Multiplicity of nontrivial solutions for an asymptotically linear nonlocal Tricomi problem

Daniela Lupo ¹, Kevin R. Payne ², *, †

Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci, 32 20133 Milano, Italy

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1. Introduction and statement of the result

In this work, we are concerned with the multiplicity of nontrivial solutions for the following nonlocal semilinear Tricomi problem, introduced in [7]:

\[ Tu \equiv -yu_{xx} - u_{yy} = Rf(u) \quad \text{in } \Omega \]
\[ u = 0 \quad \text{on } AC \cup BC \cup nESC, \]

(NST)

where \( T \equiv -y\partial^2_x \partial^2_y \) is the Tricomi operator on \( \mathbb{R}^2 \), \( R \) is the reflection operator induced by composition with the map \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( \Phi(x, y) = (-x, y) \), \( f(u) \) is an asymptotically linear term such that \( f(0) = 0 \), and \( \Omega \) is a symmetric admissible Tricomi domain (cf. Definition 2.1 of Lupo and Payne [7]). That is, \( \Omega \) is a bounded region in \( \mathbb{R}^2 \) that is symmetric with respect to the \( y \)-axis and has a piecewise \( C^2 \) boundary \( \partial\Omega = AC \cup BC \cup \sigma \), where \( \sigma \) is a symmetric with respect to the \( y \)-axis \( C^2 \) arc in the elliptic region \( y > 0 \), with endpoints on the \( x \)-axis at \( A = (-x_0, 0) \) and \( B = (x_0, 0) \) and \( AC \) and \( BC \) are two characteristic arcs for the Tricomi operator in the hyperbolic region of negative and positive slope issuing from \( A \) and \( B \), respectively, which meet at the point \( C \) on the \( y \)-axis. It should be noted that \( u \equiv 0 \) is always a solution to (NST). We refer the reader to [7] for the rationale concerning this problem, the interest

* Corresponding author. Tel.: +39-02-26602280; fax: +39-02-70630346.

E-mail addresses: danlup@mate.polimi.it (D. Lupo), paynek@mate.polimi.it (K.R. Payne).

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² Supported by a CNR grant; program for foreign mathematicians.

† Current address: Dipartimento di Matematica, Università di Milano, Via C. Saldini 50, 20133 Milano, Italy.
in variational methods for mixed elliptic–hyperbolic type equations, and for references to relevant literature on the Tricomi operator and its importance.

First of all, let us specify in which sense we are looking for generalized solutions. For $h \in \mathcal{H}$, we denote by $C^\infty_{AC}(\Omega)$ the set of all smooth functions on $\Omega$ such that $u \equiv 0$ on $\Gamma$, by $W^1_{AC}(\Omega)$ the closure of $C^\infty_{AC}(\Omega)$ with respect to the $W^{1,2}(\Omega)$ norm, and by $W^{-1}_{AC}(\Omega)$ the dual of $W^1_{AC}(\Omega)$.

Definition 1.1. One says that $u \in W^1_{AC,\Omega}$ is a generalized solution of (NST) if

$$Tu = Rf(u) \quad \text{in } L^2(\Omega),$$

and there exists a sequence $\{u_j\} \subset C^\infty_{AC,\Omega}(\Omega)$ such that

$$\lim_{j \to \infty} \|u_j - u\|_{W^1_{AC,\Omega}} = 0 \quad \text{and} \quad \lim_{j \to \infty} \|Tu_j - Rf(u_j)\|_{W^{-1}_{AC,\Omega}} = 0.$$

Let $T_{AC}$ be the unique continuous extension of $T$ from $C^\infty_{AC,\Omega}(\Omega)$ into $W^1_{AC,\Omega}$ and note that $R$ induces an isometric isomorphism between $W^1_{AC,\Omega}$ and $W^1_{BC,\Omega}$. In [7], following ideas in [2], it is shown that, given a symmetric admissible domain $\Omega$, the reflected Tricomi operator

$$R_{T_{AC}} : D(R_{T_{AC}}) = W_A \subset W^1_{AC,\Omega} \to L^2(\Omega),$$

admits a continuous left inverse

$$K = (RT_{AC})^{-1} : L^2(\Omega) \to W_A \subset W^1_{AC,\Omega}$$

for which $K$ is compact as an operator on $L^2(\Omega)$. Denote by $\mu_k^+$ the sequence of positive eigenvalues of $K$ decreasing to zero (cf. Sections 3 and 2 of [6] for more details). Suppose that

(f1) $f \in C^0(\mathbb{R}, \mathbb{R})$, $f$ is invertible and $f(0) = 0$.

In addition, suppose that $a$ and $b$ are real numbers such that $a \not\in \sigma((RT_{AC})^{-1})$ and $b \in (\mu^+_{j+1}, \mu^+_j)$ for some $j \geq 1$. After writing $f(s) = s + f_\infty(s) = s + f_0(s)$,

(f2) $f_\infty$ is bounded and $\lim_{s \to 0} f_0(s)/s = 0$, and

(f3) $-s^2/b \leq s f_0(s) \leq 0$, $s \in \mathbb{R}$.

In [6] we have proved the following result.

Theorem A. Let $\Omega$ be a symmetric admissible Tricomi domain and assume that $f$ satisfies (f1)–(f3). Then, if $a > \mu^+_j$, the problem (NST) admits at least one nontrivial generalized solution.

In this paper we want to show how, if the nonlinearity “interacts” more with the spectrum of the inverse operator $K = (RT_{AC})^{-1}$, then one obtains more nontrivial solutions. More precisely, the following result holds.

Theorem B. Let $\Omega$ be a symmetric admissible Tricomi domain and assume that $f$ satisfies (f1)–(f3). Suppose that $a \not\in \sigma((RT_{AC})^{-1})$ and that for some $j \geq 2$,
\( \mu_{j+1}^+ < b < \mu_j^+ < \mu_{j-1}^+ < a. \) Then there exists a value \( b^* \in (\mu_{j+1}^+, \mu_j^+) \) such that for every \( b \in (b^*, \mu_j^+) \) the problem (NST) admits at least two nontrivial solutions.

The result will be obtained utilizing the dual variational method, as in [6,7], combined with the following critical point theorem of Marino, Micheletti, and Pistoia on the existence of distinct linking critical points. More precisely, consider a Hilbert space \( H \) decomposed into \( H = W \oplus Z \), with \( \dim Z < + \infty \). Within \( Z \) consider a proper subspace \( Z_1 \) and two elements \( e_1 \in Z_1 \) and \( e_2 \in (Z \ominus Z_1) \) with \( \|e_1\| = \|e_2\| = 1 \). Define, for any \( Y \) subspace of \( H \),

\[
B_\rho(Y) := \{ u \in Y \mid \|u\| \leq \rho \}
\]

and by \( \partial B_\rho(Y) \) the boundary of \( B_\rho(Y) \) relative to \( Y \). Furthermore, for any \( e \in H \), consider

\[
Q_\rho(Y, e) := \{ u + ae \in Y \oplus \mathbb{R}e \mid u \in Y \ a \geq 0, \ \|u + ae\| \leq \rho \}
\]

and denote by \( \partial Q_\rho(Y, e) \) its boundary relative to \( Y \oplus \mathbb{R}e \).

**Theorem 1.2** (cf. Theorem 8.4 of Marino et al. [8]). Suppose that \( I \in C^1(H, \mathbb{R}) \) satisfies the (PS) condition. In addition, assume that there exist \( \rho_i, R_i \), for \( i = 1, 2 \), such that \( 0 < \rho_i < R_i \) and

\[
\inf_{\partial Q_{R_i}(W, e_i)} I > \sup_{\partial B_{\rho_i}(Z)} I,
\]

\[
\inf_{Q_{R_i}(W, e_i)} I > - \infty,
\]

\[
\inf_{\partial Q_{R_2}(W \oplus Z_1, e_2)} I > \sup_{\partial B_{\rho_2}(Z \ominus Z_1)} I,
\]

\[
\inf_{Q_{R_2}(W \oplus Z_1, e_2)} I > - \infty.
\]

If \( R_1 < R_2 \), then there exist at least three critical levels of \( I \). Moreover, the critical levels satisfy the following inequalities:

\[
\sup_{B_{\rho_2}(Z)} I \geq c_1 \geq \inf_{Q_{R_1}(W, e_1)} I > \sup_{\partial B_{\rho_1}(Z)} I \geq c_2 \geq \inf_{Q_{R_1}(W, e_1)} I
\]

\[
> \inf_{\partial Q_{R_2}(W \oplus Z_1, e_2)} I \geq \sup_{\partial B_{\rho_2}(Z \ominus Z_1)} I \geq c_3 \geq \inf_{Q_{R_2}(W \oplus Z_1, e_2)} I.
\]

**Remark 1.3.** (1) Since \( \dim Z < + \infty \), one has \( \sup_{B_{\rho_1}(Z)} I < + \infty \).

(2) Theorem 2.1 of [5] gives (utilizing a result of [4]) the analog to Theorem 1.2 above in the case \( \dim Z = + \infty \); (cf. also [3, Theorem 2.1] for a simpler statement for \( Z_1 = \mathbb{R}e_1 \)).

2. Setting of the problem

Following ideas contained in [6,7], we translate the problem of finding nontrivial solutions of (NST) into the problem of finding nontrivial critical points of the dual
action functional associated to (NST); the rough idea behind the use of a dual variational method is the following. Let $f$ be an invertible function and denote by $g = f^{-1}$ its inverse. The statement

there exists $u_0 \in W^{1}_{AC∪e}$ such that $u_0 \neq 0$ and $T_{AC}u_0 = Rf(u_0)$ in $L^2(Ω)$

is equivalent to

there exists $u_0 \in W^{1}_{AC∪e}$ such that $u_0 \neq 0$ and $RT_{AC}u_0 = f(u_0)$ in $L^2(Ω)$.

Hence, if we are able to find a $v_0 \in L^2(Ω)$ such that

$g(v_0) = Kv_0$,

where $K = (RT_{AC})^{-1}$ is compact on $L^2(Ω)$ (cf. Section 2 of [7] for a precise description of the linear theory), then $u_0 = Kv_0$ is a solution of our problem.

Therefore, our goal will be to set-up a variational formulation of the dual problem. To this end, define $J : L^2(Ω) \rightarrow \mathbb{R}$ as

$$J(v) = \int_{Ω} G(v) \, dx \, dy - \frac{1}{2} \int_{Ω} vKv \, dx \, dy, \quad (2.1)$$

where $G(v) = \int_{0}^{v} g(t) \, dt$ denotes the primitive of $g$ such that $G(0) = 0$. If $g$ satisfies the growth condition (for well-known properties of Nemistkii operators cf. [1])

$$|g(t)| \leq K_1 + K_2 |t| \quad (2.2)$$

it is easy to check that $J \in C^1(L^2(Ω), \mathbb{R})$ and that

$$J'(v)[w] = \int_{Ω} g(v)w \, dx \, dy - \int_{Ω} Kwv \, dx \, dy. \quad (2.3)$$

Therefore, critical points of $J$ will be weak solutions of the dual problem; let us remark explicitly that the key point in proving (2.3) is the symmetry of $K$.

3. The critical point result

We consider the case of nonlinearity $f$ which is asymptotically linear, both in zero and at infinity, where the slope in zero is different from the slope at infinity. More precisely, first we recall that since $K = (RT_{AC})^{-1}$ is a compact linear operator on $L^2(Ω)$ which is injective, nonsurjective, and self-adjoint the spectrum of $K$ is $\{0\} \cup \{μ_k\}$, where $μ_k$ is a sequence of eigenvalues of finite multiplicity whose only possible accumulation point is zero. Furthermore, it is shown in Proposition 2.1 of [6] that there exist infinitely many positive and infinitely many negative eigenvalues, which we will denote by $μ_k^\pm$. As pointed out in Remark 2.3 of [6], it is possible to associate to $μ_k^\pm$ a complete orthonormal basis in $L^2(Ω)$.

The solutions will be found as the preimage of two distinct linking critical points which are obtained by applying Theorem 1.2 to the dual action functional $J$. Hence, we have to show that $J$ satisfies an appropriate geometrical condition (cf. Lemmas 3.2 and 3.4) and a suitable compactness condition (cf. Lemma 3.1).

To begin with, one can check that the hypotheses $(fi), i = 1, 2, 3$, imply that the inverse $g = f^{-1}$ satisfies
\((g1)\) \(g \in C^0(\mathbb{R}, \mathbb{R})\), \(g\) is invertible, \(g(0) = 0\), \(g(t) = at + g_\infty(t) = bt + g_0(t)\), where the perturbations \(g_0\) and \(g_\infty\) belong to \(C^0(\mathbb{R}, \mathbb{R})\) and satisfy

\((g2)\) \(g_\infty\) is bounded and \(\lim_{t \to 0} g_0(t)/t = 0\), and

\((g3)\) \(t g_0(t) \geq 0\), \(t \in \mathbb{R}\).

One then notes that \((g1)\)–\((g3)\) imply that the primitives \(G_0(t) = \int_0^t g_0(\tau) d\tau\) and \(G_\infty(t) = \int_0^t g_\infty(\tau) d\tau\) satisfy, for every \(t \in \mathbb{R}\),

\[|G_\infty(t)| \leq M|t| \quad \text{and} \quad 0 \leq G_0(t) \leq \frac{1}{2}(a - b)t^2 + \tilde{M}|t|,\]

where \(\tilde{M} = aM_\infty\) (denoting by \(M_\infty\) the positive constant such that \(|f_\infty(s)| \leq M_\infty\), for every \(s \in \mathbb{R}\)), and

\[G_0(t) = \frac{1}{2}(a - b)t^2 + G_\infty(t).\]

Furthermore, one has

\[\lim_{t \to 0} \frac{G_0(t)}{t^2} = 0 \quad \text{and} \quad G_\infty(t) \geq \frac{(b - a)}{2}t^2, \quad t \in \mathbb{R},\]

and

\[\lim_{t \to \pm\infty} \frac{2G(t)}{t^2} = a \quad \text{and} \quad \lim_{t \to 0} \frac{2G(t)}{t^2} = b.\]

Let us now consider the dual action functional \(J\) defined in (2.1)

\[J(v) = \int_\Omega G(v) dx \, dy - \frac{1}{2} \int_\Omega vKv dx \, dy\]

\[= \int_\Omega G_0(v) dx \, dy + \frac{b}{2} \int_\Omega v^2 dx \, dy - \frac{1}{2} \int_\Omega vKv dx \, dy\]

\[= \int_\Omega G_\infty(v) dx \, dy + \frac{a}{2} \int_\Omega v^2 dx \, dy - \frac{1}{2} \int_\Omega vKv dx \, dy,\]

where the use of one of the last two equivalent decompositions will be made as suits the situation. It is clear that the growth condition on \(g_\infty\) implies (2.2) and hence \(J \in C^1(L^2(\Omega), \mathbb{R})\).

As a final preparation, let \(L^2(\Omega) = V \oplus V^\perp\), where \(V = \bigoplus_{k=1}^J V_k\) is a direct sum of eigenspaces \(V_k = \{v \in L^2(\Omega) \mid K\mu_k v\}\), while \(V^\perp\) is the orthogonal complement of \(V\) in \(L^2(\Omega)\).

From now on, we will denote by \(\|v\| = \|v\|_{L^1(\Omega)}\). Keeping in mind the ordering \(0 < \cdots < \mu_j^+ < \cdots < \mu_1^+\), it is easy to see that one has

\[-\int_\Omega vKv dx \, dy \leq -\mu_j^+ \|v\|^2 \quad \text{for every} \ v \in V,\]

\[-\int_\Omega vKv dx \, dy \geq -\mu_1^+ \|v\|^2 \quad \text{for every} \ v \in V\]

and

\[-\int_\Omega vKv dx \, dy \geq -\mu_j^+ \|v\|^2 \quad \text{for every} \ v \in V^\perp.\]
Lemma 3.1. Let $\Omega$ be a symmetric admissible Tricomi domain and assume that $g$ satisfies (g1), (g2) and (g3). Then $J$ satisfies the Palais–Smale condition on $L^2(\Omega)$.

Proof. Consider a Palais–Smale sequence; that is $\{v_n\}$ such that

$$
g_\infty(v_n) + av_n - Kv_n \to 0 \quad \text{in } L^2(\Omega) \quad (3.8)$$

and

$$|J(v_n)| \leq M.$$ 

One has that $\{v_n\}$ must be bounded in $L^2(\Omega)$. If not, one has that $\|v_n\| \to +\infty$. Then, one has $\frac{g_\infty(v_n)}{\|v_n\|} \to 0$ in $L^2(\Omega)$ since (g2) yields

$$
\frac{\int_\Omega g_\infty^2(v_n) \, dx \, dy}{\|v_n\|^2} \leq \frac{\int_\Omega \tilde{M}^2 \, dx \, dy}{\|v_n\|^2} \leq \tilde{M} |\Omega|.
$$

Now, since $v_n/\|v_n\|$ is a norm-bounded sequence, by the compactness of $K$, there is a $w \in L^2(\Omega) \setminus \{0\}$ such that $K(v_n/\|v_n\|) \to w$ in $L^2(\Omega)$, where one passes to another subsequence if needed. Then by (3.8), one must have $av_n/\|v_n\| \to w$ in $L^2(\Omega)$, and thus by the continuity of $K$ we have $aw - Kw = 0$ in $L^2(\Omega)$. This gives a contradiction since $a \not\in \sigma((RTAC)^{-1})$. Hence, $v_n$ must be bounded. Then, keeping in mind that (3.8) can be written as

$$g(v_n) - Kv_n \to 0 \quad \text{in } L^2(\Omega)$$

and that $K$ is compact, one has that, at least for a subsequence, $g(v_{n_k}) \to z$ in $L^2(\Omega)$ for some $z \in L^2(\Omega)$. Therefore, $f(g(v_{n_k})) = v_{n_k} \to f(z)$ in $L^2(\Omega)$ since $f$ is a continuous Nemitskii operator and hence the result.

We now consider the geometrical conditions. Denoting by

$$\partial B_\rho(V) = \{v \in V \mid \|v\| = \rho\}$$

and, letting $e^+_j \in V_j$ such that $\|e^+_j\| = 1$, by

$$\partial Q_R(V^\perp, e^+_j) = \{w + \alpha e^+_j \mid \alpha > 0 \text{ and } \|w + \alpha e^+_j\| = R\} \cup \{w \in V^\perp \mid \|w\| \leq R\},$$

the following lemmas provide a geometrical structure suitable for applying Theorem 1.2 to the dual functional $J$. \[\square\]

Lemma 3.2. Let $\Omega$ be a symmetric admissible Tricomi domain and assume that $g$ satisfies (g1)–(g3). If $a > \mu_j^+$, there exist $\rho_1 > 0$ and $R_1 > \rho_1$ such that the following inequality holds:

$$\sup_{\partial B_\rho(V)} J(v) < \inf_{\partial Q_{R_1}(V^\perp, e^+_j)} J(v). \quad (3.9)$$

Proof. We give a brief sketch of the proof which is contained in [7]. We want to show that it is possible to choose a $\rho_1$ small enough to make the $\sup$ strictly negative and to choose $R_1$ large enough to make the $\inf$ nonnegative. Then, choosing $R_1$ even
larger, if need be so that $R_1 > \rho_1$, will complete the lemma. To estimate the $\sup$ from above, we begin by noting that inequality (3.5) applied to arbitrary $v \in V$ yields

$$J(v) = \frac{b}{2} \int \Omega v^2 \, dx \, dy + \int \Omega G_0(v) \, dx \, dy - \frac{1}{2} \int \Omega v K v \, dx \, dy$$

$$\leq \frac{1}{2} (b - \mu^+_j) \|v\|^2 + \int \Omega G_0(v) \, dx \, dy.$$ 

If we show that

$$\lim_{\|v\| \to 0} \frac{\int \Omega G_0(v) \, dx \, dy}{\|v\|^2} = 0,$$  \tag{3.10}$$

then we will obtain the claim: there exists a $\rho_1 > 0$ such that

$$J(v) \leq \frac{1}{4} (b - \mu^+_j) \|v\|^2 < 0 \quad \text{for any } v \in V, \|v\| = \rho_1.$$  \tag{3.11}$$

In fact, it is shown in Lemma 4.5 of [7], that

$$\lim_{\|v\|_{W^{1,2}} \to 0} \frac{\int \Omega G_0(v) \, dx \, dy}{\|v\|^2_{W^{1,2}}} = 0,$$ 

and, since all norms are equivalent on $V$ with $\dim V < +\infty$, the desired limit property on $G_0$ follows.

On the other hand, to estimate the $\inf$ from below, we begin by observing that inequality (3.7) and the lower bound on $G_0$ in (3.1) yields, for arbitrary $w \in V^\perp$

$$J(w) \geq \int \Omega G_0(w) \, dx \, dy + \frac{b}{2} \int \Omega w^2 \, dx \, dy - \frac{1}{2} \mu^+_{j+1} \|w\|^2$$

$$\geq \frac{1}{2} (b - \mu^+_{j+1}) \|w\|^2 \geq 0.$$  \tag{3.12}$$

Furthermore, for arbitrary $w + \alpha e_j^+ \in V^\perp \oplus \mathbb{R}e_j^+$, the inequality (3.7) together with the lower bound on $G_\infty$ in (3.1) yields

$$J(w + \alpha e_j^+) = \frac{a}{2} \|w\|^2 + \frac{a}{2} \alpha^2 + \int \Omega G_\infty(w + \alpha e_j^+) \, dx \, dy$$

$$- \frac{1}{2} \int \Omega w Kw \, dx \, dy - \frac{\alpha^2}{2} \mu_j^+$$

$$\geq \frac{1}{2} (a - \mu^+_{j+1}) \|w\|^2 + \frac{\alpha^2}{2} (a - \mu^+_{j+1})$$

$$- \tilde{M} |\Omega|^{1/2} (\|w + \alpha e_j^+\|).$$  \tag{3.13}$$

Now, by recalling that $w$ is orthogonal to $e_j^+$, we can set $R^2 = \|w + \alpha e_j^+\|^2 = \|w\|^2 + \alpha^2$ and from (3.13) taking into account that $a > \mu^+_{j} > \mu^+_{j+1}$, we get

$$J(w + \alpha e_j^+) \geq \frac{1}{2} (a - \mu^+_{j}) R^2 - \tilde{M} |\Omega|^{1/2} R.$$  \tag{3.14}$$
It is clear that it is possible to choose $R$ sufficiently large to make the right-hand side of (3.14) strictly positive, and hence to choose, if need be, a bigger $R_1$ so that $R_1 > \rho_1$ and hence the inequality (3.9) follows.

Remark 3.3. Inequality (3.9) of Lemma 3.2 gives the first inequality in the hypotheses of Theorem 1.2 with the choices $H = L^2(\Omega)$, $Z = V$, $W = V^\perp$ and $e_1 = e^*_1$. Moreover, inequalities (3.12) and (3.13) yield the second inequality in the hypotheses of Theorem 1.2 and hence Lemma 3.1 completes the proof of Theorem A.

Now we will show that if two eigenvalues fall between $a$ and $b$, then one can construct a second linking geometry that is suitable for the application of Theorem 1.2.

Lemma 3.4. Let $\Omega$ be a symmetric admissible Tricomi domain and assume that $g$ satisfies (g1)–(g3). If for some $j \geq 2$, $\mu^+_{j+1} < b < \mu^-_{j} < \mu^-_{j-1} < a$ then there exist $\rho_2 > 0$, $R_2 > \rho_2$ and a value $b^* \in (\mu^+_{j+1}, \mu^-_{j})$ such that for every $b \in (b^*, \mu^-_{j})$ the following inequality holds:

$$
\sup_{\partial B_\rho(V \ominus V_j)} J(v) < \inf_{\partial Q_{\rho_2}(V^\perp \ominus V, e^*_j)} J(v). \tag{3.15}
$$

Proof. Since $V \ominus V_j = \bigoplus_{k=1}^{j-1} V_k$, one can choose a $\rho_2$, independently of $b$, such that for every $v \in V \ominus V_j$ with $\|v\| = \rho_2$ one has

$$
J(v) \leq \frac{1}{4}(\mu^+_{j} - \mu^-_{j-1})\rho_2^2 < 0. \tag{3.16}
$$

Indeed, for every $v \in V \ominus V_j$ one has

$$
J(v) = \frac{b}{2} \int_\Omega v^2 \, dx \, dy + \int_\Omega G_0(v) \, dx \, dy - \frac{1}{2} \int_\Omega vKv \, dx \, dy
$$

$$
\leq \frac{1}{2} (b - \mu^-_{j-1})\|v\|^2 + \int_\Omega G_0(v) \, dx \, dy
$$

$$
\leq \frac{1}{2} (\mu^+_{j} - \mu^-_{j-1})\|v\|^2 + \int_\Omega G_0(v) \, dx \, dy,
$$

and hence, by using the same reasoning that leads to the inequality (3.11), one can fix a $\rho_2$ independently of $b$ so that (3.16) holds.

On the other hand, let $e^*_j \subset V_{j-1}$ be such that $\|e^*_j\| = 1$. Then, for every $w \in V^\perp$, $v \in V_j$ and for every $\beta \in \mathbb{R}$, keeping in mind that $\mu^+_{j+1} < \mu^+_{j} < \mu^-_{j-1}$ and that $w$, $v$ and $e^*_j$ are mutually orthogonal, by choosing $R^2 = \|w+v+\beta e^*_{j-1}\|^2 = \|w\|^2 + \|v\|^2 + \beta^2$, one gets

$$
J(w + v + \beta e^*_{j-1}) = \frac{a}{2} \|w\|^2 + \frac{a}{2} \|v\|^2 + \frac{a}{2} \beta^2 + \int_\Omega G_\infty(w + v + \beta e^*_{j-1}) \, dx \, dy
$$

$$
- \frac{1}{2} \int_\Omega wKw \, dx \, dy - \frac{\mu^+_{j}}{2} \|v\|^2 - \frac{1}{2} \beta^2 \mu^-_{j-1},
$$

and hence the inequality (3.9) of Lemma 3.2 gives the first inequality in the hypotheses of Theorem 1.2 with the choices $H = L^2(\Omega)$, $Z = V$, $W = V^\perp$ and $e_1 = e^*_j$. Moreover, inequalities (3.12) and (3.13) yield the second inequality in the hypotheses of Theorem 1.2 and hence Lemma 3.1 completes the proof of Theorem A.
\[ \geq \frac{1}{2}(a - \mu_{j+1}^-)\|w\|^2 + \frac{1}{2}(a - \mu_j^+)\|v\|^2 + \frac{\beta^2}{2}(a - \mu_j^{-1}) \]

and hence

\[ J(w + v + \beta e_j^{-1}) \geq \frac{1}{2}(a - \mu_j^{-1})R^2 - \tilde{M}|\Omega|^{1/2}R. \] (3.17)

Hence, one can choose an \( \tilde{R} \) large enough to ensure that the right-hand side of (3.17) strictly positive. Therefore, with \( R_2 > \max\{\tilde{R},R_1,\rho_2\} \), one has

\[ J(w + v + \beta e_j^{-1}) > 0 \] (3.18)

for every \((w,v)\in V^\perp \oplus V_j\) and for every \(\beta \in \mathbb{R}\) such that \(\|w + v + \beta e_j^{-1}\| = R_2\).

On the other hand, for every \(w \in V^\perp\) and every \(v \in V_j\), taking into account (3.3) and \(\mu_{j+1}^- < \mu_j^+\), one has

\[ J(w + v) = \int_\Omega G_\infty(w + v)\,dx\,dy + \frac{a}{2} \int_\Omega (w + v)^2\,dx\,dy \]

\[ - \int_\Omega (w + v)K(w + v)\,dx\,dy \]

\[ \geq \frac{(b - a)}{2}(\|w\|^2 + \|v\|^2) + \frac{a}{2}(\|w\|^2 + \|v\|^2) - \frac{\mu_{j+1}^-}{2}\|w\|^2 - \frac{\mu_j^+}{2}\|v\|^2 \]

\[ \geq \frac{1}{2}(b - \mu_j^+)\|w\|^2 + \|v\|^2]. \]

Therefore, for every \(w + v \in V^\perp \oplus V_j\) with \(0 \leq \|w\|^2 + \|v\|^2 \leq R_2^2\), one gets

\[ J(w + v) \geq \frac{1}{2}(b - \mu_j^+)\|w\|^2 + \|v\|^2 \geq \frac{1}{2}(b - \mu_j^+)R_2^2. \] (3.19)

From (3.19), since \(b - \mu_j^+ \neq 0\) when \(b \neq \mu_j^+\), it is possible to fix a value \(b^* \in (\mu_{j+1}, \mu_j^+)\) such that for every \(b \in (b^*, \mu_j^+)\) and for every \(w + v \in V^\perp \oplus V_j\), with \(\|w + v\| \leq R_2\), one has

\[ J(w + v) \geq \frac{1}{6}(\mu_j^+ - \mu_{j-1}^-)\rho_2^2. \] (3.20)

From (3.16), (3.18) and (3.20) one gets

\[ \sup_{\partial B_{\rho_2}(V \oplus V_j)} J(v) \leq \frac{1}{4}(\mu_j^+ - \mu_{j-1}^-)\rho_2^2 < \frac{1}{6}(\mu_j^+ - \mu_{j-1}^-)\rho_2^2 \leq \inf_{\partial Q_{\rho_1}(V^\perp \oplus V_j,e_j^{-1})} J(v), \]

and hence the result. \(\square\)

**Remark 3.5.** Inequality (3.15) gives the third inequality in the hypotheses of Theorem 1.2 with the choices \(H = L^2(\Omega), Z = V, W = V^\perp, Z_1 = V_j, e_1 \in V_j\) and \(e_2 = e_j^{-1}\). Moreover, the fourth inequality in the hypotheses of Theorem 1.2 follows from the estimates of \(J\) from below that lead to inequalities (3.17)–(3.19).
Proof of Theorem B. Suppose that, for some $j \geq 2, \mu_j^{+} < b < \mu_{j+1}^{+} < \mu_{j-1}^{+} < a$ and that $f$ satisfies (f1)–(f3). Then $g = f^{-1}$ satisfies (g1)–(g3); hence, the functional $J$ satisfies, by Lemmas 3.1, 3.2 and 3.4 and Remarks 3.3 and 3.5, the hypotheses of Theorem 1.2. Therefore, there will exist two distinct nontrivial critical points $v_1$ and $v_2$ of $J$. Hence $u_i = Kv_i \in W^{1,\infty} \cup \cap$ for $i = 1, 2$ are distinct nontrivial generalized solutions to (NST). □

References