\infty]$ and $\mathcal{D} \subset \mathbb{R}^n$ a directional cone. Its construction and the proof that $\mathcal{M}$-monotonicity of $\mathcal{F}$ for some $\mathcal{M}$ implies monotonicity for a member $\mathcal{M}(\gamma, \mathcal{D}, R)$ of the fundamental family.

5. The Zero Maximum Principle for dual monotonicity cones $\widetilde{\mathcal{M}}$.

Statement of the Zero Maximum Principle (ZMP) and its role in the attempt to prove the Comparison Principle for $\mathcal{F}$-subharmonic, superharmonic pairs when $\mathcal{F}$ is $\mathcal{M}$-monotone. Notion of a strict approximator (of zero) for $\mathcal{M}$ and the theorem that its existence ensures the validity of the (ZMP) for the dual cone $\widetilde{\mathcal{M}}$ to $\mathcal{M}$. Theorem on the validity of the (ZMP) for each $\mathcal{M}$ in the fundamental family $\mathcal{M}(\gamma, \mathcal{D}, R)$ (with restrictions on Ω if R is finite).

6. The Comparison Principle for $\mathcal{M}$-monotone subequations.

The Jet Addition Lemma and its corollary that the Comparison Principle in the equivalent form (CP)' ((ZMP) for sums $u + v$ of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$-subharmonics) holds for twice differentiable pairs (u, v). Lemma that (CP)' holds for semi-continuous pairs (u, v) follows from the (ZMP) and the Subharmonic Addition Theorem (SAT) $\mathcal{F}(\Omega) + \widetilde{\mathcal{F}}(\Omega) \subset \widetilde{\mathcal{M}}(\Omega)$. Proof of the (SAT): reduction to a local result, suitable localization, truncating approximations and sup convolution regularization and the properties (MAX), (DL), (Perron), (Translation) of F-subharmonics to reduce the (SAT) to the case of semi-convex functions. Reduction of the validity of the Almost Everywhere Theorem (AET). Proof of the (AET): lemma on upper contact jets for semi-convex functions, notion of global upper contact points and the lemma of Jensen-Slodkowski.

7. Improvements and limitations.

Improvements with additional monotonicity by four types of monotonicity cones which contain the cone $\mathcal{M}(R)$ with R finite (and which led to domain restrictions for comparison). Lemma on radial polynomial approximators. Failure of (CP) on arbitrary small balls for certain subequation constraint sets: definition of the relevant subequations and illustration of the failure of (CP) and the Maximum Principle (MP) by the failure of uniqueness for the Dirichlet problem (DP). Proof that the maximal monotonicity cone has no interior and hence no strict approximators (of zero) can be found.

8. Subequation constraint sets and nonlinear operators.

Motivation of transporting the potential theoretic results for subequation constraint sets $\mathcal{F} \subset \mathcal{J}^2$ to PDEs $F(u, Du, D^2) = 0$ associated to operators F. Definition of compatible operator-subequation pairs $(F, \mathcal{F})$ and elementary examples and non-examples. M-monotonicity of pairs $(F, \mathcal{F})$ and proper elliptic pairs. Topological tameness of an operator F: definition and discussion on the pathologies that it rules out. Statement of the theorem on topological tameness. The Comparison Principle (CP) for compatible pairs. Definition of $\mathcal{F}$-admissible (viscosity) sub and super solutions of the PDE defined by $F(J) = c$ for each $c \in \mathbb{R}$. Theorem on the Correspondence Principle for compatible proper elliptic pairs $(F, \mathcal{F})$ for which F is topologically tame. Canonical operators F for $\mathcal{M}$-monotone subequations $\mathcal{F}$. The Structure Theorem for $\mathcal{M}$-monotone subequations $\mathcal{F}$. Definition of the canonical operator for F for $\mathcal{F}$ (determined by a fixed jet $J_0 \in \operatorname{Int} \mathcal{M}$). Theorem on canonical operators including structural properties, topological tameness and Lipschitz regularity of F. Proof by the Structure Theorem and graphing the boundaries $\partial \mathcal{F}$ and $\partial \mathcal{M}$ over hyperplanes transversal to J_0. Some calculus facts about canonical operators and examples.

9. The Comparison Principle for nonlinear operators.

Illustration of the Correspondence Principle for classes of operators. Proper elliptic gradient-free operators. Definition of compatible proper elliptic gradient free pairs $(F, \mathcal{F})$. Theorem on the validity of the Comparison Principle for such pairs. Examples in both constrained and unconstrained cases and generalizations. Failure to realize $F(r, A) = \det(A)r$ in a compatible pair $(F, \mathcal{F})$. Remarks about "generalized equations" in the sense of Harvey-Lawson. Comparison for compatible pairs with some strict monotonicity: unconstrained cases of a) degenerate elliptic operators, Lipschitz in the gradient with strict monotonicity in r and b) strictly elliptic operators which are proper and Lipschitz in the gradient. Improvements for uniformly elliptic equations. Improvements for $F(r,p,A) = \mathfrak{g}(A)h(r,P)$ with $\mathfrak{g}$ a Gårding polynomial and suitable h.

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