Our current goal is to define the Lebesgue measure on the higher-dimensional euclidean space $\mathbb{R}^n$, and to reduce the computations to integrals in lower dimensions. In order to do this, we first present the theory of integration on product measure spaces and the fundamental theorems of Tonelli and Fubini on the equality of iterated integrals.

4.1. Product measure spaces.

Definition 4.1. Let $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{Y}, \mathcal{N})$ be measurable spaces. A subset of $\mathcal{X} \times \mathcal{Y}$ of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$ is called a measurable rectangle. We define the product $\sigma$-algebra in $\mathcal{X} \times \mathcal{Y}$ as the $\sigma$-algebra generated by the collection of measurable rectangles and we denote it by $\mathcal{M} \times \mathcal{N}$.

Next, given two measure spaces $(\mathcal{X}, \mathcal{M}, \mu)$ and $(\mathcal{Y}, \mathcal{N}, \nu)$ we wish to construct a measure on $\mathcal{M} \times \mathcal{N}$ such that the measure of any measurable rectangle $A \times B$ equals $\mu(A)\nu(B)$.

Throughout this section, we denote by $\mathcal{A}$ the collection of finite unions of disjoint measurable rectangles in $\mathcal{X} \times \mathcal{Y}$.

Lemma 4.2. The following properties hold:

(i) $\mathcal{A}$ is an algebra;

(ii) if $\mathcal{M}(\mathcal{A})$ denotes the $\sigma$-algebra generated by $\mathcal{A}$, then $\mathcal{M}(\mathcal{A}) = \mathcal{M} \times \mathcal{N}$.

Proof. In order to prove that $\mathcal{A}$ is an algebra we need to show that it is closed under the complement and finite unions of sets in $\mathcal{A}$. Given two measurable rectangles $A_1 \times B_1$ and $A_2 \times B_2$, we preliminary observe that $(A_1 \times B_1) \setminus (A_2 \times B_2) = (A_1 \setminus A_2) \times (B_1 \setminus B_2)$ is also a measurable rectangle.

Now, given a measurable rectangle $A \times B$ we have that

$$\mathcal{c}(A \times B) = (\mathcal{X} \times \mathcal{c}B) \cup (\mathcal{c}A \times B),$$

that is, $\mathcal{c}(A \times B)$ is finite union of disjoint measurable rectangles, hence in $\mathcal{A}$. Next, let $A_1 \times B_1$ and $A_2 \times B_2$ be measurable rectangles $\mathcal{A}$ not necessarily disjoint (otherwise we have nothing to prove). By the first part, $\mathcal{A} \ni \mathcal{c}(A_2 \times B_2) = \bigcup_{j=1}^n (E_j \times F_j)$, where $E_j \times F_j$ are disjoint measurable rectangles. Then,

$$(A_1 \times B_1) \setminus (A_2 \times B_2) = (A_1 \setminus A_2) \times (B_1 \setminus B_2) = \left((A_1 \setminus E_1) \times (B_1 \setminus F_1)\right) \cup \cdots \cup \left((A_1 \setminus E_n) \times (B_1 \setminus F_n)\right),$$

hence a finite union of disjoint measurable rectangles, which belongs to $\mathcal{A}$. Therefore,

$$(A_1 \times B_1) \cup (A_2 \times B_2) = \left((A_1 \times B_1) \cap \mathcal{c}(A_2 \times B_2)\right) \cup (A_2 \times B_2) \in \mathcal{A}.$$

By induction, we obtain that

$$(A_1 \times B_1) \cup \cdots \cup (A_n \times B_n) \in \mathcal{A},$$
that is, \( A \) is closed under finite unions. Finally, let \( A \times B \in A \), that is, \( A \times B = \cup_{j=1}^{n} (E_j \times F_j) \), where \( E_j \times F_j \) are disjoint measurable rectangles. Then,

\[
\mathcal{C}(A \times B) = \mathcal{C}\left(\bigcup_{j=1}^{n} (E_j \times F_j)\right) = \mathcal{C}(E_1 \times F_1) \cap \cdots \cap \mathcal{C}(E_n \times F_n) = \mathcal{C}(\mathcal{X} \times \mathcal{F}_1) \cap \cdots \cap \mathcal{C}(\mathcal{X} \times \mathcal{F}_n) \cap \cdots \cap ((\mathcal{X} \times \mathcal{F}_1) \cup (\mathcal{X} \times \mathcal{F}_n)) = \bigcap_{j=1}^{n} \{ (A_j \times B_j) \cap (A'_j \times B'_j) : A_j, A'_j = \mathcal{X} \text{ or } \mathcal{F}_\ell, B_j, B'_j = F\}. \]

Hence, \( \mathcal{C}(A \times B) \) is union of intersections of disjoint measurable rectangles, hence it is union of disjoint measurable rectangles. This shows that \( A \) is an algebra, i.e. \( (i) \).

In order to prove \( (ii) \), observe that \( \mathcal{M} \times \mathcal{N} \) contains the measurable rectangles, hence their finite unions. Thus, \( \mathcal{M} \times \mathcal{N} \) contains \( A \) and therefore the \( \sigma \)-algebra generated by it, that is, \( \mathcal{M}(A) \). Conversely, \( \mathcal{M}(A) \) contains all the measurable rectangles, hence the \( \sigma \)-algebra generated by them, that is, \( \mathcal{M} \times \mathcal{N} \). This completes the proof. \( \square \)

Now, let \( (\mathcal{X}, \mathcal{M}, \mu) \) and \( (\mathcal{Y}, \mathcal{N}, \nu) \) be measure spaces and let \( A \) be the algebra of finite unions of disjoint measurable rectangles. If \( E \subseteq \mathcal{X} \times \mathcal{Y} \) and \( E = \cup_{j=1}^{n} A_j \times B_j \) is an element of \( A \) (that is, \( A_j \times B_j \) are disjoint measurable rectangles, \( j = 1, \ldots, n \)), we set

\[
\rho(E) = \sum_{j=1}^{n} \mu(A_j) \nu(B_j). \quad (14)
\]

**Lemma 4.3.** With the hypotheses as above, \( \rho \) is a premeasure on \( A \).

**Proof.** First of all, we need to show that \( \rho \) is well defined, that is, for any \( \mathcal{D} \in A \), the value of \( \rho(E) \) is independent of the decomposition of \( \mathcal{D} \) as finite unions of disjoint measurable rectangles. Observe that, if \( A \times B \) is a measurable rectangle and \( A \times B = \cup_{j=1}^{m} A'_j \times B'_j \), where \( A'_j \times B'_j \) are disjoint measurable rectangles, we have

\[
\chi_A(x)\chi_B(y) = \chi_{A \times B}(x, y) = \sum_{j=1}^{m} \chi_{A'_j \times B'_j}(x, y) = \sum_{j=1}^{m} \chi_{A'_j}(x)\chi_{B'_j}(y). \]

We now integrate w.r.t. \( x \) on both sides of the equalities above and see that

\[
\mu(A)\chi_B(y) = \int \chi_A(x)\chi_B(y) \, d\mu(x) = \sum_{j=1}^{m} \int \chi_{A'_j}(x)\chi_{B'_j}(y) \, d\mu(x) = \sum_{j=1}^{m} \mu(A'_j)\chi_{B'_j}(y). \]

We proceed by integrating w.r.t. \( y \) and obtain

\[
\rho(A \times B) = \mu(A)\nu(B) = \int \mu(A)\chi_B(y) \, d\nu(y) = \int \sum_{j=1}^{m} \mu(A'_j)\chi_{B'_j}(y) \, d\nu(y) = \sum_{j=1}^{m} \mu(A'_j)\nu(B'_j). \quad (15)
\]

This shows that \( \rho(A \times B) = \sum_{j=1}^{m} \mu(A'_j)\nu(B'_j) \) for any decomposition of \( A \times B \) as finite unions of disjoint measurable rectangles. This easily implies that \( \rho \) is well defined. Now, \( \rho \) is finitely additive by construction.
We need to show \( \rho \) is countably additive, that is, if \( \{A_j \times B_j\} \) is a collection of disjoint measurable rectangles whose union is in \( \mathcal{A} \), then

\[
\rho\left( \bigcup_{j=1}^{\infty} (A_j \times B_j) \right) = \sum_{j=1}^{\infty} \rho(A_j \times B_j).
\]

By the finite additivity of \( \rho \), we first show that it suffices to consider the case in which \( \bigcup_{j=1}^{\infty} (A_j \times B_j) \) is a single measurable rectangle. The argument is analogous to the one in the proof of Lemma 3.8, but we repeat it for sake of completeness. If \( A \times B = \bigcup_{k=1}^{n} (A^{(k)} \times B^{(k)}) \), with \( A^{(k)} \times B^{(k)} \) disjoint measurable rectangles, it is possible to find subcollections \( \{(A_j^{(k)} \times B_j^{(k)})\}, k = 1, 2, \ldots, n, \) of \( \{(A_j \times B_j)\} \) such that, for each \( k = 1, 2, \ldots, n, \)

\[
A^{(k)} \times B^{(k)} = \bigcup_{j=1}^{\infty} (A_j^{(k)} \times B_j^{(k)}).
\]

If we know that for each \( k \), \( \rho(A^{(k)} \times B^{(k)}) = \sum_{j=1}^{\infty} \rho(A_j^{(k)} \times B_j^{(k)}) \), the conclusion then follows from the finite additivity of \( \rho \).

Thus, let us show (16) when \( \bigcup_{j=1}^{\infty} (A_j \times B_j) = (A \times B) \). We proceed as in the argument leading to(15). We have that \( \chi_A(x) \chi_B(y) = \sum_{j=1}^{\infty} \chi_{A_j}(x) \chi_{B_j}(y) \). Integrating first w.r.t. \( x \) we have

\[
\mu(A)\chi_B(y) = \int \chi_A(x) \chi_B(y) \, d\mu(x) = \sum_{j=1}^{\infty} \mu(A_j) \chi_{B_j}(y).
\]

Next we integrate w.r.t. \( y \) and obtain

\[
\rho\left( \bigcup_{j=1}^{\infty} (A_j \times B_j) \right) = \rho(A \times B) = \mu(A)\nu(B)
\]

\[
= \int \mu(A) \chi_B(y) \, d\nu(y) = \int \sum_{j=1}^{\infty} \mu(A_j) \chi_{B_j}(y) \, d\nu(y)
\]

\[
= \sum_{j=1}^{m} \mu(A_j)\nu(B_j).
\]

This shows that \( \rho \) is a premeasure and we are done. \( \square \)

**Definition 4.4.** Let \( (X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu) \) be measure spaces, \( \mathcal{M} \times \mathcal{N} \) the product \( \sigma \)-algebra. We define an outer measure \( (\mu \times \nu)^* \) on \( \mathcal{P}(X \times Y) \) as

\[
(\mu \times \nu)^*(E) = \inf \left\{ \sum_{j=1}^{\infty} \rho(A_j) : E \subseteq \bigcup_{j=1}^{\infty} A_j, \ A_j \in \mathcal{A} \text{ for all } j \right\}.
\]

Lemma 3.5 shows that indeed \( (\mu \times \nu)^* \) is an outer measure, that \( \mathcal{A} \) is contained in the \( \sigma \)-algebra of \( (\mu \times \nu)^* \)-measurable sets, and \( (\mu \times \nu)^* \) restricted to \( \mathcal{A} \) coincides with \( \rho \). As a consequence, we also obtain that \( \mathcal{M} \times \mathcal{N} \) is contained in the \( \sigma \)-algebra of \( (\mu \times \nu)^* \)-measurable sets.

Furthermore, if we also assume that \( (X, \mathcal{M}, \mu) \) and \( (Y, \mathcal{N}, \nu) \) are \( \sigma \)-finite, Thm. 3.7 shows that there exists a measure defined on the \( \sigma \)-algebra of \( (\mu \times \nu)^* \)-measurable sets.
Thus, we define the product measure \( \mu \times \nu \) on the product \( \sigma \)-algebra \( \mathcal{M} \times \mathcal{N} \)

\[
(\mu \times \nu)(E) = (\mu \times \nu)^*(E), \quad E \in \mathcal{M} \times \mathcal{N}.
\]

Finally, \( \mu \times \nu \) is the unique measure on \( \mathcal{M} \times \mathcal{N} \) such that \( (\mu \times \nu)|_\mathcal{A} = \rho \).

We observe that the \( \sigma \)-algebra of \( (\mu \times \nu)^* \)-measurable sets equals \( (\mathcal{M} \times \mathcal{N}) \) union the subsets of sets of \( (\mu \times \nu)^* \)-measure 0.

### 4.2. Integration on product measure spaces.

Having constructed the product measure space \((\mathcal{X} \times \mathcal{Y}, \mathcal{M} \times \mathcal{N}, \mu \times \nu)\) we now have all the results about the integration theory in an abstract measure space at our disposal. However, we wish to relate the calculus of integrals on \(\mathcal{X} \times \mathcal{Y}\) to the integrals on \(\mathcal{X}\) and on \(\mathcal{Y}\). This is the goal of the current section.

Thus, let the measure spaces \((\mathcal{X}, \mathcal{M}, \mu)\) and \((\mathcal{Y}, \mathcal{N}, \nu)\) be given. Given a set \(E \in \mathcal{M} \times \mathcal{N}\) we define its \(x\)-section as

\[
E_x = \{ y \in \mathcal{Y} : (x, y) \in E \}
\]

and its \(y\)-section as

\[
E^y = \{ x \in \mathcal{X} : (x, y) \in E \}.
\]

If \(f\) is a function on \(\mathcal{X} \times \mathcal{Y}\) we define the \(x\)-section as the function \(f_x(y) = f(x, y)\) and the \(y\)-section as \(f^y(x) = f(x, y)\). We now have

**Proposition 4.5.** With the notation above, the following hold true:

(i) if \(E \in \mathcal{M} \times \mathcal{N}\) then, for every \(x \in \mathcal{X}\), \(E_x \in \mathcal{N}\) and, for every \(y \in \mathcal{Y}\), \(E^y \in \mathcal{M}\);

(ii) if \(f\) is a function on \(\mathcal{X} \times \mathcal{Y}\) that is \(\mathcal{M} \times \mathcal{N}\)-measurable, for every \(x \in \mathcal{X}\), \(f_x\) is \(\mathcal{N}\)-measurable and, for every \(y \in \mathcal{Y}\), \(f^y\) is \(\mathcal{M}\)-measurable.

**Proof.** This is easy. Let

\[
\mathcal{R} = \{ E \in \mathcal{M} \times \mathcal{N} : E_x \in \mathcal{N} \text{ and } y \in \mathcal{Y} \text{ for all } x \in \mathcal{X}, y \in \mathcal{Y} \}.
\]

Then, \(\mathcal{R}\) contains all measurable rectangles \(A \times B\), since

\[
(A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \quad \text{and} \quad (A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B. \end{cases}
\]

Next, notice that \(\mathcal{R}\) is a \(\sigma\)-algebra, since

\[
\left( \bigcup_{j=1}^{+\infty} E_j \right)_x = \bigcup_{j=1}^{+\infty} E_{j,x} \quad \text{and} \quad \left( \bigcup_{j=1}^{+\infty} E_j \right)^y = \bigcup_{j=1}^{+\infty} E_{j,y},
\]

and also

\[
\left( ^c E \right)_x = ^c \left( E_x \right) \quad \text{and} \quad \left( ^c E \right)^y = \left( ^c E^y \right).
\]

Therefore, \(\mathcal{R}\) contains the \(\sigma\)-algebra generated by the measurable rectangles, hence the \(\sigma\)-algebra generated by \(\mathcal{A}\), which equals \(\mathcal{M} \times \mathcal{N}\). Hence, \(\mathcal{R} = \mathcal{M} \times \mathcal{N}\), and this shows (i).

In order to show (ii), it suffices notice that \((f^{-1}(U))_x = (f_x)^{-1}(U)\) and \((f^{-1}(U))^y = (f^y)^{-1}(U)\).

Next, we need the following notion.

**Definition 4.6.** A collection \(\mathcal{C}\) of subsets of a set \(\mathcal{X}\) is called a monotone class if it is closed with respect to countable increasing unions and countable decreasing intersections.
Clearly, a $\sigma$-algebra is a monotone class, and in particular $\mathcal{P}(\mathcal{X})$ is a monotone class. It is easy to see that intersections of monotone classes is still a monotone class. Hence, given any $\mathcal{E} \subseteq \mathcal{P}(\mathcal{X})$ there exists the smallest monotone class containing $\mathcal{E}$; namely the intersection of all monotone classes containing $\mathcal{E}$, family that is not empty since it contains $\mathcal{P}(\mathcal{X})$. We denote by $\mathcal{C}(\mathcal{E})$ such monotone class and we call it the monotone class generated by $\mathcal{E}$.

**Theorem 4.7. (Monotone Class Lemma)** Let $\mathcal{A}$ be an algebra of subsets of $\mathcal{X}$. Then, $\mathcal{M}(\mathcal{A}) = \mathcal{C}(\mathcal{A})$, that is, the monotone class generated by $\mathcal{A}$ coincides with the $\sigma$-algebra generated by $\mathcal{A}$.

*Proof.* (†) See [Folland], Monotone Class Lemma 2.35. □

We use the Monotone Class Lemma to prove the following result, which is fundamental for the proof of the Tonelli–Fubini theorem that will follow.

**Proposition 4.8.** Let $(\mathcal{X}, \mathcal{M}, \mu)$, $(\mathcal{Y}, \mathcal{N}, \nu)$ be $\sigma$-finite measure spaces. Let $E \in \mathcal{M} \times \mathcal{N}$. Then, for every $x \in \mathcal{X}$, the function $y \mapsto \mu(E^y)$ is $\mathcal{N}$-measurable and for every $y \in \mathcal{Y}$, the function $x \mapsto \nu(E_x)$ is $\mathcal{M}$-measurable. Moreover,

$$ (\mu \times \nu)(E) = \int_{\mathcal{X}} \nu(E_x) \, d\mu(x) = \int_{\mathcal{Y}} \mu(E^y) \, d\nu(y). $$

*Proof.* We first assume that $\mu$ and $\nu$ are finite measures, that is, $\mu(\mathcal{X}), \nu(\mathcal{Y}) < +\infty$. Let $\mathcal{C}$ be the collection of all sets $E \in \mathcal{M} \times \mathcal{N}$ for which the conclusions in the statement are true. We claim that all measurable rectangles $A \times B$ belong to $\mathcal{C}$. Indeed, by (17) it follows that

$$ \mu((A \times B)^y) = \begin{cases} \mu(A) & \text{if } y \in B \\ 0 & \text{if } y \notin B \end{cases} = \mu(A) \chi_B(y), $$

so that the function $y \mapsto \mu((A \times B)^y)$ is $\nu$-measurable. Analogously,

$$ \nu((A \times B)_x) = \begin{cases} \nu(B) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} = \nu(B) \chi_A(x), $$

so that the function $x \mapsto \nu((A \times B)_x)$ is $\mu$-measurable. Moreover,

$$ (\mu \times \nu)(A \times B) = \mu(A) \nu(B) = \int_{\mathcal{Y}} \mu(A) \chi_B(y) \, d\nu(y) = \int_{\mathcal{Y}} \mu((A \times B)^y) \, d\nu(y), $$

and also

$$ (\mu \times \nu)(A \times B) = \mu(A) \nu(B) = \int_{\mathcal{X}} \nu(B) \chi_A(x) \, d\mu(x) = \int_{\mathcal{X}} \nu((A \times B)_x) \, d\mu(x). $$

This proves the claim. By finite additivity, also finite unions of disjoint measurable rectangles also verify the conclusions in the statement, so that $\mathcal{C}$ contains the algebra $\mathcal{A}$ of finite unions of disjoint measurable rectangles.

If we show that $\mathcal{C}$ is a monotone class, then $\mathcal{C}$ would contain the monotone class generated by the algebra $\mathcal{A}$, that, by the Monotone Class Lemma, coincides with $\mathcal{M} \times \mathcal{N}$. This would imply the result, under the assumption that both $\mu$ and $\nu$ are finite measures. In order to show that $\mathcal{C}$ is a monotone class, we need to show that it closed under countable increasing unions and countable decreasing intersections of sets. Suppose then that $\{E_n\} \subseteq \mathcal{C}$ and $E_1 \subseteq E_2 \subseteq \cdots$. Set $E = \bigcup_{n=1}^{+\infty} E_n$. Then $\{\mu(E_n^y)\}$ is an increasing sequence of non-negative, $\mathcal{N}$-measurable
functions converging to $\mu(E^y)$. Thus, $\mu(E^y)$ is also $\mathcal{N}$-measurable, and by the MCT we have that
\[
\int_y \mu(E^y) \, d\nu = \lim_{n \to +\infty} \int_y \mu(E_n^y) \, d\nu = \lim_{n \to +\infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E).
\]
By switching the roles of $\mu$ and $\nu$ and arguing in the same way, we also obtain that
\[
\int_{\mathcal{X}} \nu(E_x) \, d\mu = (\mu \times \nu)(E).
\]
Thus, $\mathcal{C}$ is closed under countable increasing unions.

On the other hand, suppose that $\{E_n\} \subseteq \mathcal{C}$ and $E_1 \supseteq E_2 \supseteq \cdots$. Set $E = \cap_{n=1}^{+\infty} E_n$. Here we are going to use the assumption that $\mu$ and $\nu$ are finite measures. We have that $\{\mu(E_n^y)\}$ is a decreasing sequence of non-negative, $\mathcal{N}$-measurable functions converging to $\mu(E^y)$. Thus, $\mu(E^y)$ is also $\mathcal{N}$-measurable. Moreover,
\[
0 \leq \mu(E_n^y) \leq \mu(\mathcal{X}) \in L^1(\nu),
\]
since $\mu(\mathcal{X}), \nu(\mathcal{Y}) < +\infty$. Thus, we can apply the DCT and obtain
\[
\int_{\mathcal{Y}} \mu(E^y) \, d\nu = \lim_{n \to +\infty} \int_{\mathcal{Y}} \mu(E_n^y) \, d\nu = \lim_{n \to +\infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E).
\]
Again, by switching the roles of $\mu$ and $\nu$ and arguing in the same way, we have
\[
\int_{\mathcal{X}} \nu(E_x) \, d\mu = (\mu \times \nu)(E).
\]
Thus, $\mathcal{C}$ is closed under countable decreasing intersections and $\mathcal{C}$ is a monotone class. This proves the assertion in the case $\mu, \nu$ are finite measures.

Suppose now that $(\mathcal{X}, \mathcal{M}, \mu)$, $(\mathcal{Y}, \mathcal{N}, \nu)$ are $\sigma$-finite measure spaces. Let $\{X_j\} \subseteq M$, $\{Y_j\} \subseteq N$ be increasing sequences of sets such that $\bigcup_{j=1}^{+\infty} X_j = \mathcal{X}$, $\bigcup_{j=1}^{+\infty} Y_j = \mathcal{Y}$. Let $E$ be any set in $\mathcal{M} \times \mathcal{N}$. The previous argument applies to the set $E_j = E \cap (X_j \times Y_j)$ to give that for every $j$, the functions $y \mapsto \mu((E_j)^y)$ and $x \mapsto \nu((E_j)_x)$ are $\mathcal{N}$-measurable and $\mathcal{M}$-measurable, resp., for every $y \in Y_j$, $x \in X_j$, resp. Therefore, the functions $y \mapsto \mu((E_j)^y) \chi_{Y_j}(y)$ and $x \mapsto \nu((E_j)_x) \chi_{X_j}(x)$ are $\mathcal{N}$-measurable and $\mathcal{M}$-measurable, resp., for every $y \in \mathcal{Y}$, $x \in \mathcal{X}$, resp. Moreover,\\
\[
(\mu \times \nu)(E_j) = \int_{X_j} \nu((E_j)_x) \, d\mu(x) = \int_{\mathcal{X}} \nu((E_j)_x) \chi_{X_j}(x) \, d\mu(x)
\]
\[
= \int_{Y_j} \mu((E_j)^y) \, d\nu(y) = \int_{\mathcal{Y}} \mu((E_j)^y) \chi_{Y_j}(y) \, d\nu(y).
\]
Finally, we apply the MCT to the increasing sequences $\{\nu((E_j)_x) \chi_{X_j}\}$ and $\{\mu((E_j)^y) \chi_{Y_j}\}$, that converge to $\nu(E_x)$ and $\mu(E^y)$, resp., and obtain that
\[
(\mu \times \nu)(E) = \lim_{j \to +\infty} (\mu \times \nu)(E_j)
\]
\[
= \lim_{j \to +\infty} \int_{\mathcal{X}} \nu((E_j)_x) \chi_{X_j}(x) \, d\mu = \int_{\mathcal{X}} \nu(E_x) \, d\mu
\]
\[
= \lim_{j \to +\infty} \int_{\mathcal{Y}} \mu((E_j)^y) \chi_{Y_j}(y) \, d\nu = \int_{\mathcal{Y}} \mu(E^y) \, d\nu.
\]
This completes the proof. □
Theorem 4.9. (The Tonelli–Fubini Theorem.) Let \((\mathcal{X}, \mathcal{M}, \mu), (\mathcal{Y}, \mathcal{N}, \nu)\) be \(\sigma\)-finite measure spaces.

(1) (Tonelli) Let \(f\) be a non-negative \(\mathcal{M} \times \mathcal{N}\)-measurable function on \(\mathcal{X} \times \mathcal{Y}\). Then, the functions \(g(y) = \int_{\mathcal{X}} F^y(x) \, d\mu(x)\) and \(h(x) = \int_{\mathcal{Y}} f_x(y) \, d\nu(y)\) are \(\mathcal{N}\)-measurable and \(\mathcal{M}\)-measurable, resp., and
\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d(\mu \times \nu)(x, y) = \int_{\mathcal{Y}} g(y) \, d\nu(y) = \int_{\mathcal{X}} h(x) \, d\mu(x).
\]  
(18)

Notice that (18) can also be written in terms of \textit{iterated integrals}
\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d(\mu \times \nu)(x, y) = \int_{\mathcal{Y}} \left( \int_{\mathcal{X}} f(x, y) \, d\mu(x) \right) \, d\nu(y) = \int_{\mathcal{X}} \left( \int_{\mathcal{Y}} f(x, y) \, d\nu(y) \right) \, d\mu(x).
\]

It is customary to use the notations
\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d(\mu \times \nu)(x, y) = \int_{\mathcal{X}} \int_{\mathcal{Y}} f(x, y) \, d\nu(y) \, d\mu(x) = \iint f \, d\mu \, d\nu.
\]

Proof. (1) Observe that Tonelli’s theorem reduces to Prop. 4.8 in the case of characteristic functions. By finite additivity of the integrals and measures, the result follows for non-negative simple functions. Let \(f \geq 0\) be \((\mathcal{M} \times \mathcal{N})\)-measurable. By Thm. 2.9 (1) there exists an increasing sequence \(\{s_n\}\) of non-negative simple functions, such that \(s_n \to f\) pointwise. Then, we can apply the MCT to obtain
\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d(\mu \times \nu)(x, y) = \lim_{n \to +\infty} \int_{\mathcal{X} \times \mathcal{Y}} s_n(x, y) \, d(\mu \times \nu)(x, y)
= \lim_{n \to +\infty} \int_{\mathcal{Y}} \left( \int_{\mathcal{X}} s_n(x, y) \, d\mu(x) \right) \, d\nu(y) = \int_{\mathcal{Y}} \left( \int_{\mathcal{X}} f(x, y) \, d\mu(x) \right) \, d\nu(y),
\]
(19)

and also
\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d(\mu \times \nu)(x, y) = \lim_{n \to +\infty} \int_{\mathcal{X} \times \mathcal{Y}} s_n(x, y) \, d(\mu \times \nu)(x, y)
= \lim_{n \to +\infty} \int_{\mathcal{X}} \left( \int_{\mathcal{Y}} s_n(x, y) \, d\nu(y) \right) \, d\mu(x) = \int_{\mathcal{X}} \left( \int_{\mathcal{Y}} f(x, y) \, d\nu(y) \right) \, d\mu(x).
\]
(20)

This proves (1).

In order to prove Fubini’s theorem, we proceed in an analogous fashion. For \(f \in L^1(\mu \times \nu)\), we Thm. 2.9 (2) to find a sequence \(\{\varphi_n\}\) of simple functions, such that \(0 \leq |\varphi_1| \leq |\varphi_n| \leq \cdots \leq |f|\), \(\varphi_n \to f\) pointwise. Then, we argue as in (19) and (20), using the DCT, instead of the MCT. This completes the proof of Tonelli–Fubini’s theorem.

Example 4.10. The hypotheses in the Tonelli–Fubini theorem cannot be relaxed, as the following examples show.

(1) Let \(f : (0, 1) \times (-1, 1) \to \mathbb{R}\) be given by \(f(x, y) = y/x\). Then, clearly
\[
\int_0^1 \left( \int_{-1}^1 \frac{y}{x} \, dy \right) \, dx = \int_0^1 0 \, dx = 0,
\]
holds.

\[ \int_{-1}^{1} \left( \int_{0}^{1} \frac{y}{x} \, dy \right) \, dx = \int_{-1}^{1} y \left( \int_{0}^{1} \frac{1}{x} \, dx \right) \, dy = \int_{-1}^{1} g(y) \, dy \]

where \( g(y) = +\infty \) if \( 0 < y < 1 \) and \( g(y) = -\infty \) if \( -1 < y < 0 \). This last integral does not exist and the iterated integrals are not equal.

(2) Let \( T_1 = \{(x, y) : 0 < x < 1, 0 < y < x\} \), \( T_2 = \{(x, y) : 0 < y < 1, 0 < x < y\} \), and define \( f : (0, 1) \times (0, 1) \to \mathbb{R} \) be given \( f(x, y) = \frac{1}{xy} \chi_{T_1}(x, y) - \frac{1}{xy} \chi_{T_2}(x, y) \). Then,

\[
\int_{0}^{1} \left( \int_{0}^{1} f(x, y) \, dy \right) \, dx = \int_{0}^{1} \left( \int_{0}^{x} \frac{1}{xy} \, dy \right) - \int_{x}^{1} \frac{1}{xy^2} \, dy \, dx \\
= \int_{0}^{1} \left( \frac{1}{x} + \left( \frac{1}{y} \right)^1 \right) \, dy = \int_{0}^{1} \left( \frac{1}{x} + 1 - \frac{1}{x} \right) \, dy \\
= 1 .
\]

On the other hand,

\[
\int_{0}^{1} \left( \int_{0}^{1} f(x, y) \, dx \right) \, dy = \int_{0}^{1} \left( - \int_{0}^{y} \frac{1}{y^2} \, dx + \int_{y}^{1} \frac{1}{xy^2} \, dx \right) \, dy \\
= \int_{0}^{1} \left( - \frac{1}{y} + \left( - \frac{1}{x} \right)^1 \right) \, dy = \int_{0}^{1} \left( - \frac{1}{y} - 1 + \frac{1}{y} \right) \, dy \\
= -1 .
\]

Thus, again the iterated integrals are not equal.

**Remark 4.11.** The measure \( \mu \times \nu \) is not complete, even if \((\mu, \mathcal{M})\) and \((\nu, \mathcal{N})\) are complete measures. As an example consider \((\mathcal{X}, \mu, \mathcal{M}) = (\mathbb{R}, m, \mathcal{L})\), the Lebesgue measure on \( \mathbb{R} \). Let \( A \) be any set of measure 0 and \( B \) the example of a non-measurable set constructed in Example 1.1. Then \((A \times B) \notin \mathcal{L} \times \mathcal{L}\), but it is contained in a set of \((m \times m)\)-measure 0.

This is not a significant drawback, since we can always complete the measure \((\mu \times \nu)\). However, the Tonelli–Fubini theorem takes a slightly more complicated form in the case of complete measures.

**Theorem 4.12. (The Tonelli–Fubini Theorem for complete measures)** Let \((\mathcal{X}, \mathcal{M}, \mu)\), \((\mathcal{Y}, \mathcal{N}, \nu)\) be complete \(\sigma\)-finite measure spaces. Let \((\mathcal{X} \times \mathcal{Y}, \mathcal{T}, \tau)\) be the completion of the product measure space \((\mathcal{X} \times \mathcal{Y}, \mathcal{M} \times \mathcal{N}, \mu \times \nu)\) and \(f\) be \(\mathcal{T}\)-measurable.

1-Tonelli If \( f \geq 0 \) then \( f_x \) is \(\mathcal{N}\)-measurable for \(\mu\)-a.e. \( x \in \mathcal{X} \) and \( f^y \) is \(\mathcal{M}\)-measurable for \(\nu\)-a.e. \( y \in \mathcal{Y} \) and

\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d\tau(x, y) = \int_{\mathcal{Y}} \left( \int_{\mathcal{X}} f(x, y) \, d\mu(x) \right) \, d\nu(y) = \int_{\mathcal{X}} \left( \int_{\mathcal{Y}} f(x, y) \, d\nu(y) \right) \, d\mu(x) . \quad (21)
\]

2-Fubini If \( f \in L^1(\tau) \) then \( f_x \) is \(\mathcal{N}\)-measurable for \(\mu\)-a.e. \( x \in \mathcal{X} \) and \( f^y \) is \(\mathcal{M}\)-measurable for \(\nu\)-a.e. \( y \in \mathcal{Y} \). Moreover \( f_x \in L^1(\nu) \) for \(\mu\)-a.e. \( x \in \mathcal{X} \), \( f^y \in L^1(\mu) \) for \(\nu\)-a.e. \( y \in \mathcal{Y} \) and \((21)\) holds.
Proof. (i) We sketch the proof, leaving the details to the reader.

Let \( f \) be \( T \)-measurable. Since \((T, \tau)\) is the completion of \((\mathcal{M} \times \mathcal{N}, \mu \times \nu)\), by Prop. 2.12 it follows that there exists an \((\mathcal{M} \times \mathcal{N})\)-measurable function \( g \) such that \( g = f \) \( \tau \)-a.e. Let \( h = f - g \). We can apply Tonelli–Fubini theorem to \( h \) in either case \( f \in L^+(\tau) \) or \( f \in L^1(\tau) \) and obtain that

\[
\int_{\mathcal{X} \times \mathcal{Y}} g(x, y) \, d\tau(x, y) = \int_{\mathcal{X} \times \mathcal{Y}} g(x, y) \, d(\mu \times \nu)(x, y) = \int_{\mathcal{X}} \left( \int_{\mathcal{Y}} g(x, y) \, d\mu(x) \right) \, d\nu(y) = \int_{\mathcal{X}} \left( \int_{\mathcal{Y}} g(x, y) \, d\nu(y) \right) \, d\mu(x).
\]

Hence, it suffices to show that if \( h \) is \( T \)-measurable and \( h = 0 \) \( \tau \)-a.e., then \( h_x, h_y \) are in \( L^1(\nu) \) and in \( L^1(\mu) \), hence measurable, resp., and

\[
\int_{\mathcal{Y}} h_x \, d\nu = \int_{\mathcal{X}} h^y \, d\mu = 0 \tag{22}
\]

for \( \mu \)-a.e. \( x \), and \( \nu \)-a.e. \( y \), resp. Indeed, using the observation at the end of Section 2, (22) would imply that

\[
\int_{\mathcal{X} \times \mathcal{Y}} (f(x, y) - g(x, y)) \, d\tau(x, y) = \int_{\mathcal{X} \times \mathcal{Y}} h(x, y) \, d\tau(x, y) = \int_{\mathcal{X} \times \mathcal{Y}} h(x, y) \, d(\mu \times \nu)(x, y) = \int_{\mathcal{Y}} \left( \int_{\mathcal{X}} h(x, y) \, d\mu(x) \right) \, d\nu(y) = \int_{\mathcal{X}} \left( \int_{\mathcal{Y}} h(x, y) \, d\nu(y) \right) \, d\mu(x) = 0.
\]

Thus, we prove (22). Since \( h = 0 \) \( \tau \)-a.e., there exists \( E \in \mathcal{M} \times \mathcal{N} \) such that \((\mu \times \nu)(E) = 0 \) and \( h = 0 \) on \(^cE \). We then have

\[
\int_{\mathcal{X} \times \mathcal{Y}} h(x, y) \, d\tau = \int_{\mathcal{X} \times \mathcal{Y}} h(x, y) \, d(\mu \times \nu) = \int_{E} h(x, y) \, d(\mu \times \nu).
\]

Since \((\mu \times \nu)(E) = 0 \), we have

\[
0 = \int_{\mathcal{Y}} \mu(E^y) \, d\nu = \int_{\mathcal{X}} \nu(E_x) \, d\mu
\]

which imply that

\[
\mu(E^y) = 0 \quad \nu \text{ – a.e.} \quad \text{and} \quad \nu(E_x) = 0 \quad \mu \text{ – a.e.} \ . \tag{23}
\]

Then, there exists a sequence of simple functions \( \tau \)-measurable \( \{\varphi_n\} \) such that \( 0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |h| \) and \( \varphi_n \to h \). Notice that \( \varphi_{nx} \to h_x \) and \( \varphi_{ny} \to h^y \), for all \( x, y \).

Observe also that, clearly, \( \varphi_n = 0 \) on \(^cE \), so that \( \varphi_{nx} = 0 \) on \(^c(E_x) \) and \( \varphi_{ny} = 0 \) on \(^c(E^y) \), for all \( x, y \). Thus, by (23), \( \varphi_{nx} = \chi_{E_x} \mu \text{-a.e.} \) and \( \varphi_{ny} = \chi_{E^y} \nu \text{-a.e.} \). Since \( \chi_{E_x} \) and \( \chi_{E^y} \) are \( \mathcal{N} \) and \( \mathcal{M} \)-measurable, resp., and \( \mu \) and \( \nu \) are complete, by Prop. ?? it follows that also \( \varphi_{nx} \) and \( \varphi_{ny} \) are \( \mathcal{N} \) and \( \mathcal{M} \)-measurable, resp. This easily implies that \( h_x \in L^1(\nu) \) and \( h^y \in L^1(\mu) \) for \( \mu \text{-a.e.} \) \( x \) and \( \nu \text{-a.e.} \) \( y \) and thus (22), and hence the theorem, follow. \( \Box \)
4.3. The Lebesgue integral in $\mathbb{R}^n$. Clearly, the results of the previous section on product measure spaces can be extended to the case of cartesian products of any finite number of measure spaces. We assume the validity of all these results, without formally state any of them.

We define the Lebesgue measure in $m_n$ on $\mathbb{R}^n$ as the completion of $m \times \cdots \times m$ ($n$ copies of $m$) on $\mathcal{B}_\mathbb{R} \times \cdots \times \mathcal{B}_\mathbb{R}$, or, equivalently, on $\mathcal{L} \times \cdots \times \mathcal{L}$. We denote by $\mathcal{L}_n$ the $\sigma$-algebra domain of $m_n$, and call these sets \textit{Lebesgue measurable sets} in $\mathbb{R}^n$.

In this section we prove some properties of $m_n$ that extend similar properties valid in the case $n = 1$. If $R = R_1 \times \cdots \times R_n \subseteq \mathbb{R}^n$ is a rectangle, we call $R_j$ its \textit{sides}, $j = 1, \ldots, n$.

**Theorem 4.13.** Let $E \in \mathcal{L}_n$. Then, the following properties hold true.

(i) $m_n(E) = \inf \{m_n(U) : E \subseteq U, \text{ U open}\} = \sup \{m_n(K) : K \subseteq E, \text{ K compact}\}$;

(ii) $E = A_1 \setminus N_1$, where $A_1$ is a $\mathcal{G}_n$-set and $m_n(N_1) = 0$, and also $E = A_2 \cup N_2$, where $A_2$ is an $\mathcal{F}_n$-set and $m_n(N_2) = 0$;

(iii) if $m_n(E) < +\infty$, then for every $\varepsilon > 0$ there exists a finite collection of rectangles $\{R_j\}$, $j = 1, \ldots, N$, whose sides are intervals and such that $m_n(E \triangle \cup_{j=1}^N R_j) < \varepsilon$.

**Proof.** (i) By construction of the product measure, given $E \in \mathcal{L}_n$ and $\varepsilon > 0$, there exists a collection $\{R_j\}$ of rectangles such that $E \subseteq \cup_{j=1}^N R_j$ and $\sum_{j=1}^N m_n(R_j) \leq m_n(E) + \varepsilon$. In order to prove the first equality, notice that we may assume that $m_n(E) < +\infty$, since otherwise we have nothing to prove. Then, $\sum_{j=1}^N m_n(R_j)$ is a convergent series, so that there exists $M > 0$ such that $m_n(R_j) \leq M$ for all $j$.

For each $j$ fixed, $R_j = R_j^{(1)} \times \cdots \times R_j^{(n)}$, with $R_j^{(k)} \in \mathcal{L}$, $k = 1, \ldots, n$. By Prop. 3.11, for each $j$ fixed, we can find open sets $U_j^{(k)}$ such that $R_j^{(k)} \subseteq U_j^{(k)}$ and $m(U_j^{(k)}) < m(R_j^{(k)}) + 2^{-j}\varepsilon/(NM^{n-1})$, where $N$ is a sufficiently large fixed positive integer. Then

$$R \subseteq U := \bigcup_{j=1}^{+\infty} \left( U_j^{(1)} \times \cdots \times U_j^{(n)} \right).$$

Notice that the $U_j$ are open rectangles. Hence $U$ is open, and

$$m_n(U) \leq \sum_{j=1}^{+\infty} m_n(U_j) = \sum_{j=1}^{+\infty} \prod_{k=1}^n m(U_j^{(k)}) \leq \sum_{j=1}^{+\infty} \prod_{k=1}^n \left( m(R_j^{(k)}) + \varepsilon \right)$$

$$\leq \sum_{j=1}^{+\infty} \left( m_n(R_j) + \varepsilon 2^{-j} \right)$$

$$\leq m_n(E) + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the first equality in (i) is proved. In order to prove the second equality, we proceed exactly as in the proof of (ii) in Prop. 3.11.

The proofs of (ii), which exactly the same as the ones in Prop. 3.14, is again left as an exercise.

(iii) Let $m_n(E) < +\infty$, and let $\varepsilon > 0$ be given. We can find open rectangles $\{U_j\}$ such that $\sum_{j=1}^{+\infty} m_n(U_j) \leq m_n(E) + \varepsilon/2$. Hence, $m_n(U_j) < +\infty$ for all $j$. For each $j$, the sides of $U_j$ are countable unions of disjoint open intervals, say $U_j^{(\ell)} = \bigcup_{k_j=1}^{+\infty} I_j^{(\ell)}$, $j = 1, \ldots, n$. Thus, for each $j$
fixed,
\[ U_j = \left( \bigcup_{k_1=1}^{+\infty} I_{j,k_1}^{(1)} \right) \times \cdots \times \left( \bigcup_{k_n=1}^{+\infty} I_{j,k_n}^{(n)} \right) = \bigcup_{k_1,\ldots,k_n=1}^{+\infty} I_{j,k_1}^{(1)} \times \cdots \times I_{j,k_n}^{(n)}. \]

Hence, for each \( j \), we can find a finite subunion \( V_j \) of the \( I_{j,k_1}^{(1)} \times \cdots \times I_{j,k_n}^{(n)} \)'s such that
\[ m_n(U_j) \leq m_n(V_j) + 2^{-j} \varepsilon / 2. \]

Notice that the sides of the \( V_j \)'s are finite unions of intervals. Then, if \( N \) is large enough so that \( \sum_{j=N+1}^{+\infty} m_n(U_j) < \varepsilon / 2 \) we have
\[
m_n \left( E \setminus \bigcup_{j=1}^{N} V_j \right) \leq m_n \left( \bigcup_{j=1}^{N} (U_j \setminus V_j) \cup \bigcup_{j=N+1}^{+\infty} U_j \right) \\
\leq m_n \left( \bigcup_{j=1}^{N} (U_j \setminus V_j) \right) + m_n \left( \bigcup_{j=N+1}^{+\infty} U_j \right) \\
< \varepsilon.
\]

On the other hand,
\[
m_n \left( \bigcup_{j=1}^{N} V_j \setminus E \right) \leq m_n \left( \bigcup_{j=1}^{+\infty} U_j \setminus E \right) \leq \sum_{j=1}^{+\infty} m_n(U_j) - m_n(E) < \varepsilon.
\]

This proves (iii), and we are done. \( \square \)

Next, we prove that \( m_n \) is also translation invariant.

**Proposition 4.14.** Let \( E \in \mathcal{L}_n \), and \( f \in L^+(m_n) \) or \( E \in L^1(m_n) \). For \( a \in \mathbb{R} \) define \( E + a = \{ x + a \mid x \in E \} \in \mathcal{L}_n \) and \( f_a = f(\cdot + a) \). Then, \( m_n(E + a) = m_n(E) \) and \( \int f_a \, dm_n = \int f \, dm_n \).

**Proof.** By composition of translations along each each axis, it suffices to consider the case of translations along one axis, that is, \( a = (0, \ldots, a_j, 0, \ldots, 0) \). Then, if \( E = E_1 \times \cdots \times E_n \) is a rectangle, then \( m_n(E + a) = m(E_1) \cdots m(E_j + a_j) \cdots m(E_n) = m_n(E) \), by the 1-dimensional result Prop. 3.13. Then, the result is true for the elements of algebra \( \mathcal{A} \) of finite unions of disjoints measurable rectangles. By the construction of the product measure, the invariance now follows. Notice also that in particular the class of sets of measure zero is invariant under translations.

Next, let \( f \) be Lebesgue measurable and \( f_a = f(\cdot + a) \). Let \( B \) be a Borel set in \( \mathbb{R} \) (or in \( \mathbb{C} \), if \( f \) is complex-valued). Then, \( f^{-1}(B) \) is a Lebesgue measurable set, hence \( f^{-1}(B) = E \cup N \), where \( E \) is a Borel set in \( \mathbb{R}^n \) and \( N \) is contained in a Borel set of measure 0. Then, denoting by \( \tau_a \) the traslation by \( a \in \mathbb{R}^n \), so that \( f(\cdot + a) = f \circ \tau_a \),
\[
(f_a)^{-1}(B) = \tau_a^{-1}(f^{-1}(B)) = \tau_a^{-1}(E \cup N) = (E - a) + (N - a),
\]
, that is, \( (f_a)^{-1}(B) \) is union of a Borel set and of a set of measure 0. Hence, it is Lebesgue measurable, and \( f_a \) is measurable.

Finally, the identity \( \int f_a \, dm_n = \int f \, dm_n \) holds true when \( f = \chi_E \), and \( E \) is a measurable set. Hence, it holds true for simple functions, and by the MCT and the DCT it holds true for \( f \in L^+(m_n) \) and \( f \in L^1(m_n) \), resp. \( \square \)