3.4. The Cantor set and its generalizations. In this section we introduce a noticeable example of a set in \([0, 1]\) that is uncountable and yet it has measure 0, while a variation on the same theme produces sets in \([0, 1]\) of measures arbitrarily close to 1.

**Definition 3.16.** Consider the compact interval \([0, 1]\). Let \(I_{1,0} = (1/3, 2/3)\) and \(C_1 = [0, 1] \setminus I_{1,0}\). Then \(C_1\) is union of two intervals, namely \(I_{1,0} = [0, 1/3]\) and \(I_{1,1} = [2/3, 1]\), each of length 1/3. Next, we define \(C_2\) as the interval \(C_1\) taken away the two middle thirds of each \(I_{1,j}, j = 0, 1\). Then \(C_2\) is union of 2 intervals, namely \(I_{2,0} = [0, 1/9]\), \(I_{2,1} = [2/9, 3/9]\), \(I_{2,2} = [6/9, 7/9]\), and \(I_{2,3} = [8/9, 9/9]\), each of length 1/3\(^2\). We now iterate this process. Notice that at each step we double the number of intervals. Thus, we can write

\[
C_k = \bigcup_{j=0}^{2^k-1} I_{k,j}, \quad m(I_{k,j}) = 3^{-k} \text{ for all } j = 0, \ldots, 2^k - 1,
\]

and

\[
C_{k+1} = \bigcup_{j=0}^{2^{k+1}-1} (I_{k,j} \setminus J_{k,j}) = \bigcup_{j=0}^{2^{k+1}-1} I_{k+1,j}, \quad \text{with } m(J_{k,j}) = 3^{-k}.
\]

We then construct sets \(C_k, k = 1, 2, \ldots\) such that

- \(C_1 \supseteq C_2 \supseteq \cdots\);
- each \(C_k\) is compact so that \(\bigcap_{k=1}^{+\infty} C_k \neq \emptyset\);
- each \(C_k\) is union of \(2^k\) disjoint intervals, each of length \(3^{-k}\).

These properties are all easy to check. We finally set

\[
C = \bigcap_{k=1}^{+\infty} C_k.
\]

The set \(C\) is called the **Cantor set**.

**Proposition 3.17.** Let \(C\) be the Cantor set. Then, the following properties hold true.

(i) \(C\) is compact, nowhere dense and totally disconnected;

(ii) \(C\) has the cardinality of continuum;

(iii) \(C \in \mathcal{L}\) and \(m(C) = 0\).

We recall that a set is said to be nowhere dense if its closure has empty interior and a set is totally disconnected if its connected components are single points.

**Proof.** It is clear that \(C\) is compact and non-empty, as intersection of incapsulated compact sets. For any given \(\varepsilon > 0\) and an interval \(I \subseteq [0, 1]\), let \(k\) be such that \(2^{-k} < \varepsilon\). Since \(C \subseteq C_k\) for each \(k\), and \(C_k\) is union of disjoint compact intervals of length \(3^{-k}\), \(I\) cannot be contained in \(C_k\), hence in \(C\). Thus, \(C\) does not contain any interval. This implies that \(C\) is nowhere dense and totally disconnected. This proves (i).

To show (ii), it is easy to construct an injective correspondence between the set of sequences of 0 and 1 and \(C\). Given the sets \(I_{1,0}\) and \(I_{1,1}\) we associate the values 0 and 1, resp. Splitting \(I_{1,0}\) into \(I_{2,0}\) and \(I_{2,1}\) we associate to each of them the terms \(\{0, 0\}\) and \(\{0, 1\}\), resp., and so on. At each step we have an interval \(I_{k,j}\) to which it is associated the finite sequence of length \(k\) of 0’s and 1’s, when we split it into two intervals, we associate to them the sequences of length \(k+1\) by adding a 0 to the sequence corresponding to interval on the left, and a 1 to the sequence...
corresponding to interval on the right. In this way we associate to any given sequence of 0’s and 1’s the a sequence of incapsulated compact intervals \( \{ I_{k,j} \} \), which has non-empty intersection and thus contains at least one element of \( C \). This proves (ii).

(iii) Notice that \( C = \lim_{k \to +\infty} m(C_k) \). The set \( C_{k+1} \) is obtained by removing \( 2^k \) intervals each of length \( 3^{-(k+1)} \) from \( C_k \). Then,

\[
m(C) = \lim_{k \to +\infty} m(C_{k+1}) = \lim_{k \to +\infty} \left( m(C_k) - \sum_{j=0}^{2^k-1} m(J_{k,j}) \right) = \lim_{k \to +\infty} \left( m(C_k) - \frac{2^k}{3^{k+1}} \right) \\
= \lim_{k \to +\infty} \left( m(C_{k-1}) - \frac{2^{k-1}}{3^k} - \frac{2^k}{3^{k+1}} \right) = \lim_{k \to +\infty} \left( 1 - \frac{1}{3} - \cdots - \frac{2^{k-1}}{3^k} - \frac{2^k}{3^{k+1}} \right) \\
= \lim_{k \to +\infty} 1 - \frac{1}{2} \sum_{j=1}^{k+1} \frac{2^j}{3^j} \\
= 1 - \frac{1}{2} \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 0.
\]

This proves the proposition. \( \square \)

Using a similar procedure, for any \( \varepsilon > 0 \) we can construct a nowhere dense, totally disconnected subset of \([0,1]\), of measure greater than \( 1 - \varepsilon \).

**Proposition 3.18.** Let \( \varepsilon > 0 \). Then, there exists \( \alpha > 0 \) and a set \( C^{(\alpha)} \subseteq [0,1] \), which is compact, nowhere dense, and totally disconnected set, such that \( m(C^{(\alpha)}) \geq 1 - \varepsilon \).

**Proof.** We imitate the construction of the Cantor set. Given \([0,1]\), we remove the central interval of length \( \alpha \), that is we define \( C^{(\alpha)} = [0,1] \setminus \left( \left[ \frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2} + \frac{\alpha}{2} \right] \right) \). From the resulting two intervals, we remove the central intervals, each of length \( \alpha^2 \). Then, at the step \( k+1 \), we remove from \( C^{(\alpha)}_k \) the central intervals of length \( \alpha^{k+1} \), and we set \( C^{(\alpha)} = \cap_{k=1}^{\infty} C^{(\alpha)}_k \). Notice that, if \( \alpha = \frac{1}{3} \), then \( C^{(\alpha)} = C \), the Cantor set.

The topological properties of \( C^{(\alpha)} \) are proved as in the case of \( C \). Finally, observing that the set \( C^{(\alpha)}_{k+1} \) is obtained by removing \( 2^k \) intervals each of length \( \alpha^{k+1} \) from \( C^{(\alpha)}_k \), we have

\[
m(C^{(\alpha)}) = \lim_{k \to +\infty} m(C^{(\alpha)}_{k+1}) = \lim_{k \to +\infty} \left( m(C^{(\alpha)}_k) - 2^k \alpha^{k+1} \right) \\
= \lim_{k \to +\infty} 1 - \frac{1}{2} \sum_{j=1}^{k+1} (2\alpha)^j \\
= 1 - \frac{1}{2} \cdot \frac{2\alpha}{1 - 2\alpha} \\
= 1 - \frac{3\alpha}{1 - 2\alpha}.
\]

Hence, it suffices to select \( \alpha \) so that \( \frac{1 - 3\alpha}{1 - 2\alpha} > 1 - \varepsilon \). \( \square \)
3.5. Integrals depending on a parameter. The following result is quite often very useful. We state it for a generic measure space \((X,\mathcal{M},\mu)\), although we will essentially use it in the case of the Lebesque measure \((\mathbb{R},\mathcal{L},m)\).

**Theorem 3.19.** Let \((X,\mathcal{M},\mu)\) be a measure space, \(f : X \times [a,b] \to \mathbb{R} \) (or \(\mathbb{C}\)), such that \(f(\cdot,t) \in L^1(\mu)\) for every \(t \in [a,b]\) and let 
\[
F(t) = \int_X f(x,t) \, d\mu(x).
\]

1. Suppose that there exists \(g \in L^1(\mu)\) such that \(|f(x,t)| \leq g(x)\) for all \(t \in [a,b]\) and suppose that \(\lim_{t \to t_0} f(x,t) = f(x,t_0)\). Then 
\[
\lim_{t \to t_0} F(t) = F(t_0).
\]

In particular, if \(f(x,\cdot)\) is continuous for every \(x \in X\), then \(F\) is continuous.

2. Suppose that \(\partial_t f\) exists and that there exists \(g \in L^1(\mu)\) such that \(|\partial_t f(x,t)| \leq g(x)\) for all \((x,t) \in X \times [a,b]\). Then \(F\) is differentiable and 
\[
F'(t) = \int_X \partial_t f(x,t) \, d\mu(x).
\]

We remark that in the applications this result is often used in its local form, that is, in the nbhd of a given point \(t_0\) of the parameter. It is in fact sufficient just take \([a,b]\) to be a sufficiently small nbgh of \(t_0\).

**Proof.** (1) Let \(\{n_n\} \subseteq [a,b]\) be any sequence converging to \(t_0\) and set \(f_n(x) = f(x,t_n)\). Then \(f_n \to f(x) := f(x,t_0)\) \(|f_n| \leq g \in L^1(\mu)\), and we can apply the DCT to \(\{f_n\}\) and obtain 
\[
\lim_{n \to +\infty} F(t_n) = \lim_{n \to +\infty} \int_X f_n(x) \, d\mu = \int_X f(x,t_0) \, d\mu = F(x,t_0).
\]

Since the sequence \(t_n \to t_0\) was arbitrary, the conclusion follows.

(2) Again, let \(\{n_n\} \subseteq [a,b]\) be any sequence converging to \(t_0\). Set 
\[
h_n(x) = \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0}.
\]

Then, \(\lim_{n \to +\infty} h_n(x) = \partial_t f(x,t_0)\). Hence, \(\partial_t f(\cdot,t_0)\) is measurable. Moreover, using the mean value theorem we see that 
\[
|h_n(x)| \leq \sup_{t \in [a,b]} |\partial_t f(\cdot,t)| \leq g(x).
\]

Therefore, we can apply the DCT to \(\{h_n\}\), using again the fact that the sequence \(t_n \to t_0\) was arbitrary, and obtain that 
\[
F'(t_0) = \lim_{n \to +\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \lim_{n \to +\infty} \int_X \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0} \, d\mu(x) = \lim_{n \to +\infty} \int_X h_n(x) \, d\mu
\]
\[
= \int_X \partial_t f(x,t_0) \, d\mu,
\]

as we wished to show. \(\square\)
Example 3.20. The Gamma function. For \( z \in \mathbb{C}, \) \( \text{Re} \, z > 0 \) we set
\[
\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t} \, dt.
\]
(We recall that \( t^{z-1} = e^{(z-1)\log t} \).)

Then:
1. for all \( a \in \mathbb{C}, \) \( \text{Re} \, z > 0, \) \( z\Gamma(z) = \Gamma(z+1) \);
2. \( \Gamma(1) = 1; \)
3. for all non-negative integer \( n, \) \( \Gamma(n+1) = n!; \)
4. \( \Gamma \in C^\infty((0, +\infty)). \)

Furthermore, identifying \( z = x + iy \in \mathbb{C} \) with \( (x, y) \in \mathbb{R}^2, \) the function \( \Gamma \) is \( C^\infty \) in \( \{(x, y) : x > 0\}. \)

We prove the above statements in the case \( z \) real, that is, \( z = x > 0. \) The case of the complex \( z \) is proven in exactly the same way, however we do not need in the present class.

We begin by observing that, for every fixed \( x > 0, \) \( t^{x-1}e^{-t} \in L^1((0,\infty)) \). For, \( t^{x-1}e^{-t} \leq t^{x-1} \) for \( 0 < t < 1, \) while \( t^{x-1}e^{-t} \) is easily see to be integrable on \((1, +\infty). \)

Then, integrating by parts we see that
\[
\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t} \, dt = \lim_{\varepsilon \to 0^+, b \to +\infty} \int_\varepsilon^b t^{x-1}e^{-t} \, dt
= \frac{1}{x} \left( t^x e^{-t} \right|_{\varepsilon}^b + \frac{1}{x} \int_\varepsilon^b t^x e^{-t} \, dt
= 0 + \frac{1}{x} \int_0^{+\infty} t^x e^{-t} \, dt
= \frac{\Gamma(x)}{x}.
\]

This shows (1), while (2) it is obvious. To show (3), assume by induction that \( \Gamma(n) = (n-1)!. \)

By (1) we now have that \( \Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!. \) It easily follows from Thm. 3.19 (2), and we leave the details as an exercise.

3.6. More on \( L^1(m). \) In this section we prove some more results about the Lebesgue space \( L^1(m), \) that by Thm. 2.31, we recall, is a Banach space. We recall, if \( X \) is a topological space, that the support of a continuous function \( g \) is the closure of the set \( \{x \in X : g(x) \neq 0\}. \)

We denote by \( C_c(X) \) the space of continuous functions with compact support in \( X. \)

We begin with an elementary lemma, which holds true on any measure space, but, for sake of simplicity, we state only in the case of \( (\mathbb{R}, \mathcal{L}, m). \)

Lemma 3.21. Let \( f \in L^1(m). \) For any \( \varepsilon > 0 \) there exists a set \( A \) such that \( m(\{x \in A \}) < \varepsilon \) and \( f \) is bounded on \( A. \)

Proof. First, we consider the intervals \([-n, n]\) and the sequence \( f_n = \chi_{[-n, n]} f. \) Then, \( f_n \to f \) pointwise, \( |f_n| \leq |f| \in L^1(m), \) so that \( f_n \to f \) in \( L^1(m) \) as \( n \to +\infty, \) by the DTC. It suffices to choose \( n \) so that \( \int_{|x| > n} |f| \, dm = ||f - f_n||_{L^1(m)} < \varepsilon/2. \) Next, we consider the sets \( E_m = \{x : |x| \leq n \text{ and } |f(x)| > m\}. \) Then, \( f \in L^1(m) \) implies that \( m(\cap_{m=1}^{+\infty} E_m) = 0 \) (since otherwise

\[\text{In fact, } \Gamma \text{ is differentiable in the complex sense, that is, } \Gamma \text{ is holomorphic. We will not stress this, however fundamental, aspect at the present time.}\]
By Urysohn’s Lemma there exist continuous functions \(K\) such that \(E = \{x \in \mathbb{R} : f(x) \neq g(x)\}\) for simplicity, and uniformly since \(f\) is bounded, say \(1 = 1\) and \(\varepsilon > 0\), there exists \(g \in C_c(\mathbb{R})\) such that

\[m\{x \in \mathbb{R} : f(x) \neq g(x)\} < \varepsilon.\]

If \(f\) is bounded, \(g\) can be chosen so that \(|g| \leq |f|\).

(2) The space \(C_c(\mathbb{R})\) is dense in \(L^1(m)\).

Proof. Assume first that \(f\) is bounded, say \(|f| \leq 1\), and that \(A\) is compact. Then, by Thm. 2.9 (2), there exists a sequence of simple functions \(\{\varphi_n\}\) such that \(|\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f|, \varphi_n \to f\) pointwise and uniformly on every set on which \(f\) is bounded – hence on all of \(\mathbb{R}\) in this case. Set \(\psi_1 = \varphi_1\) and \(\psi_n = \varphi_n - \varphi_{n-1}\) for \(n \geq 2\). Then, \(\sum_{k=1}^{n} \psi_k = \varphi_n\) so that \(\sum_{n=1}^{\infty} \psi_n = f\), pointwise and uniformly since \(f\) is bounded. Observe that, with the notation of Thm. 2.9, assuming \(f\) real-valued for simplicity,

\[\psi_n = \varphi_n - \varphi_{n-1} = (s_u^+ - s_u^{n-1}) - (s_u^- - s_u^{n-1}).\]

We claim that \(\psi_n = 2^{-n} \chi_{T_n}\), for a certain set \(T_n\). Indeed, assume for a moment that \(0 \leq f \leq 1\), and recall the construction of \(s_n\):

\[s_n(x) = \sum_{k=0}^{2^n-1} k2^{-n} \chi_{E_{n,k}}(x),\]

where

\[E_{n,k} = f^{-1}\left((k2^{-n}, (k + 1)2^{-n})\right),\]

and similarly for \(s_{n-1}\). (Notice that \(0 \leq k \leq 2^n - 1\) only, since \(f \leq 1\).) The function \(s_{n+1}\) was constructed by splitting the sets \(E_{n,k} = E_{n+1,2k} \cup E_{n+1,2k+1}\) as in (3), we have that \(s_{n+1} = s_n\) on \(E_{n+1,2k}\), while \(s_{n+1} - s_n = 2^{-(n+1)}\) on \(E_{n+1,2k+1}\), for each \(k = 0, \ldots, 2^n - 1\). This establishes the claim for \(s_u^\pm - s_u^{n-1}\), hence for \(\psi_n\).

Next, fix an open set \(U\) such that \(A \subseteq U\) and \(\overline{U}\) is compact. Given \(\varepsilon > 0\) there exist sets \(K_n, U_n\), where \(K_n\) is compact, \(U_n\) is open, and such that

\[K_n \subseteq T_n \subseteq U_n \subseteq U \quad \text{and} \quad m(U_n \setminus K_n) < \varepsilon 2^{-n}.\]

By Urysohn’s Lemma there exist continuous functions \(h_n\) such that \(0 \leq h_n \leq 1, h_n = 1\) on \(K_n\), and \(h_n = 0\) on \(\overline{U_n}\). Define

\[g(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x).\]

\(^5\)See e.g. [Rudin, Real and Complex Analysis, 3rd Ed., McGraw–Hill Editor]
Then, the series converges uniformly, \( g \) is continuous, and it is 0 outside \( U \), which is is bounded, so that \( \text{supp}(g) \) is compact. Moreover, observe that \( h_n = 1 \) on \( K_n \), so that \( 2^{-n} h_n = \psi_n \) on \( K_n \), and that \( h_n = 0 = \psi_n \) on \( ^cU_n \). Hence,

\[
\{ x : g(x) \neq f(x) \} \subseteq \bigcup_{n=1}^{+\infty} U_n \setminus K_n
\]

and

\[
m(\{ x : g(x) \neq f(x) \}) \leq \sum_{n=1}^{+\infty} m(U_n \setminus K_n) < \varepsilon.
\]

This proves (1) when \( |f| \leq 1 \) and \( A \) is compact.

Now, let \( A \) be any set such that \( m(A) < +\infty \). Then, given any \( \varepsilon > 0 \), there exist \( U, K \), with \( K \) compact, \( U \) open \( K \subseteq A \subseteq U \), and such that \( m(U \setminus K) < \varepsilon / 2 \). Then, by the argument above, we can find a continuous function with support in \( K \) such that

\[
m(\{ x \in K : g(x) \neq f(x) \}) < \varepsilon / 2.
\]

Since also \( f \) vanishes outside \( U \)

\[
m(\{ x : g(x) \neq f(x) \}) = m(\{ x \in U : g(x) \neq f(x) \})
\]

\[
\leq m(\{ x \in K : g(x) \neq f(x) \}) + m(U \setminus K)
\]

\[
< \varepsilon.
\]

Thus, the conclusion holds true when \( f \) is bounded.

Finally, we remove the assumption that \( f \) is bounded. For each \( n \), let \( B_n = \{ x : |f(x)| \leq n \} \). Since \( f \) is real valued, \( \bigcup_{n=1}^{+\infty} B_n = \mathbb{R} \), so that \( ^cB_1 \supseteq ^cB_2 \supseteq \cdots \), and \( \cap_{n=1}^{+\infty} ^cB_n = \emptyset \). Thus, \( m(^cB_n) \to 0 \) as \( n \to +\infty \). Then, given \( \varepsilon > 0 \) we can find an integer \( n \) such that \( m(^cB_n) \leq \varepsilon / 2 \) and then \( g \) continuous with compact support contained in \( B_n \) such that \( m(\{ x : g(x) \neq (\chi_{B_n} f)(x) \}) < \varepsilon / 2 \). Therefore,

\[
m(\{ x : g(x) \neq f(x) \}) \leq m(\{ x \in B_n : g(x) \neq f(x) \}) + m(^cB_n)
\]

\[
< \varepsilon.
\]

This proves (1).

In order to prove (2), given \( f \in L^1(m) \), given any \( \varepsilon > 0 \), by Lemma 3.21 there exists \( A \subseteq \mathbb{R} \) such that \( m(A) < +\infty \) and \( \int_A |f| \, dm < \varepsilon / 2 \) and \( f \) is bounded on \( A \), say \( |f| \leq M \) on \( A \). Part (1) now gives that there exists \( g \) continuous and with support in \( A \) and such that \( m(A_1) < \varepsilon / 4M \), where \( A_1 = \{ x \in A : g(x) \neq f(x) \} \), and \( |g| \leq |f| \). Then

\[
\| f - g \|_{L^1(m)} \int_{A_1} |f - g| \, dm \int_{A \setminus A_1} |f - g| \, dm + \int_A |f - g| \, dm
\]

\[
\leq \varepsilon / 2 + \varepsilon / 2.
\]

The proof is now complete. \( \square \)