AN INTRODUCTION TO MEASURE THEORY AND THE LEBESGUE INTEGRAL

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A † denotes the proofs and the parts that are not strictly required for the exam for the a.a. 2016/17.
In these notes we present a concise introduction to abstract measure theory and to the
Lebesgue integral in euclidean spaces. These notes should be considered only as a support
for the preparation for the exam, and not as detailed introduction to the subject.

1. First elements of measure theory

We would like to introduce a notion of measure as a function \( \mu \) that assigns to every subset
\( E \) of \( \mathbb{R}^n \) a value \( \mu(E) \in [0, +\infty] \) in such a way the following conditions are satisfied:

(i) if \( E_1, \ldots, E_k \ldots \) are disjoint subsets, then \( \mu(\bigcup_k E_k) = \sum_k \mu(E_k) \);

(ii) if \( F \) is congruent to \( E \), that is, obtained from \( E \) by a translation, rotation or reflection,
then \( \mu(F) = \mu(E) \);

(iii) \( \mu(Q) = 1 \), where \( Q = [0, 1]^n \) denotes the unit cube.

Unfortunately, these three conditions are mutually incompatible, as the next example shows.

Example 1.1. Assume that \( n = 1 \). Let us introduce an equivalent relation \( \sim \) in \([0, 1)\) by setting
\( x \sim y \) if \( x - y \) is rational. Let \( N \) be a subset of \([0, 1)\) containing exactly one element for each
equivalent class.\(^1\) Let \( R = \mathbb{Q} \cap [0, 1) \) and for \( r \in R \) define
\[
N_r = \left\{ x + r : x \in N \cap [0, 1 - r) \right\} \cup \left\{ x + r : -1 x \in N \cap [1 - r, 1) \right\} .
\]

Then \( N_r \subseteq [0, 1) \) and each \( x \in [0, 1) \) belongs to exactly one \( N_r \). Indeed, let \( y \in N \) be the element
that is in the same equivalent class as \( x \). If \( x \geq y \), then \( x \in N_r \), where \( x - y = r \in R \), while
if \( x < y \), then \( x \in N_y \), where \( x - y + 1 = r \in R \). Thus, \( x \in N_r \) for some \( r \in R \). On the other
hand, if \( x \in N_r \cap N_s \), then we would have \( x - r \) (or \( x - r + 1 \)) and \( x - s \) (or \( x - s + 1 \)) as distinct elements in \( N \) but belonging to the same equivalent class, against our choice of \( N \).

Suppose then that \( \mu : \mathcal{P}(\mathbb{R}) \to [0, +\infty] \) satisfy (i)-(iii). By (i) and (ii) we have that
\[
\mu\left( \left\{ x + r : x \in N \cap [0, 1 - r) \right\} \right) + \mu\left( \left\{ x + r : -1 x \in N \cap [1 - r, 1) \right\} \right) = \mu(N_r)
\]
for every \( r \in R \). Moreover, since \([0, 1)\) is the disjoint union of the \( N_r \) and these sets are countably
many, by (ii) and (iii) we have
\[
1 = \mu([0, 1)) = \mu\left( \bigcup_{r \in R} N_r \right) = \sum_{r \in R} \mu(N_r) = \sum_{r \in R} \mu(N) .
\]

But this is impossible, since the right-hand-side equals \( +\infty \) is \( \mu(N) > 0 \), or it equals 0 if
\( \mu(N) = 0 \).

This example easily generalizes to the case \( n > 1 \).

1.1. Measure spaces. Led by the previous example, we try to define such function \( \mu \) on a
domain which is strictly contained in \( \mathcal{P}(\mathbb{R}^n) \) but still satisfying (i)-(iii). To this end we introduce
the following definitions.

Definition 1.2. Given a set \( \mathcal{X} \) we call algebra a non-empty collection \( \mathcal{A} \) of subsets of \( \mathcal{X} \) that
is closed under finite unions and complements, that is, if the following conditions are satisfied:

(i) if \( E_1, \ldots, E_m \in \mathcal{A} \), then \( \bigcup_{k=1}^m E_k \in \mathcal{A} \);

(ii) if \( E \in \mathcal{A} \), then \( ^c E \in \mathcal{A} \).

\(^1\)To obtain this, we need to assume the validity of the axiom of choice.
Notice that, if \( E \in \mathcal{A} \), \( X = E \cup cE \in \mathcal{A} \), so that also \( \emptyset = cX \in \mathcal{A} \).

A non-empty collection \( \mathcal{A} \) of subsets of \( X \) is called a \( \sigma \)-algebra if it is an algebra and it is closed under countable unions. In other words, if the following conditions are satisfied:

(i') if \( E_1, E_2, \ldots \in \mathcal{A} \), then \( \bigcup_{k=1}^{+\infty} E_k \in \mathcal{A} \);

(ii') if \( E \in \mathcal{A} \), then \( cE \in \mathcal{A} \).

Notice that a \( \sigma \)-algebra \( \mathcal{A} \) is closed under countable intersections, since \( \bigcap_{j=1}^{+\infty} E_j = c\left( \bigcup_{j=1}^{+\infty} cE_j \right) \in \mathcal{A} \).

We observe that an algebra \( \mathcal{A} \) that is closed under countable unions of disjoint subsets of \( X \), is a \( \sigma \)-algebra. Indeed, given a sequence \( \{E_k\} \) of elements in \( \mathcal{A} \), define

\[
F_1 = E_1, \quad F_k = E_k \setminus \left( \bigcup_{j=1}^{k-1} E_j \right) = E_k \cap c\left( \bigcup_{j=1}^{k-1} E_j \right).
\]

Then the \( F_k \) are disjoint and \( \bigcup_{k=1}^{+\infty} E_k = \bigcup_{k=1}^{+\infty} F_k \), which belongs to \( \mathcal{A} \) by assumption.

**Example 1.3.** Let \( X \) be any set.

1. \( \mathcal{P}(\mathcal{X}) \) is a \( \sigma \)-algebra;
2. if \( \{\emptyset, \mathcal{X}\} \) is a \( \sigma \)-algebra;
3. if \( \mathcal{X} \) is uncountable and we set \( \mathcal{A} = \{ E : E \text{ is countable, or } cE \text{ is countable} \} \), then \( \mathcal{A} \) is a \( \sigma \)-algebra.

It is easy to see that the intersection of \( \sigma \)-algebras is again a \( \sigma \)-algebra. Therefore, the following definition makes sense.

**Definition 1.4.** Given any subset \( \mathcal{E} \) of \( \mathcal{P}(\mathcal{X}) \), we call \( \mathcal{M}(\mathcal{E}) \) the \( \sigma \)-algebra generated by \( \mathcal{E} \) as the smallest \( \sigma \)-algebra containing \( \mathcal{E} \), that is, the intersection of all \( \sigma \)-algebras containing \( \mathcal{E} \).

Observe that \( \mathcal{P}(\mathcal{X}) \) is a \( \sigma \)-algebra containing \( \mathcal{E} \), so the above intersection is not empty. The following lemma is elementary, but it deserves its own statement.

**Lemma 1.5.** If \( \mathcal{A} \) is a \( \sigma \)-algebra, and \( \mathcal{E} \subseteq \mathcal{M}(\mathcal{F}) \subseteq \mathcal{A} \), then \( \mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F}) \).

**Proof.** Since \( \mathcal{M}(\mathcal{F}) \) is a \( \sigma \)-algebra containing \( \mathcal{E} \), it contains the smallest \( \sigma \)-algebra containing \( \mathcal{E} \), i.e. \( \mathcal{M}(\mathcal{E}) \). \( \square \)

The above definition leads us to the following fundamental notion.

**Definition 1.6.** If \( \mathcal{X} \) is a topological space, we call the Borel \( \sigma \)-algebra in \( \mathcal{X} \), and we denote it by \( \mathcal{B}_{\mathcal{X}} \), the \( \sigma \)-algebra generated by the collections of open sets in \( \mathcal{X} \), and a set \( E \in \mathcal{B}_{\mathcal{X}} \) a Borel set in \( \mathcal{X} \).

Observe that if \( F \) is a closed set, then \( F \in \mathcal{B}_{\mathcal{X}} \), so are countable unions of closed sets, their complements, countable unions of such sets, etc.

We call a countable intersection of open sets a \( G_\delta \)-set, and countable union of closed sets an \( F_\sigma \)-set.

If \( \mathcal{X} = \mathbb{R} \) is endowed with its natural topology, we denote by \( \mathcal{B}_\mathbb{R} \) the \( \sigma \)-algebra of Borel sets in \( \mathbb{R} \) (with respect to this topology). It is going to play a fundamental role in remainder of these notes.

The following are elementary consequences of the definition.

\(^2\)In this case we say that \( E \) is co-countable.
Proposition 1.7. The σ-algebra $B_\mathbb{R}$ is generated by each of the following sets:

1. the collection of open intervals $\mathcal{E}_1 = \{(a,b) : a < b\}$;
2. the collection of closed intervals $\mathcal{E}_2 = \{[a,b] : a < b\}$;
3. the collection of half-open intervals $\mathcal{E}_3 = \{(a,b] : a < b\}$, or $\mathcal{E}_3' = \{[a,b) : a < b\}$;
4. the collection of open rays $\mathcal{E}_4 = \{(-\infty, b) : b \in \mathbb{R}\}$, or $\mathcal{E}_4' = \{(a, +\infty) : a \in \mathbb{R}\}$;
5. the collection of closed rays $\mathcal{E}_5 = \{(-\infty, b] : b \in \mathbb{R}\}$, or $\mathcal{E}_5' = \{[a, +\infty) : a \in \mathbb{R}\}$.

Proof. The elements of the $\mathcal{E}_j$ and $\mathcal{E}_j'$ are all open, or closed, or intersection of open and closed sets, so that each of the σ-algebras in (1)-(5) is contained in $B_\mathbb{R}$. Conversely, it is clear that $\mathcal{M}(\mathcal{E}_1) \supseteq B_\mathbb{R}$ since each open set is countable union of elements of $\mathcal{E}_1$. In all remaining cases, it suffices to show that open sets are contained in $\mathcal{M}(\mathcal{E}_j)$ or $\mathcal{M}(\mathcal{E}_j')$. By symmetry, we consider only the cases of $\mathcal{M}(\mathcal{E}_j)$, $j = 1, \ldots, 5$. Clearly, $(a, b) = \bigcup_{k=1}^{+\infty} \left[a - \frac{b-a}{2^k}, b - \frac{b-a}{2^k}\right]$, so (2) follows. Next, $(a, b) = \bigcup_{k \in \mathbb{N}} (a, b - \frac{b-a}{2^k}]$, where $\mathbb{N}$ is chosen so that $a < b - \frac{b-a}{2^k}$. This shows (3). To prove (4) we consider $[b, +\infty) = \bigcup_{j=1}^{+\infty} (-\infty, b)$ and $(-\infty, a) \cap [b, +\infty) = [b, a)$, if $b < a$. Thus, (4) follows from the case (3) (for $\mathcal{E}_4'$). Case (5) is analogous and left to the reader. □

A similar description can be given for the σ-algebra of Borel sets in $\mathbb{R}^n$. If $\mathcal{X}_1, \ldots, \mathcal{X}_n$ is a collection of non-empty sets, $\mathcal{X} = \prod_{j=1}^{n} \mathcal{X}_j$, we denote by $\pi_j : \mathcal{X} \to \mathcal{X}_j$ the coordinate maps. If $\mathcal{M}_j$ is a σ-algebra on $\mathcal{X}_j$, $j = 1, \ldots, n$, we set

$$\bigotimes_{j=1}^{n} \mathcal{M}_j = \mathcal{M}\left(\{\pi_j^{-1}(E_j) : E_j \in \mathcal{M}_j, j = 1, \ldots, n\}\right)$$

that is, the σ-algebra generated by $\pi_j^{-1}(E_j)$ where $E_j \in \mathcal{M}_j$, $j = 1, \ldots, n$.

Proposition 1.8. We have that

$$B_{\mathbb{R}^n} = \bigotimes_{j=1}^{n} B_\mathbb{R}.$$

Proof. (†) Let $E_j \in B_\mathbb{R}$, $j = 1, \ldots, n$. Then $\pi_1^{-1}(E_1) = E_1 \times \mathbb{R} \times \cdots \mathbb{R}$, and similarly for $\pi_j^{-1}(E_j)$. Then $\bigotimes_{j=1}^{n} E_j = \cap_{j=1}^{n} \pi_j^{-1}(E_j)$. Hence, $\bigotimes_{j=1}^{n} B_\mathbb{R}$ contains the σ-algebra generated by $\{\prod_{j=1}^{n} E_j : E_j \text{ open in } \mathbb{R} \} = \mathcal{F}$, i.e. $\bigotimes_{j=1}^{n} B_\mathbb{R} \supseteq \mathcal{M}(\mathcal{F})$. On the other hand, for each $j = 1, \ldots, n$, the set $\{E_j \subseteq \mathbb{R} : \pi_j^{-1}(E_j) \in \mathcal{M}(\mathcal{F})\}$ is a σ-algebra, that contains $B_\mathbb{R}$. Hence,

$$\bigotimes_{j=1}^{n} B_\mathbb{R} = \mathcal{M}\left(\{\prod_{j=1}^{n} E_j : E_j \text{ open in } \mathbb{R} \}\right).$$

This implies that $\bigotimes_{j=1}^{n} B_\mathbb{R} \subseteq B_{\mathbb{R}^n}$.

To see the reverse inclusion, let $E$ be any open set in $\mathbb{R}^n$ and consider its points with rational coordinates, which is a dense subset in $E$. Take a ngbh, contained in $E$, of any such point that is a cartesian product of open sets in $\mathbb{R}$ in each coordinate. Then $E$ is union of open sets that are of the form $\prod_{j=1}^{n} E_j$ with $E_j$ open in $\mathbb{R}$. Thus,

$$B_{\mathbb{R}^n} \subseteq \mathcal{M}\left(\{\prod_{j=1}^{n} E_j : E_j \text{ open in } \mathbb{R} \}\right) = \bigotimes_{j=1}^{n} B_\mathbb{R}.$$

This proves the proposition. □
1.2. Measures. Let $\mathcal{X}$ be a set and $\mathcal{M}$ a $\sigma$-algebra in $\mathcal{P}(\mathcal{X})$. The pair $(\mathcal{X}, \mathcal{M})$ is called a measurable space.

**Definition 1.9.** Given a measurable space $(\mathcal{X}, \mathcal{M})$, a function $\mu : \mathcal{M} \to [0, +\infty]$ is called a measure on $\mathcal{X}$ if

(i) $\mu(\emptyset) = 0$;

(ii) if $E_1, E_2, \ldots \in \mathcal{M}$, are disjoint, then $\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \sum_{j=1}^{+\infty} \mu(E_j)$.

The triple $(\mathcal{X}, \mathcal{M}, \mu)$ is called a measure space.

Property (ii) is called countable additivity, in contrast with an analogous property

(ii') if $E_1, E_2, \ldots, E_m \in \mathcal{M}$, are disjoint, then $\mu\left(\bigcup_{j=1}^{m} E_j\right) = \sum_{j=1}^{m} \mu(E_j)$,

which will be called finite additivity. Notice that a measure $\mu$ satisfies also (ii') since we may take $E_k = \emptyset$ for all $k \geq 1$.

Given a measure space $(\mathcal{X}, \mathcal{M}, \mu)$, $\mu$ is said to be finite if $\mu(\mathcal{X}) < +\infty$ and is said to be $\sigma$-finite if there exists a collection of sets $\{E_j\}$, with $E_j \in \mathcal{M}$, $\bigcup_{j=1}^{+\infty} E_j = \mathcal{X}$ and such that $\mu(E_j) < +\infty$. In other words, $\mu$ is said to be $\sigma$-finite if $\mathcal{X}$ is countable union of sets of finite measure. Our treatment will essentially concern only with $\sigma$-finite measures.

Here are some simple examples of measure spaces.

**Example 1.10.**

1. Let $\mathcal{X}$ be any set, and let $\mathcal{M} = \mathcal{P}(\mathcal{X})$. Define the counting measure on $\mathcal{X}$ by setting

$$\mu(E) = \begin{cases} +\infty & \text{if } E \text{ is infinite} \\ m & \text{if } E \text{ contains exactly } m \text{ elements} \\ 0 & \text{if } E = \emptyset. \end{cases}$$

Then $\mu$ is a measure. Notice that $\mu(\{x\}) = 1$ for all $x \in \mathcal{X}$, and that it is $\sigma$-finite if $\mathcal{X}$ is countable, and it is finite if $\mathcal{X}$ is finite.

2. A particular case of (1) is when $\mathcal{X}$ is countable, or more in particular, equals $\mathbb{N}$ or $\mathbb{Z}$.

3. Instead, a generalization of (1) is the following. Let $f : \mathcal{X} \to [0, +\infty]$ be given and define

$$\mu(E) = \sum_{x \in E} f(x).$$

Since $f(x)$ is non-negative, there is no ambiguity in the above definition even if $E$ is uncountable. For, if there exist uncountably many $x \in E$ such that $f(x) > 0$, then there exists $n \in \mathbb{N}$ for which $E_n = \{x \in E : f(x) > 1/n\}$ is infinite. Then, $\sum_{x \in E} f(x) \geq \sum_{x \in E_n} f(x) = +\infty$. If there exist at most countably many $x \in E$ such that $f(x) > 0$, then the sum becomes a series, and the notion of convergence is the standard one. It is easy to check again that such $\mu$ is a measure.

A particular case is when $f$ is the function that equals 1 at a given $x_0 \in \mathcal{X}$ and it is 0 anywhere else. Such measure, is called the Dirac delta at $x_0$.

4. if $\mathcal{X}$ is uncountable, let $\mathcal{A}$ be the $\sigma$-algebra of countable or co-countable sets. Define

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{if } E \text{ is co-countable}. \end{cases}$$
Again, it is easy to check that such $\mu$ is a measure on $\mathcal{A}$.

The first elementary properties of measures are given in the next result.

**Proposition 1.11.** Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space. Then, the following properties hold true.

(i) **Monotonicity:** If $E, F \in \mathcal{M}$ with $E \subseteq F$, then $\mu(E) \leq \mu(F)$.

(ii) **Subadditivity:** If $\{E_j\} \subseteq \mathcal{M}$, then

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j \right) \leq \sum_{j=1}^{+\infty} \mu(E_j).$$

(iii) **Continuity from below:** If $\{E_j\} \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$, then

$$\lim_{j \to +\infty} \mu(E_j) = \mu\left(\bigcup_{j=1}^{+\infty} E_j \right).$$

(iv) **Continuity from above:** If $\{E_j\} \subseteq \mathcal{M}$ and $E_1 \supseteq E_2 \supseteq \cdots$ and $\mu(E_1) < +\infty$ then

$$\lim_{j \to +\infty} \mu(E_j) = \mu\left(\bigcap_{j=1}^{+\infty} E_j \right).$$

**Proof.** (i) Since $F \supseteq E$, we have that

$$\mu(F) = \mu\left((F \setminus E) \cup (F \cap E)\right) = \mu\left((F \setminus E) \cup E\right) = \mu(F \setminus E) + \mu(E) \geq \mu(E).$$

(ii) Given $\{E_j\}$, we define $\{F_j\}$ as in (1), that is, we set

$$F_1 = E_1, \quad F_k = E_k \setminus \left(\bigcup_{j=1}^{k-1} E_j\right) \text{ for } k \geq 2.$$

Since the $\{F_j\}$ are disjoint and $\bigcup_{j=1}^{+\infty} F_j = \bigcup_{j=1}^{+\infty} E_j$, by (i) we have that

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \mu\left(\bigcup_{j=1}^{+\infty} F_j\right) = \sum_{j=1}^{+\infty} \mu(F_j) \leq \sum_{j=1}^{+\infty} \mu(E_j).$$

(iii) Let now $\{E_j\} \subseteq \mathcal{M}$ and $E_1 \subseteq E_2 \subseteq \cdots$. Setting $E_0 := \emptyset$, we have that the sets $E_j \setminus E_{j-1}$ are disjoint and $\bigcup_{j=1}^{+\infty} E_j \setminus E_{j-1} = \bigcup_{j=1}^{+\infty} E_j$. Then,

$$\mu\left(\bigcup_{j=1}^{+\infty} E_j\right) = \sum_{j=0}^{+\infty} \mu\left(E_j \setminus E_{j-1}\right) = \lim_{n \to +\infty} \sum_{j=0}^{n} \mu\left(E_j \setminus E_{j-1}\right) = \lim_{n \to +\infty} \mu(E_n).$$

(iv) Set $F_j = E_1 \setminus E_j$. Then $F_1 \subseteq F_2 \subseteq \cdots$, and $\mu(E_1) = \mu(F_j) + \mu(E_j)$, since $E_j$ and $F_j$ are disjoint, and $\bigcup_{j=1}^{+\infty} F_j = E_1 \setminus \bigcap_{j=1}^{+\infty} E_j$. Finally, using (iii) we have

$$\mu(E_1) = \mu\left(\bigcap_{j=1}^{+\infty} E_j\right) + \lim_{j \to +\infty} \mu(F_j) = \mu\left(\bigcap_{j=1}^{+\infty} E_j\right) + \lim_{j \to +\infty} \mu(E_1) - \mu(E_j).$$

Since $\mu(E_1) < +\infty$, we can substract it from both side of the equation to obtain (iv). \qed
We remark that the assumption \( \mu(E_1) < +\infty \) could be replaced by \( \mu(E_n) < +\infty \) for some \( n \), but the finiteness of some \( \mu(E_n) \) is necessary. For, consider the case of \((\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)\), where \( \mu \) is the counting measure, and the sets \( E_j = \{ n : n \geq j \} \). Then, \( \mu(E_j) = +\infty \) for all \( j \) but \( \mu(\bigcap_{j=1}^{+\infty} E_j) = \mu(\emptyset) = 0 \).

Let \((\mathcal{X}, \mathcal{M}, \mu)\) be a measure space. A very important class of sets, is the class of sets of measure zero, also called null sets, that is, the sets \( E \) such that \( \mu(E) = 0 \). By countable subadditivity, a countable union of null sets is again a null set. By monotonicity, if \( E \in \mathcal{M} \), \( F \subseteq E \), \( F \in \mathcal{M} \) and \( \mu(E) = 0 \), then also \( \mu(F) = 0 \). But, in general, it is not true that \( F \in \mathcal{M} \).

**Definition 1.12.** A measure space \((\mathcal{X}, \mathcal{M}, \mu)\) is said to be complete if \( \mathcal{M} \) contains all subsets of null sets, that is, for every \( E \in \mathcal{M} \) with \( \mu(E) = 0 \) and \( F \subseteq E \), we have \( F \in \mathcal{M} \).

The next result shows that any measure space can be extended to a complete measure space.

**Theorem 1.13.** Let \((\mathcal{X}, \mathcal{M}, \mu)\) be a measure space and let \( \overline{\mathcal{M}} = \mathcal{M} \cup \mathcal{N} \), where \( \mathcal{N} = \{ E \in \mathcal{P}(\mathcal{X}) : \text{there exists } F \in \mathcal{M} \text{ with } \mu(F) = 0 \} \). If \( E \cup N \in \overline{\mathcal{M}} \), with \( E \in \mathcal{M} \) and \( N \in \mathcal{N} \), define

\[
\overline{\mu}(E \cup N) = \mu(E).
\]

Then \((\mathcal{X}, \overline{\mathcal{M}}, \overline{\mu})\) is a complete measure space, called the completion of \((\mathcal{X}, \mathcal{M}, \mu)\).

Observe that the completion of \((\mathcal{X}, \mathcal{M}, \mu)\) is uniquely determined, in the sense that once \( \overline{\mathcal{M}} \) is constructed, there exists a unique measure \( \overline{\mu} \) on \( \overline{\mathcal{M}} \) that is complete and restricted to \( \mathcal{M} \) coincides with \( \mu \).

**Proof.** We begin by showing that \( \overline{\mathcal{M}} \) is a \( \sigma \)-algebra. If \( \{E_j\} \subseteq \overline{\mathcal{M}} \), then \( E_j = E_j \cup N_j \), where \( E_j \in \mathcal{M} \) and \( N_j \in \mathcal{N} \), for all \( j \). Since \( N_j \subseteq \mathcal{N} \), there exists \( F_j \in \mathcal{M} \) such that \( N_j \subseteq F_j \) and \( \mu(F_j) = 0 \). Therefore, \( \bigcup_{j=1}^{+\infty} N_j \subseteq \bigcup_{j=1}^{+\infty} F_j =: F \), where \( \mu(F) \leq \sum_{j=1}^{+\infty} \mu(F_j) = 0 \). Therefore, \( \bigcup_{j=1}^{+\infty} N_j \subseteq \mathcal{N} \). Hence,

\[
\bigcup_{j=1}^{+\infty} E_j = \bigcup_{j=1}^{+\infty} (E_j \cup N_j) = \left( \bigcup_{j=1}^{+\infty} E_j \right) \cup \left( \bigcup_{j=1}^{+\infty} N_j \right) \in \mathcal{M} \cup \mathcal{N} = \overline{\mathcal{M}}.
\]

Next, let \( E' = E \cup N \in \mathcal{M} \cup \mathcal{N} \). Then there exists \( F \in \mathcal{M} \) such that \( N \subseteq F \) and \( \mu(F) = 0 \). Then, \( E \cup N = (E \cup F) \cap (\complement F \cup N) \) and

\[
\complement(E \cup N) = \complement(E \cup F) \cup (F \setminus N).
\]

Notice that \( \complement(E \cup F) \in \mathcal{M} \), while \( F \setminus N \subseteq F \), with \( \mu(F) = 0 \), so that \( F \setminus N \in \mathcal{N} \). This shows that \( \overline{\mathcal{M}} \) is a \( \sigma \)-algebra.

We now show that \( \overline{\mu} \) is well defined and is a complete measure. If \( E_1 \cup N_1 = E_2 \cup N_2 \) with \( E_j \in \mathcal{M} \) and \( N_j \in \mathcal{N} \), with \( N_j \subseteq F_j \) and \( \mu(F_j) = 0 \), \( j = 1, 2 \), then \( E_1 \subseteq E_2 \cup F_2 \) and \( \mu(E_1) \leq \mu(E_2) + \mu(F_2) = \mu(E_2) \). Thus, \( \mu(E_1) \leq \mu(E_2) \). Arguing in the same way we obtain the reverse inequality so that \( \mu(E_1) = \mu(E_2) \). Therefore, \( \overline{\mu} \) is well defined.

Next, let \( \{E_j \cup N_j\} \) be a sequence of disjoint sets in \( \overline{\mathcal{M}} \). If \( F_j \in \mathcal{M} \), \( N_j \subseteq F_j \) and \( \mu(F_j) = 0 \), then

\[
\overline{\mu}\left( \bigcup_{j=1}^{+\infty} (E_j \cup N_j) \right) = \overline{\mu}\left( \left( \bigcup_{j=1}^{+\infty} E_j \right) \cup \left( \bigcup_{j=1}^{+\infty} N_j \right) \right) = \mu\left( \bigcup_{j=1}^{+\infty} E_j \right) = \sum_{j=1}^{+\infty} \mu(E_j) = \sum_{j=1}^{+\infty} \overline{\mu}(E_j \cup N_j).
\]

It follows that \( \overline{\mu} \) is a measure, and it is clear that it is complete. \( \square \)
2. Abstract integration theory

In order to begin our approach to the theory of integration, we need to discuss the notion of measurable functions.

2.1. Measurable functions. In analogy with the definition of continuous functions, as morphisms between topological spaces, we have the following definition.

**Definition 2.1.** Let $(\mathcal{X}, \mathcal{M}), (\mathcal{Y}, \mathcal{N})$ be measurable spaces, and $f : \mathcal{X} \to \mathcal{Y}$ be given. We say that $f$ is $(\mathcal{M}, \mathcal{N})$-measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

It is clear that if $f : \mathcal{X} \to \mathcal{Y}$ is $(\mathcal{M}, \mathcal{N})$-measurable and $g : \mathcal{Y} \to \mathcal{Z}$ is $(\mathcal{N}, \mathcal{O})$-measurable, then $g \circ f : \mathcal{X} \to \mathcal{Z}$ is $(\mathcal{M}, \mathcal{O})$-measurable. Also, observe that if $\mathcal{N}$ is a $\sigma$-algebra in $\mathcal{P}(\mathcal{Y})$, then $\{f^{-1}(E) : E \in \mathcal{N}\}$ is a $\sigma$-algebra in $\mathcal{P}(\mathcal{X})$. In fact, $\cup_j f^{-1}(E_j) = f^{-1}(\cup_j E_j)$ and $c(f^{-1}(E)) = f^{-1}(cE)$, and the conclusion follows.

**Proposition 2.2.** Let $f : \mathcal{X} \to \mathcal{Y}$ be $(\mathcal{M}, \mathcal{N})$-measurable and suppose that $\mathcal{N}$ is generated by $\mathcal{E}$. Then $f$ is $(\mathcal{M}, \mathcal{N})$-measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Hence, if $\mathcal{X}, \mathcal{Y}$ are topological spaces, and $f : \mathcal{X} \to \mathcal{Y}$ continuous, then $f$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

**Proof.** First of all, we observe that the implication “only if” is trivial. Conversely, suppose that $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$. It is easy to check that $\{E \subseteq \mathcal{Y} : f^{-1}(E) \in \mathcal{M}\}$ is a $\sigma$-algebra in $\mathcal{P}(\mathcal{Y})$ that contains $\mathcal{E}$. Hence, it contains the $\sigma$-algebra generated by $\mathcal{E}$, that is, it contains $\mathcal{N}$ and $f$ is $(\mathcal{M}, \mathcal{N})$-measurable. The second conclusion now is obvious. 

**Definition 2.3.** If $(\mathcal{X}, \mathcal{M})$ is a measurable space, then $f : \mathcal{X} \to \mathbb{R}$ is said to be measurable if it is $(\mathcal{M}, \mathcal{B}_\mathbb{R})$-measurable.

It is however convenient to consider functions that take value in the extended reals, that is, in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\} = [-\infty, +\infty]$. In this case, the $\sigma$-algebra of Borel sets $\mathcal{B}_{\overline{\mathbb{R}}}$ is defined as $\{E \subseteq \overline{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_\mathbb{R}\}$. Then we say that $f : \mathcal{X} \to \overline{\mathbb{R}}$ is measurable if it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

**Proposition 2.4.** Let $(\mathcal{X}, \mathcal{M})$ be a measurable space, and let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be given. Then the following conditions are equivalent.

(i) $f$ is measurable, that is, it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$-measurable.

(ii) $f^{-1}([-\infty, b]) \in \mathcal{M}$ for every $b \in \mathbb{R}$.

(iii) $f^{-1}([-\infty, b]) \in \mathcal{M}$ for every $b \in \mathbb{R}$.

(iv) $f^{-1}([a, +\infty]) \in \mathcal{M}$ for every $a \in \mathbb{R}$.

(v) $f^{-1}([a, +\infty]) \in \mathcal{M}$ for every $a \in \mathbb{R}$.

**Proof.** If $f : \mathcal{X} \to \mathbb{R}$, the conclusions follow at once from Prop.’s 1.7 and 2.2. If $f$ takes values in $[-\infty, +\infty]$, the conclusions are a simple consequence of Def. 2.3.

**Lemma 2.5.** Let $(\mathcal{X}, \mathcal{M})$ be a measurable space, and let $f : \mathcal{X} \to \mathbb{C}$ be given. Then $f$ is measurable if and only if $\text{Re} f$ and $\text{Im} f$ are measurable.
Proof. (i) We identify $f$ with $(f_1, f_2) : \mathcal{X} \to \mathbb{R}^2$. Observe that the coordinate maps $\pi_j : \mathbb{R}^2 \to \mathbb{R}$, $j = 1, 2$ are measurable. Since composition of measurable functions is measurable and $f_1 = \Re f = \pi_1 \circ f$, $f_2 = \Im f = \pi_2 \circ f$, $f$ measurable implies $\Re f, \Im f$ are measurable.

Conversely, suppose $\Re f$ and $\Im f$ are measurable. By Prop. 1.8 we know that $\mathcal{B}_C = \mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$. Then, if $E \in \mathcal{B}_C$, $E = E_1 \times E_2$, with $E_1, E_2 \in \mathcal{B}_\mathbb{R}$ and $f^{-1}(E) = f_1^{-1}(E_1) \cap f_2^{-1}(E_2)$ that is in $\mathcal{M}$ by assumption. □

We remark that when we require that $f$ takes values in $\mathbb{R}$ we include the case of $f$ having finite values.

**Theorem 2.6.** Let $(\mathcal{X}, \mathcal{M})$ be a measurable space.

(i) If $f, g : \mathcal{X} \to \mathbb{R}$ are measurable, then $f + g$, $fg$ are measurable.

(ii) If $f_j : \mathcal{X} \to \mathbb{R}$ are measurable, $j = 1, 2, \ldots$, then

$$g_1 = \sup_j f_j, \quad g_2 = \inf_j f_j, \quad g_3 = \limsup_{j \to +\infty} f_j, \quad g_4 = \liminf_{j \to +\infty} f_j$$

are measurable.

(iii) $f, g : \mathcal{X} \to \mathbb{R}$ are measurable, then

$$\max(f, g), \quad \min(f, g)$$

are measurable.

(iv) If $f_j : \mathcal{X} \to \mathbb{C}$ are measurable, $j = 1, 2, \ldots$, and $g(x) = \lim_{j \to +\infty} f_j(x)$ exists, then $g$ is measurable.

Proof. (i) Observe that $f + g = S \circ F$, where $F(x) = (f(x), g(x))$ and $S : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is the sum-function, i.e. $S(z + w) = z + w$. Arguing as in Lemma 2.5 we see that $F$ is measurable. Since $S$ is continuous is measurable, and then $f + g$ is measurable. Replacing $S$ by $P$, where $P(z, w) = zw$, we obtain the measurability of $fg$.

(ii) Notice that

$$g_1^{-1}((a, +\infty)) = \{x \in \mathcal{X} : \sup_j f_j(x) > a\} = \bigcup_{j=1}^{+\infty} f_j^{-1}((a, +\infty)),$$

while

$$g_2^{-1}([-\infty, b)) = \{x \in \mathcal{X} : \inf_j f_j(x) < b\} = \bigcup_{j=1}^{+\infty} f_j^{-1}([-\infty, b)).$$

The measurability of $g_1$ and $g_2$ follows from Prop. 2.4.

Next, we observe that

$$g_3(x) = \limsup_{j \to +\infty} f_j(x) = \inf_k \left( \sup_{j \geq k} f_j(x) \right)$$

Then $h_k(x) = \sup_{j < k} f_j(x)$ are measurable, so is $\inf_k h_k(x) = g_3(x)$. The argument for $g_4$ is analogous, since

$$g_4(x) = \liminf_{j \to +\infty} f_j(x) = \sup_k \left( \inf_{j \geq k} f_j(x) \right).$$

\[3\text{We recall that, given a sequence } \{a_n\}, \limsup_n a_n = s^*, \text{ where } s^* \text{ is sup of the limit points of } \{a_n\}. \text{ Then, } s^* = \inf_n (\sup_{n \geq k} a_n) = \lim_{k \to \infty} (\sup_{n \geq k} a_n). \text{ Analogously, if we set } s_* = \liminf_n a_n, \text{ then } s_* = \sup_n (\inf_{n \geq k} a_n) = \lim_{k \to \infty} (\inf_{n \geq k} a_n).\]
Finally, (iii) and (iv) are trivial consequences of (ii) \hfill \square

**Corollary 2.7.** If \( f : \mathcal{X} \to \mathbb{R} \) is measurable, then \( f_+ = \max(f, 0) \) and \( f_- = -\min(f, 0) \) are measurable. If \( g : \mathcal{X} \to \mathbb{C} \) is measurable, then \( |g| \) and \( \text{sgn} \, g := g/|g| \) are measurable.

We point out that \( f = f_+ - f_- \), \( |f| = f_+ + f_- \), and \( g = \text{sgn} \, g \cdot |g| \).

**Proof.** We only need to prove the statements for \( |g| \) and \( \text{sgn} \, g \). But these are elementary and we leave the details to the reader. \hfill \square

**Definition 2.8.** Let \((\mathcal{X}, \mathcal{M})\) be a measurable space, and let \( E \in \mathcal{M} \). We define the **characteristic function of** \( E \) **as the function**

\[
\chi_E(x) = \begin{cases} 
1 & \text{if } x \in E \\
0 & \text{otherwise} \end{cases}
\]

We call a **simple function** a finite linear combination with complex coefficients of characteristic function of measurable sets \( E_j \)

\[
f(x) = \sum_{j=1}^{n} c_j \chi_{E_j}(x). \tag{2}
\]

Observe that simple functions can characterized as the measurable functions whose range is finite (that is, they attain at most finitely many values). We remark that the representation (2) of a simple function \( f \) is not unique, but it becomes unique if we write

\[
f(x) = \sum_{j=1}^{m} d_j \chi_{F_j}(x).
\]

where \( F_j = f^{-1}(\{d_j\}) \), \( d_j \neq 0 \), and \( \{d_1, \ldots, d_m\} = f(\mathcal{X}) \setminus \{0\} \) is the range of \( f \) taken away the value 0.

Notice also that finite sums and products of simple functions are again simple functions.

The main result of this section is that measurable can be suitable approximated with simple functions, as the next result shows.

**Theorem 2.9.** Let \((\mathcal{X}, \mathcal{M})\) be a measurable space.

1. Let \( f : \mathcal{X} \to [0, +\infty] \) be a measurable function. Then there exists an increasing sequence of non-negative measurable functions \( 0 \leq s_1 \leq s_2 \leq \cdots \leq f \) such that \( s_n(x) \to f(x) \) as \( n \to +\infty \) and the convergence is uniform on all sets where \( f \) is bounded.

2. Let \( f : \mathcal{X} \to \mathbb{C} \) be a measurable function. Then there exists a sequence of complex-valued simple measurable functions \( \{\varphi_n\} \) such that \( 0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f| \) such that \( \varphi_n(x) \to f(x) \) as \( n \to +\infty \) and the convergence is uniform on all sets where \( f \) is bounded.

**Proof.** (1) For \( n = 0, 1, 2, \ldots \) and \( k \) integer, \( 0 \leq k \leq 2^{2n} - 1 \) we define the sets

\[
F_n = f^{-1}((2^n, +\infty]) \quad E_{n,k} = f^{-1}((k2^{-n}, (k+1)2^{-n}]],
\]

and set

\[
s_n(x) = 2^n \chi_{F_n}(x) + \sum_{k=0}^{2^n-1} k2^{-n} \chi_{E_{n,k}}(x).
\]
Clearly \( \{s_n\} \) are measurable and non-negative. In order to check that \( \{s_n\} \) is increasing, notice that
\[
(k2^{-n}, (k + 1)2^{-n}] = (2k2^{-(n+1)}, (2k + 2)2^{-(n+1)}] \\
= (2k2^{-(n+1)}, (2k + 1)2^{-(n+1)}] \cup ([2k + 1)2^{-(n+1)}), (2k + 2)2^{-(n+1)}].
\]
This implies that
\[
E_{n,k} = E_{n+1,2k} \cup E_{n+1,2k+1}.
\]
Moreover,
\[
(2^n, 2^{n+1}] = \bigcup_{j=2^{2n+1}}^{2^{2(n+1)}-1} (j2^{-(n+1)}, (j + 1)2^{-(n+1)}],
\]
so that
\[
F_n \setminus F_{n+1} = \bigcup_{j=2^{2n+1}}^{2^{2(n+1)}-1} E_{n+1,j}.
\]
Therefore, using (3) and (4),
\[
s_n(x) = 2^n \chi_{F_n}(x) + \sum_{k=0}^{2^{2n}-1} k2^{-n} \left( \chi_{E_{n+1,2k}}(x) + \chi_{E_{n+1,2k+1}}(x) \right)
\]
\[
= 2^n \left( \chi_{F_{n+1}}(x) + \chi_{F_n \setminus F_{n+1}}(x) \right) + \sum_{k=0}^{2^{2n}-1} 2k2^{-(n+1)} \left( \chi_{E_{n+1,2k}}(x) + \chi_{E_{n+1,2k+1}}(x) \right)
\]
\[
\leq 2^n \left( \chi_{F_{n+1}}(x) + \chi_{F_n \setminus F_{n+1}}(x) \right) + \sum_{k=0}^{2^{2n}-1} 2k2^{-(n+1)} \chi_{E_{n+1,2k}}(x) + \sum_{k=0}^{2^{2n}-1} (2k + 1)2^{-(n+1)} \chi_{E_{n+1,2k+1}}(x)
\]
\[
= 2^n \chi_{F_{n+1}}(x) + 2^n \sum_{j=2^{2n+1}}^{2^{2(n+1)}-1} \chi_{E_{n+1,j}}(x) + \sum_{\ell=0}^{2^{2n+1}-1} \ell2^{-(n+1)} \chi_{E_{n+1,\ell}}(x)
\]
\[
\leq 2^{n+1} \chi_{F_{n+1}}(x) + 2^{2(n+1)} \sum_{j=2^{2n+1}}^{2^{2(n+1)}-1} j2^{-(n+1)} \chi_{E_{n+1,j}}(x) + \sum_{\ell=0}^{2^{2n+1}-1} \ell2^{-(n+1)} \chi_{E_{n+1,\ell}}(x)
\]
\[
= s_{n+1}(x).
\]
Hence, \( \{s_n\} \) is monotone increasing. It is clear that \( s_n(x) \leq f(x) \) for all \( x \). Finally, notice that for \( x \notin F_n, \ 0 \leq f(x) - s_n(x) \leq 2^{-n} \). Then, \( s_m \to f \) uniformly on each set \( cF_n = \{ x : f(x) \leq 2^n \} \).
On \( F = \cap_n F_n, \ f(x) = +\infty \) and \( s_n(x) \to +\infty \) for \( x \in F \). This proves (1).

(2) now follows easily from (1). If \( f : \mathcal{X} \to \mathbb{C} \), we write \( f = u + iv, \ u = u_+ - u_-, \ v = v_+ - v_- \).
Then, there exists increasing sequences \( \{s_n^+\} \) and \( \{s_n^-\} \) of non-negative simple functions such that
\[
s_n^+ \to u_+, \quad \text{and} \quad s_n^- \to v_+, \quad \text{as in (1). Then, setting}
\]
\[
\varphi_n = s_n^+ - s_n^- + i(s_n^+ - s_n^-)
\]
we have that $\varphi_n \to u_+ - u_- + i(v_+ - v_-) = f$, pointwise, and uniformly on all sets where $f$ is bounded. Finally, it also holds that
\[
|\varphi_n| = [(s_n^+ - s_n^-)^2 + (\sigma_n^+ - \sigma_n^-)^2]^{1/2} = [(s_n^+)^2 + (s_n^-)^2 + (\sigma_n^+)^2 + (\sigma_n^-)^2]^{1/2}
\leq [(s_{n+1}^+)^2 + (s_{n+1}^-)^2 + (\sigma_{n+1}^+)^2 + (\sigma_{n+1}^-)^2]^{1/2}
\leq |\varphi_{n+1}|
\]
for all $n$, where we have used the obvious fact that $g_+g_- = 0$ for all $g$. \hfill \Box

**Definition 2.10.** Given a measure space $(X, \mathcal{M}, \mu)$, we say that a property $(P)$ is valid $\mu$-a.e. (or, simply, a.e. if the measure $\mu$ is understood by the context), if there exists a set $F \in \mathcal{M}$, with $\mu(F) = 0$ such that the property $(P)$ holds on $^c F$.

One convenience of working with complete measures is the following result.

**Proposition 2.11.** Let $(X, \mathcal{M}, \mu)$ be a measure space. The following implications are true if and only if the measure $\mu$ is complete.

1. If $f$ is measurable and $g = f$ $\mu$-a.e., then $g$ is measurable.
2. If $f_n$ are measurable, $n = 1, 2, \ldots$ and $f_n \to f$ pointwise $\mu$-a.e., then $f$ is measurable.

*Proof.* (†) Exercise. \hfill \Box

On the other hand, we have the following result.

**Proposition 2.12.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If $f$ is $\overline{\mathcal{M}}$-measurable, then there exists a $\mathcal{M}$-measurable function such that $f = g$ $\overline{\mu}$-a.e.

*Proof.* The proof is indicative of a method that will be often used. Suppose first $f$ is a characteristic function of an $\overline{\mathcal{M}}$-measurable set $E = E \cup N$, with $E \in \mathcal{M}$, $N \subseteq F$, $\mu(F) = 0$. Then $f = \chi_E^\overline{\mu} = \chi_E$ $\overline{\mu}$-a.e. and the conclusion holds true in this case. In the same way we see that the conclusion follows in the case of a simple function $f = \sum_{j=1}^n c_j \chi_{E_j}$.

In the general case, given $f$ an $\overline{\mathcal{M}}$-measurable function, by Thm. 2.9 we can find a sequence of $\overline{\mathcal{M}}$-measurable functions $\{\varphi_n\}$ pointwise converging to $f$. For each $n$, let $\psi_n$ be an $\mathcal{M}$-measurable function, $\psi_n = \varphi_n$ except on a set $N_n \in \overline{\mathcal{M}}$ with $\overline{\mu}(N_n) = 0$. Then, there exist $F_n \in \mathcal{M}$, $F_n \supseteq N_n$ and $\mu(F_n) = 0$. Set $F = \cup_n F_n$ and define $g = \lim_{n \to +\infty} \chi_{^cF} \psi_n$. Then $g$ is $\mathcal{M}$-measurable by the previous Prop. 2.11 (2), and $g = f$ on $^c F$, that is, $\overline{\mu}$-a.e. \hfill \Box

### 2.2. Integration of non-negative functions

We begin the construction of the integral with the simple functions.

**Definition 2.13.** Let $(X, \mathcal{M}, \mu)$ be a measure space. We define
\[
L^+ = \{ f : X \to [0, +\infty], \mathcal{M} \text{ - measurable} \}.
\]

If $s \in L^+$ is a simple function $s = \sum_{j=1}^n c_j \chi_{E_j}$, we define the integral of $s$ with respect to $\mu$,
\[
\int s \, d\mu = \sum_{j=1}^n c_j \mu(E_j),
\]
(with the standard convention that $0 \cdot \infty = 0$). More generally, if $E \in \mathcal{M}$ we also define

$$
\int_E s \, d\mu = \int \chi_E s \, d\mu = \sum_{j=1}^{n} c_j \mu(E \cap E_j).
$$

We will use the following notation to denote the integral of $s$ w.r.t. $\mu$:

$$
\int s \, d\mu = \int \chi s \, d\mu = \sum_{j=1}^{n} c_j \mu(E \cap E_j).
$$

and similarly in the case of $\int_E s \, d\mu$.

The following properties are easy to check.

**Proposition 2.14.** Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space and $s, \sigma$ be non-negative simple functions on $\mathcal{X}$. Then, the following properties hold true.

(i) If $c \geq 0$, then $\int cs \, d\mu = c \int s \, d\mu$;

(ii) $\int (s + \sigma) \, d\mu = \int s \, d\mu + \int \sigma \, d\mu$;

(iii) if $s \leq \sigma$, then $\int s \, d\mu \leq \int \sigma \, d\mu$;

(iv) the map $\mathcal{M} \ni E \mapsto \int_{E} s \, d\mu$ is a measure.

We are now in the position to define the integral of non-negative measurable functions.

**Definition 2.15.** Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space. We define

$$
L^+ = \{ f : \mathcal{X} \to [0, +\infty], \mathcal{M} - \text{measurable} \}.
$$

If $f \in L^+$ we set

$$
\int f \, d\mu = \sup \{ \int s \, d\mu : 0 \leq s \leq f, \text{ simple} \}.
$$

For any $E \in \mathcal{M}$ we also define

$$
\int_E f \, d\mu = \int f\chi_E \, d\mu.
$$

It follows at once that, if $c \geq 0$ and $f \in L^+$, then $cf \in L^+$ and $\int cf \, d\mu = c \int f \, d\mu$. It also follows at once that if $f, g \in L^+$ and $f \leq g$, then

$$
\int f \, d\mu \leq \int g \, d\mu.
$$

The next result is a fundamental building block of this theory.

**Theorem 2.16. (Monotone Convergence Theorem.)** Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space \{f_n\} be given such that $f_n : \mathcal{X} \to [0, +\infty]$, $0 \leq f_1 \leq f_2 \leq \cdots$. Setting $f(x) = \lim_{n \to +\infty} f_n(x)$, we have

$$
\int f \, d\mu = \lim_{n \to +\infty} \int f_n \, d\mu.
$$

Notice that the conclusion can we written in the suggestive form

$$
\int \left( \lim_{n \to +\infty} f_n \right) \, d\mu = \lim_{n \to +\infty} \int f_n \, d\mu.
$$
Proof. Since \(0 \leq f_1(x) \leq f_2(x) \leq \cdots\) is a monotone sequence the limit function \(f\) exists, and it is measurable by Thm. 2.6 (iv). Moreover, by the mononicity of the integral, property (8),
\[
\int f_1 \, d\mu \leq \int f_2 \, d\mu \leq \cdots,
\]
that is, the sequence \(\{ \int f_n \, d\mu \}\) is increasing, so that it has a limit and since \(\int f_n \, d\mu \leq \int f \, d\mu\) for all \(n\), by the comparison test
\[
\lim_{n \to +\infty} \int f_n \, d\mu \leq \int f \, d\mu.
\]

In order to prove the reverse inequality, let \(s\) be a simple function such that \(0 \leq s \leq f\) and let \(0 < \alpha < 1\). Since \(\lim_{n \to +\infty} f_n(x) = f(x)\), setting
\[
E_n = \{ x : f_n(x) \geq \alpha s(x) \},
\]
we have that \(E_1 \subseteq E_2 \subseteq \cdots\), and \(\bigcup_{n=1}^{+\infty} E_n = \mathcal{X}\). Therefore, by the monontonicity of the integral,
\[
\alpha \int_{E_n} s \, d\mu \leq \int_{E_n} f_n \, d\mu = \int \chi_{E_n} f_n \, d\mu \leq \int f_n \, d\mu. \tag{9}
\]
By Prop. 2.14 (iv), since \(\{E_n\}\) is an increasing sequence of measurable sets whose union is \(\mathcal{X}\), \(\lim_{n \to +\infty} \int_{E_n} s \, d\mu = \int s \, d\mu\), therefore, passing to the limit in (9) we obtain
\[
\alpha \int s \, d\mu \leq \lim_{n \to +\infty} \int f_n \, d\mu.
\]
This holds for all \(0 < \alpha < 1\) so that
\[
\int s \, d\mu \leq \lim_{n \to +\infty} \int f_n \, d\mu.
\]
Passing to the supremum on the left hand side for all simple functions \(s\) with \(0 \leq s \leq f\) we obtain
\[
\int f \, d\mu \leq \lim_{n \to +\infty} \int f_n \, d\mu,
\]
and we are done. \(\square\)

Corollary 2.17. (1) Let \(f, g \in L^+\). Then
\[
\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu.
\]
(2) If \(f_n \in L^+, n = 1, 2, \ldots\), then
\[
\int \left( \sum_{n=1}^{+\infty} f_n \right) \, d\mu = \sum_{n=1}^{+\infty} \int f_n \, d\mu
\]
Notice that part (1) is just the statement of the additivity of the integral. In the present theory, in order to obtain such property we had to recourse to the Monotone Convergence Theorem (that we will abbreviate as MCT in what follows).
Proof. Let \( \{s_n\}, \{\sigma_n\} \) be sequences of non-negative simple functions, monotonically converging to \( f \) and \( g \), respectively. Then, \( \{s_n + \sigma_n\} \) is a sequence of non-negative simple functions, monotonically converging to \( f + g \). By the MCT,

\[
\int (f + g) \, d\mu = \lim_{n \to +\infty} \int (s_n + \sigma_n) \, d\mu = \lim_{n \to +\infty} \int s_n \, d\mu + \lim_{n \to +\infty} \int \sigma_n \, d\mu = \int f \, d\mu + \int g \, d\mu.
\]

This proves (1). In order to prove (2), notice that by (1) and induction we have that

\[
\int \left( \sum_{n=1}^{N} f_n \right) \, d\mu = \sum_{n=1}^{N} \int f_n \, d\mu,
\]

for every \( N \). Observe that \( \{ \sum_{n=1}^{\infty} f_n \} \) is a sequence of functions in \( L^+ \) that converges monotonically to \( \sum_{n=1}^{\infty} f_n \). Hence, applying the MCT again we obtain

\[
\int \left( \sum_{n=1}^{\infty} f_n \right) \, d\mu = \lim_{n \to +\infty} \int \left( \sum_{n=1}^{N} f_n \right) \, d\mu = \lim_{n \to +\infty} \sum_{n=1}^{N} \int f_n \, d\mu = \sum_{n=1}^{}\int f_n \, d\mu,
\]

as we wished to show. \( \square \)

**Corollary 2.18.** Let \( f \in L^+ \). Then \( \int f \, d\mu = 0 \) if and only if \( f = 0 \) \( \mu \)-a.e. Therefore, if \( f, g \in L^+ \) and \( f = g \) a.e., then

\[
\int f \, d\mu = \int g \, d\mu.
\]

**Proof.** Assume first the \( f \) is simple, \( f = \sum_j c_j \chi_{E_j} \). Then \( f = 0 \) clearly implies \( \int f \, d\mu = 0 \), while if \( 0 = \int f \, d\mu = \sum_j c_j \mu(E_j) \), it must be either \( c_j = 0 \) or \( \mu(E_j) = 0 \), for each \( j \). In any case, \( f = 0 \) \( \mu \)-a.e.

Let now \( f \in L^+ \) and assume first \( f = 0 \) \( \mu \)-a.e. If \( s \) is simple, \( 0 \leq s \leq f \), then \( s = 0 \) \( \mu \)-a.e., \( \int s \, d\mu = 0 \), and passing to the supremum of such functions we obtain that \( \int f \, d\mu = 0 \). Finally, suppose \( \int f \, d\mu = 0 \). Then

\[
\{ x : f(x) > 0 \} = \bigcup_{n=1}^{\infty} \{ x : f(x) > 1/n \} =: \bigcup_{n=1}^{\infty} E_n,
\]

Since \( E_1 \subseteq E_2 \subseteq \cdots \),

\[
\mu \left( \{ x : f(x) > 0 \} \right) = \lim_{n \to +\infty} \mu(E_n).
\]

If \( f \) is not equal to 0 a.e., then it must be \( \mu(E_n) > 0 \) for some \( n \), but then

\[
\int f \, d\mu \geq \int_{E_n} f \, d\mu \geq \int_{E_n} \frac{1}{n} \, d\mu = \frac{1}{n} \mu(E_n) > 0,
\]

a contradiction. Then, \( \int f \, d\mu = 0 \) implies \( f = 0 \) \( \mu \)-a.e., and we are done. \( \square \)

The conclusion of the MCT holds true also if we just assume that \( \{f_n\} \) is a sequence in \( L^+ \) monotonically convergent a.e.

**Corollary 2.19.** Let \( \{f_n\} \) is a sequence in \( L^+ \) monotonically convergent a.e. to \( f \in L^+ \), then

\[
\int f \, d\mu = \lim_{n \to +\infty} \int f_n \, d\mu.
\]
Proof. Let $E \in \mathcal{M}$ be such that $\mu(E) = 0$, and $f_1(x) \leq f_2(x) \leq \cdots$ and $\lim_{n \to +\infty} f_n(x) = f(x)$ for $x \in ^c E$. Setting $g_n = \chi_{^c E} f_n$ and $g = \chi_{^c E} f$, we observe that $g_n = f_n$ a.e. and $g = f$ a.e. so that $\int g_n \, d\mu = \int f_n \, d\mu$, for all $n$ and $\int g \, d\mu = \int f \, d\mu$, by Cor. 2.18.

We may apply the MCT to $\{g_n\}$ and obtain that $\int f \, d\mu = \int g \, d\mu = \lim_{n \to +\infty} \int g_n \, d\mu = \lim_{n \to +\infty} \int f_n \, d\mu$, as we wish to show. □

In the MCT the key assumption, besides the non-negativity of the functions, was the monotonicity of the sequence. Such monotonicity guaranteed the existence of the limits. In the general case, we have the following.

**Theorem 2.20. (Fatou’s Lemma.)** Let $\{f_n\} \subseteq L^+$, then

$$\int \left( \liminf_{n \to +\infty} f_n \right) \, d\mu \leq \liminf_{n \to +\infty} \int f_n \, d\mu.$$  

Proof. As in the proof of Thm. 2.6 (iv), if we set $g_k(x) = \inf_{j \leq k} f_j(x)$, for $k = 1, 2, \ldots$, we have that $\{g_k\}$ is an increasing sequence in $L^+$ and $\lim_k g_k = \liminf_{n \to +\infty} f_n$. Moreover, $g_k \leq f_k$ so that

$$\lim_{k \to +\infty} \int g_k \, d\mu \leq \liminf_{n \to +\infty} \int f_n \, d\mu.$$  

Therefore, by the MCT,

$$\int \left( \liminf_{n \to +\infty} f_n \right) \, d\mu = \int \left( \lim_{k \to +\infty} g_k \right) \, d\mu = \lim_{k \to +\infty} \int g_k \, d\mu \leq \liminf_{n \to +\infty} \int f_n \, d\mu,$$

as we wished to show. □

The following result follows at once from Fatou’s Lemma and Cor. 2.19.

**Corollary 2.21.** Let $\{f_n\} \subseteq L^+$ and suppose $f_n \to f$ a.e. Then

$$\int f \, d\mu \leq \liminf_{n \to +\infty} \int f_n \, d\mu.$$  

Proof. Let $E \in \mathcal{M}$ be a set such that $f_n \to f$ in $E$ and $\mu(^c E) = 0$. Set $g_n = \chi_E f_n$ and $g = \chi_E f$. Then $g_n \to g$, $f_n = g_n$ a.e. and $f = g$ a.e. By Fatou’s Lemma,

$$\int f \, d\mu = \int g \, d\mu \leq \liminf_{n \to +\infty} \int g_n \, d\mu = \liminf_{n \to +\infty} \int f_n \, d\mu.$$  

□

**Remark 2.22.** We observe that the previous result cannot be improved to have equality even if we assume that the limit of the sequence $\{\int f_n \, d\mu\}$ exists.

Indeed, consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where $\mu$ is the counting measure. Let $s_n$ be the numerical sequence $\{s_{n,k}\}$ that is equal to 1 if $k = n$ and equals 0 otherwise. Then the sequence of functions on $\mathbb{N}$ $\{s_n\}$ converges to 0 pointwise, but

$$\int_{\mathbb{N}} s_n \, d\mu = \sum_{k=1}^{+\infty} s_{n,k} = 1$$

for all $n$. Therefore,

$$0 = \int_{\mathbb{N}} s \, d\mu \leq \lim_{n \to +\infty} \int_{\mathbb{N}} s_n \, d\mu = 1.$$
Remark 2.23. We also observe that there exists a version of the MCT for decreasing sequences \( \{f_n\} \), but one needs to assume that \( \int f_1 d\mu < +\infty \). Precisely, if \( \{f_n\} \subseteq L^+, f_1 \geq f_2 \geq \cdots \), and \( \int f_1 d\mu < +\infty \), then

\[
\int f d\mu = \lim_{n \to +\infty} \int f_n d\mu.
\]

Proof. We first observe that if \( 0 \leq g \leq f \), then \( f - g \geq 0 \) and

\[
\int (f - g) d\mu = \int f d\mu - \int g d\mu.
\]

Next, since \( \{f_n\} \) is monotone and non-negative, the limit \( f \) exists it is non-negative, and \( f_n \geq f \) for each \( n \). Set \( g_n = f_1 - f_n \). Then \( \{g_n\} \) is a sequence of non-negative functions such that \( g_n \leq g_{n+1} \) for all \( n \) and converging to \( f_1 - f \). Applying the MCT we obtain

\[
\int f_1 d\mu - \int f d\mu = \int (f_1 - f) d\mu = \int g d\mu = \lim_{n \to +\infty} \int g_n d\mu = \lim_{n \to +\infty} \left( \int f_1 d\mu - \int f_n d\mu \right).
\]

Therefore, since \( \int f_1 d\mu < +\infty \) we can subtractive from both sides to obtain

\[
\int f d\mu = \lim_{n \to +\infty} \int f_n d\mu,
\]

as we wished to prove. \( \square \)

2.3. Integration of complex-valued functions. We now pass to consider generic real, or complex, valued functions. If \((\mathcal{X}, \mathcal{M}, \mu)\) is a measure space, \( f : \mathcal{X} \to \mathbb{C} \) is measurable, and \( u = \text{Re } f, v = \text{Im } f \), we write

\[
f = u_+ - u_- + i(v_+ - v_-).
\]

Then, \( u_\pm, v_\pm : \mathcal{X} \to [0, +\infty) \). If \( f \) is real-valued, we simply write \( f = f_+ - f_- \).

Definition 2.24. Given a measure space \((\mathcal{X}, \mathcal{M}, \mu)\), given \( f : \mathcal{X} \to \mathbb{C} \) is measurable, we say that \( f \) is absolutely integrable, or simply integrable, if

\[
\int |f| d\mu < +\infty;
\]

equivalently, if

\[
0 \leq \int u_\pm d\mu, \int v_\pm d\mu < +\infty.
\]

For \( f \) integrable we set

\[
\int f d\mu = \int u_+ d\mu - \int u_- d\mu + i \left( \int v_+ d\mu - \int v_- d\mu \right).
\]

In particular, if \( f \) is real-valued, we have

\[
\int f d\mu = \int f_+ d\mu - \int f_- d\mu.
\]

Remark 2.25. Notice that, if \( f \) is integrable, \( f = u + iv \), then \( 0 \leq u_\pm, v_\pm \leq |f| \) so that

\[
0 \leq \int u_\pm d\mu, \int v_\pm d\mu \leq \int |f| d\mu < +\infty.
\]
Conversely, if \( 0 \leq \int u_+ d\mu, \int v_+ d\mu < +\infty \), then \(|f| \leq |u| + |v| = u_+ + u_- + v_+ + v_-\), so that
\[
\int |f| d\mu \leq \int u_+ d\mu + \int u_- d\mu + \int v_+ d\mu + \int v_- d\mu < +\infty .
\]

Moreover, the set of integrable functions is a complex vector space, as it is easy to check, and for integrable \( f, g \) and \( a, b \in \mathbb{C} \) we have
\[
\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu .
\]

**Proposition 2.26.** The following properties hold true.

(i) If \( f \) is integrable, then \(|\int f d\mu| \leq \int |f| d\mu\).

(ii) If \( f, g \) are integrable, then \( \int_E f d\mu = \int_E g d\mu \) for every \( E \in \mathcal{M} \) if and only if \(|f - g| = 0\) a.e. if and only if \( \int |f - g| d\mu = 0\).

**Proof.** (i) If \( \int f = 0 \) the result is trivial. Next, if \( f \) is real
\[
|\int f d\mu| = |\int f_+ d\mu - \int f_- d\mu| \leq \int f_+ d\mu + \int f_- d\mu = \int |f| d\mu .
\]
If \( f \) is complex-valued, and \( \int f \neq 0 \), from Cor. 2.7 that \( \text{sgn } f, |f| \) are measurable and we set \( \alpha = |\int f d\mu|/|\int f d\mu| \). Then
\[
|\int f d\mu| = \alpha \int f d\mu = \int \alpha f d\mu
\]
so that, in particular \( \int \alpha f d\mu \) is non-negative. Therefore,
\[
|\int f d\mu| = \text{Re} \int \alpha f d\mu = \int \text{Re}(\alpha f) d\mu \leq \int |\alpha f| d\mu = \int |f| d\mu ,
\]
since \(|\alpha| = 1\).

(ii) If \( f, g \in L^1(\mu) \), from Cor. 2.18 we know that \( \int |f - g| d\mu = 0 \) if and only if \( |f - g| = 0 \) a.e., that is, if and only if \( f = g \) a.e. Now, if \( \int |f - g| d\mu = 0 \), then, for every \( E \in \mathcal{M} \),
\[
|\int_E f d\mu - \int_E g d\mu| \leq \int \chi_E |f - g| d\mu \leq \int |f - g| d\mu = 0 ,
\]
so that
\[
\int_E f d\mu = \int_E g d\mu \quad \text{for every } E \in \mathcal{M} .
\]
Finally, suppose that the above condition holds, and assume for simplicity that \( f \) and \( g \) are real-valued. Consider the function \( f - g = m \) and its decomposition \( m = m_+ - m_- \). Then, by assumption we have that \( \int_E m d\mu = 0 \) for all \( E \in \mathcal{M} \). In particular if we take \( E_+ = \{ x : m(x) \leq 0 \} \), then we have
\[
0 = \int_{E_+} m d\mu = \int_{E_+} m_+ d\mu - \int_{E_+} m_- d\mu ,
\]
since \( m_+ = 0 \) on \( ^c E_+ \). Thus, \( m_+ = 0 \) a.e. and, with a similar argument we obtain also \( m_- = 0 \) a.e. Hence, \( m = 0 \) a.e., i.e. \( f = g \) a.e., as we wished to show. \( \square \)
From now on we will identify functions that differ only on a set of measure 0. Indeed, we have the following definition.

**Definition 2.27.** Given a measure space \((X, \mathcal{M}, \mu)\), we consider the equivalent relation \(\sim\) on the space of integrable saying that \(f \sim g\) if \(f = g\) a.e. We define the space \(L^1(\mu)\) as the space of equivalent classes of integrable function modulo the relation \(\sim\) and we define the norm \(\|f\|_{L^1(\mu)} = \int |f| \, d\mu\).

**Remark 2.28.** We need to check that the above definition gives in fact a norm. It is clear that \(\|f\|_{L^1(\mu)} \geq 0\) and that \(f = 0\) in \(L^1(\mu)\) (that is, \(f = 0\) a.e.) implies \(\|f\|_{L^1(\mu)} = 0\). Conversely, if \(0 = \|f\|_{L^1(\mu)} = \int |f| \, d\mu\), then Cor. 2.18 implies that \(|f| = 0\) a.e., i.e. \(f = 0\) a.e. and \(f = 0\) in \(L^1\). The symmetry and triangular inequality of the norm follow easily.

In Thm. 2.31 we will show that \(L^1\) is a complete normed space, that is, a Banach space.

Therefore, we have a norm on \(L^1(\mu)\), hence a metric, given by the expression

\[ d(f, g) = \|f - g\|_{L^1(\mu)}. \]

Then, given \(\{f_n\} \subseteq L^1(\mu)\) we say that \(f_n \to f\) in \(L^1(\mu)\) (or, simply in \(L^1\) if the measure \(\mu\) is understood) if

\[ \|f_n - f\|_{L^1} \to 0 \quad \text{as} \quad n \to +\infty. \]

**Theorem 2.29. (Dominated Convergence Theorem.)** Let \((X, \mathcal{M}, \mu)\) be a measure space and \(\{f_n\} \subseteq L^1(\mu)\). Suppose that \(f_n \to f\) pointwise a.e. and that there exists a non-negative \(g \in L^1(\mu)\) such that \(g \geq |f_n|\) for all \(n\). Then, \(f_n \to f\) in \(L^1(\mu)\) and

\[ \int f \, d\mu = \lim_{n \to +\infty} \int f_n \, d\mu. \]

**Proof.** Since \(|f_n| \leq g\) for all \(n\), since \(f_n \to f\) pointwise a.e., it follows that \(f\) is measurable and, by the comparison theorem, that \(|f| \leq g\). Therefore, \(f \in L^1(\mu)\).

Observing that \(|f_n - f| \leq 2g\), we apply Fatou’s Lemma to the sequence \(\{2g - |f_n - f|\}\) and obtain

\[ \int 2g \, d\mu \leq \liminf_{n \to +\infty} \int 2g - |f_n - f| \, d\mu = \int 2g \, d\mu + \liminf_{n \to +\infty} \left( - \int |f_n - f| \, d\mu \right) = \int 2g \, d\mu - \limsup_{n \to +\infty} \int |f_n - f| \, d\mu. \]

Since \(\int 2g \, d\mu < +\infty\), we can subtract it from both sides of the inequality and obtain that

\[ \limsup_{n \to +\infty} \int |f_n - f| \, d\mu \leq 0. \]

This implis that \(\lim_{n \to +\infty} \int |f_n - f| \, d\mu = 0\), that is, \(f_n \to f\) in \(L^1(\mu)\). Finally,

\[ \left| \int f_n \, d\mu - \int f \, d\mu \right| \leq \int |f_n - f| \, d\mu \to 0 \]

and the last conclusion follows. \(\square\)
2.4. **The space $L^1(\mu)$**. Our next goal is to show that $L^1(\mu)$ is a Banach space, that is it is complete in its norm. We recall that a normed space $(X, \| \cdot \|_X)$ is **complete** if it is complete as a metric space w.r.t the metric $d(f, g) = \| f - g \|_X$. We also recall that a series $\sum_n f_n$ of elements in $X$ is said to be **absolutely convergent** in $X$ if the numerical series $\sum_n \| f_n \|_X$ is convergent. We have the following result.

**Theorem 2.30.** A normed space $(X, \| \cdot \|_X)$ is a Banach space if and only if every absolutely convergent series $\sum_n f_n$ is convergent in $X$.

**Proof.** Suppose first $X$ is a Banach space and that $\sum_n f_n$ is an absolutely convergent series. Let $\{s_N\}$ be the sequence of partial sums, that is, $s_N = \sum_{n=1}^{+\infty} f_n$. Then, given $\varepsilon > 0$, there exists $n_\varepsilon$ such that for $n \geq M < N$,

$$d(s_N, s_M) = \| s_N - s_M \|_X = \left\| \sum_{n=M+1}^{N} f_n \right\|_X \leq \sum_{n=M+1}^{N} \| f_n \|_X < \varepsilon.$$  

Hence, $\{s_N\}$ is a Cauchy sequence in $X$, that converges, since $X$ is complete.

Conversely, suppose every absolutely convergent series is convergent in $X$. Let $\{g_n\}$ be a Cauchy sequence in $X$. Hence, for every $k = 1, 2, \ldots$, there exists an integer $N_k$ such that if $m, n \geq N_k$, $d(g_n, g_m) = \| g_n - g_m \|_X < 2^{-k}$. Notice that we may assume that $N_{k+1} > N_k$ for all $k$. Then, the subsequence $\{g_{N_k}\}$ is such that $\| g_{N_{k+1}} - g_{N_k} \|_X < 2^{-k}$. Then, $\sum_{k=1}^{+\infty} \| g_{N_{k+1}} - g_{N_k} \|_X$ converges, so that by assumption, $\sum_{k=1}^{+\infty} g_{N_{k+1}} - g_{N_k}$ converges to an element $g \in X$. But, the series $\sum_{k=1}^{+\infty} g_{N_{k+1}} - g_{N_k}$ is telescopic and the partial sum

$$\sum_{k=1}^{M} g_{N_{k+1}} - g_{N_k} = g_{N_{M+1}} - g_{N_1},$$  

so that $g_{N_k} - g_{N_1} \to g$ as $k \to +\infty$. Hence, the original sequence must converge too, and the conclusion follows. $\square$

**Theorem 2.31.** Let $\{f_n\}$ be a sequence of functions in $L^1(\mu)$ such that $\sum_{n=1}^{+\infty} \| f \|_{L^1(\mu)} < +\infty$. Then, there exists $f \in L^1(\mu)$ such $\sum_{n=1}^{+\infty} f_n$ converges to $f$ in $L^1(\mu)$ and

$$\int f \, d\mu = \sum_{n=1}^{+\infty} \int f_n \, d\mu.$$  

As a consequence, $L^1(\mu)$ is a Banach space. Finally, the simple functions are dense in $L^1(\mu)$.

**Proof.** By Cor. 2.17 (ii) we know that

$$\int \sum_{n} |f_n| \, d\mu = \sum_{n} \int |f_n| \, d\mu \leq \sum_{n=1}^{+\infty} \| f_n \|_{L^1(\mu)} < +\infty.$$  

Then, setting $g = \sum_{n=1}^{+\infty} |f_n|$, we have $g \in L^1(\mu)$, and hence it is finite a.e. This implies that $\sum_{n=1}^{+\infty} f_n$ converges but on a set of measure 0, call $f$ such limit. We apply the DCT to the sequence of partial sums $s_n = \sum_{k=1}^{n} f_k$. Clearly, $s_n \to f$ pointwise, and also

$$|s_n| = \left| \sum_{k=1}^{n} f_k \right| \leq \sum_{k=1}^{n} |f_k| = g.$$  

Hence, the DCT applies to give that \( s_n \to f \) in \( L^1(\mu) \) and \( \lim_{n \to +\infty} \int s_n \, d\mu = \int f \, d\mu \), that is
\[
\int f \, d\mu = \lim_{n \to +\infty} \int \left( \sum_{k=1}^{n} f_k \, d\mu \right) = \lim_{n \to +\infty} \sum_{k=1}^{n} \int f_k \, d\mu = \sum_{n=1}^{+\infty} \int f_n \, d\mu.
\]
In particular, every absolutely convergent series also converges in \( L^1(\mu) \), hence \( L^1(\mu) \) is complete.

Finally, let \( f \in L^1(\mu) \). Then, \(|f|\) is finite a.e. and the sequence of simple functions \( \{\varphi_n\} \) in Thm. 2.9 (2), is such that \( \varphi_n \to f \) pointwise, and \(|\varphi_n| \leq |f|\). Therefore, we can apply the DCT to obtain that \( \varphi_n \to f \) in \( L^1(\mu) \).

We conclude this part by comparing \( L^1(\mu) \) with \( L^1(\bar{\mu}) \). If \( f \in L^1(\bar{\mu}) \) and \( g \) is given by by Prop. 2.12, then \( g = f \bar{\mu}\text{-a.e. and } g \in L^1(\mu) \). Conversely, if \( g \in L^1(\mu) \), then \( g \) is also \( \mathcal{M} \)-measurable and \( \int |g| \, d\mu = \int |g| \, d\bar{\mu} \). Then \( g \in L^1(\bar{\mu}) \). Therefore, we have shown that the spaces \( L^1(\mu) \) and \( L^1(\bar{\mu}) \) can be identified and we will do in what follows.

3. The Lebesgue measure on \( \mathbb{R} \)

Having established the definition and main properties of abstract integrals, we now go back to the construction of measures and in particular of the Lebesgue measure and of other Borel measures in \( \mathbb{R} \), and eventually in \( \mathbb{R}^n \).

3.1. Outer measures. Given a set \( \mathcal{X} \), an outer measure on \( \mathcal{X} \) is a set function \( \mu^* \) defined on the set of parts of \( \mathcal{X} \), \( \mathcal{P}(\mathcal{X}) \) such that

(i) \( \mu^*(\emptyset) = 0 \);

(ii) if \( A \subseteq B \subseteq \mathcal{X} \), then \( \mu^*(A) \leq \mu^*(B) \);

(iii) \( \mu^*( \bigcup_{j=1}^{+\infty} A_j ) \leq \sum_{j=1}^{+\infty} \mu^*(A_j) \), for all \( \{A_j\} \subseteq \mathcal{X} \).

The reason for this definition is that outer measures arise naturally when one has a family of elementary sets and a notion of measure for such sets.

**Proposition 3.1.** Let \( \mathcal{E} \) be a collection of sets in \( \mathcal{P}(\mathcal{X}) \) such that \( \emptyset, \mathcal{X} \in \mathcal{E} \), and let \( \rho : \mathcal{E} \to [0, +\infty] \) such that \( \rho(\emptyset) = 0 \) be given. Define
\[
\mu^*(A) = \inf \left\{ \sum_{j=1}^{+\infty} \rho(E_j) : A \subseteq \bigcup_{j=1}^{+\infty} E_j, \ E_j \in \mathcal{E} \right\}.
\]
Then \( \mu^* \) is an outer measure.

The key example to keep in mind is when \( \mathcal{E} \) is the collection of (all, or a subclass of) intervals \( I \) in \( \mathbb{R} \), and \( \rho(I) \) is the length of \( I \), which could be \( +\infty \) if \( I \) is a ray.

**Proof.** First of all we observe that the definition makes sense, since for any \( A \subseteq \mathcal{X} \) there exists a covering of \( A \) with sets in \( \mathcal{E} \), e.g. \( \{\mathcal{X}, \emptyset, \emptyset, \ldots\} \), where \( \emptyset, \mathcal{X} \in \mathcal{E} \). Also, it is clear that \( \mu^*(\emptyset) = 0 \), since \( \emptyset \) can be covered by \( \{\emptyset, \emptyset, \ldots\} \). Property (ii) in the definition of an outer measure is satisfied since, if \( A \subseteq B \), and \( \{E_j\} \) is a covering of \( B \) with sets in \( \mathcal{E} \), \( \{E_j\} \) is also a covering of \( A \). Thus, the infimum in the definition of \( \mu^*(A) \) is taken with respect a larger collection of coverings than in the case of \( \mu^*(B) \); hence (ii) holds.
Finally, let \( \{ A_j \} \subseteq \mathcal{P}(\mathcal{X}) \). Fix \( \varepsilon > 0 \). Then, by the properties of the infimum of real numbers, for each \( j \), there exists a covering \( \{ E_{j,k} \} \) of sets in \( \mathcal{E} \) of \( A_j \) such that
\[
\sum_{k=1}^{+\infty} \rho(E_{j,k}) < \mu^*(A_j) + \varepsilon 2^{-j}.
\]
Therefore, \( \{ E_{j,k} \} \) for \( j, k = 1, 2, \ldots \) is a covering of \( \bigcup_{j=1}^{+\infty} A_j \) by sets in \( \mathcal{E} \) such that
\[
\mu^* \left( \bigcup_{j=1}^{+\infty} A_j \right) = \inf \left\{ \sum_n \rho(E_n) : \left( \bigcup_{j=1}^{+\infty} A_j \right) \subseteq \bigcup_{n=1}^{+\infty} E_n \right\} \leq \sum_{j,k=1}^{+\infty} \rho(E_{k,j})
\]
\[
\leq \sum_{j=1}^{+\infty} \left( \mu^*(A_j) + \varepsilon 2^{-j} \right)
\]
\[
= \sum_{j=1}^{+\infty} \mu^*(A_j) + \varepsilon.
\]
Since \( \varepsilon > 0 \) was arbitrary, (iii) follows. \( \square \)

The key step to pass from an outer measure to an actual measure is based on the following definition.

**Definition 3.2.** Let \( \mu^* \) be an outer measure on a set \( \mathcal{X} \). We say that a set \( E \subseteq \mathcal{X} \) is \( \mu^* \)-measurable if
\[
\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c),
\]
for all \( A \subseteq \mathcal{X} \).

Observe that we always have the inequality
\[
\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c),
\]
by subadditivity. Then, in order to prove the \( \mu^* \)-measurability of a set \( E \), we only need to prove the reverse inequality
\[
\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c),
\]
for all sets \( A \subseteq \mathcal{X} \). This inequality is trivial if \( \mu^*(A) = +\infty \). Therefore, \( E \subseteq \mathcal{X} \) is \( \mu^* \)-measurable if and only if
\[
\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \text{for all } A \text{ with } \mu^*(A) < +\infty.
\]

The key step to pass from an outer measure to an actual measure is the following result, known as Carathéodory’s theorem.

**Theorem 3.3. (Carathéodory’s Theorem.)** Let \( \mathcal{X} \) be a set and \( \mu^* \) an outer measure on \( \mathcal{X} \). Let \( \mathcal{M} \) be the collection of \( \mu^* \)-measurable sets, and \( \mu \) the restriction of \( \mu^* \) to \( \mathcal{M} \). Then \( \mathcal{M} \) is a \( \sigma \)-algebra, and \( \mu \) is a complete measure on \( \mathcal{M} \).

**Proof.** We first observe that \( E \in \mathcal{M} \) implies that also \( E^c \in \mathcal{M} \) since the condition (10) is symmetric in \( E \) and \( E^c \). In order to show that \( \mathcal{M} \) is a \( \sigma \)-algebra, it suffices to show that it is
closed under countable unions of disjoint sets – see the comment regarding formula (1). To this end, we first show that \( \mathcal{M} \) is closed under finite unions. Let \( E, F \in \mathcal{M} \) and let \( A \subseteq \mathcal{X} \). Then
\[
\mu^*(A) = \mu^*((A \cap E)) + \mu^*((A \cap cE)) = \mu^*((A \cap E) \cap F) + \mu^*((A \cap E) \cap cF) + \mu^*((A \cap cE) \cap F) + \mu^*((A \cap cE) \cap cF).
\]
We observe that
\[
E \cup F = (E \cap F) \cup (E \cap cF) \cup (cE \cap F)
\]
so that, using subadditivity
\[
\mu^*(E \cup F) = \mu^*((E \cup F) \cap E) + \mu^*((E \cup F) \cap cE) = \mu^*(E) + \mu^*(F),
\]
so that \( \mu^* \) is finitely additive in \( \mathcal{M} \). In order to show that \( \mathcal{M} \) is a \( \sigma \)-algebra, it suffices to show that it is closed under countable union of disjoint subsets. Then, let \( \{E_j\} \subseteq \mathcal{M} \) be a sequence of disjoint sets. Setting \( F = \bigcup_{j=1}^{+\infty} E_j \), we wish to show that \( F \) is \( \mu^* \)-measurable.

To this end, set \( F_n = \bigcup_{j=1}^{n} E_j \), and let \( A \subseteq \mathcal{X} \). Then
\[
\mu^*(A \cap F_n) = \mu^*(A \cap F_n \cap E_n) + \mu^*(A \cap F_n \cap cE_n) = \mu^*(A \cap E_n) + \mu^*(A \cap F_{n-1}),
\]
as it is easy to check. Arguing by induction we then obtain that
\[
\mu^*(A \cap F_n) = \sum_{j=1}^{n} \mu^*(A \cap E_j).
\]
It follows that
\[
\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap cF_n) \geq \sum_{j=1}^{n} \mu^*(A \cap E_j) + \mu^*(A \cap cF),
\]
for every \( n \geq 1 \). Letting \( n \to +\infty \) we obtain
\[
\mu^*(A) \geq \sum_{j=1}^{+\infty} \mu^*(A \cap E_j) + \mu^*(A \cap cF) \geq \mu^* \left( \bigcup_{j=1}^{+\infty} (A \cap E_j) + \mu^*(A \cap cF) \right)
\]
\[
= \mu^*(A) + \mu^*(A \cap cF) = \mu^*(A).
\]
Thus, the above ones are all equalities and \( F \) is \( \mu^* \)-measurable, as we wanted to show. Moreover, taking \( A = F \) in the above equalities, we obtain
\[
\mu^* \left( \bigcup_{j=1}^{+\infty} E_j \right) = \mu^*(F) = \sum_{j=1}^{+\infty} \mu^*(E_j),
\]
that is, \( \mu^* \) is countable additive on \( \mathcal{M} \).
Finally, we need to show that $\mathcal{M}$ is complete. Suppose that $\mu^*(N) = 0$ and let $A \subseteq X$ be any set. Then,

$$\mu^*(A) \leq \mu^*(A \cap N) + \mu^*(A \cap cN) = \mu^*(A \cap cN) \leq \mu^*(A),$$

so that $N \in \mathcal{M}$ and $\mathcal{M}$ is complete. $\square$

The application of Carathéodory’s theorem we have in mind deals with the notion of premeasure.

**Definition 3.4.** Given a set $X$ and an algebra $\mathcal{A}$ of subsets of $X$, a set function $\rho : \mathcal{A} \rightarrow [0, +\infty]$ is called a premeasure if

(i) $\rho(\emptyset) = 0$;

(ii) if $\{A_j\}$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_{j=1}^{+\infty} A_j \in \mathcal{A}$, then

$$\rho\left(\bigcup_{j=1}^{+\infty} A_j\right) = \sum_{j=1}^{+\infty} \rho(A_j).$$

Notice that a premeasure satisfies the monotonicity condition: if $E, F \in \mathcal{A}$, $E \subseteq F$, then $\rho(E) \leq \rho(F)$. Indeed, writing $F = E \cup (F \cap cE)$, the conclusion follows easily.

**Lemma 3.5.** Let $X$ be a set, $\mathcal{A}$ an algebra of subsets of $X$ and $\rho : \mathcal{A} \rightarrow [0, +\infty]$ a premeasure, and let $\mu^*$ be given by

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{+\infty} \rho(E_j) : A \subseteq \bigcup_{j=1}^{+\infty} E_j, \ E_j \in \mathcal{E} \right\}.$$  \hspace{1cm} (11)

Then $\mu^*$ is an outer measure such that the following properties hold true:

(i) $\mu^*|_A = \rho$;

(ii) every set $E \in \mathcal{A}$ is $\mu^*$-measurable.

**Proof.** By Prop. 3.1 we know that $\mu^*$ is an outer measure. (i) Let $E \in \mathcal{A}$. Then $E \subseteq \bigcup_{j=1}^{+\infty} E_j$, where $E_1 = E$ and $E_j = \emptyset$ for $j = 2, 3, \ldots$. Then

$$\mu^*(E) \leq \sum_{j=1}^{+\infty} \rho(E_j) = \rho(E).$$

Thus, it suffices to prove the reverse inequality. Let $E \in \mathcal{A}$ and let $\{E_j\}$ be any covering of $E$ with sets in $\mathcal{A}$, i.e. $E_j \in \mathcal{A}$, $j = 1, 2, \ldots$, and $E \subseteq \bigcup_{j=1}^{+\infty} E_j$. Set $A_n = E_n \setminus \bigcup_{j=1}^{n-1} E_j$. Then the $A_n$’s are disjoint elements of $\mathcal{A}$, as it is easy to check, and their union is $E$. Therefore, by the definition of premeasure,

$$\rho(E) = \sum_{n=1}^{+\infty} \rho(A_n) \leq \sum_{n=1}^{+\infty} \rho(E_n).$$

Taking the infimum on the right hand side over the coverings of $E$ by sets in $\mathcal{A}$, it follows that $\rho(E) \leq \mu^*(E)$ for $E \in \mathcal{A}$. This proves (i).
Lemma 3.6. Let $E \in A$. In order to show that $E$ is $\mu^*$-measurable, it suffices to prove (10). Let $B \subseteq \mathcal{X}$. Given $\varepsilon > 0$, there exists a collection $\{A_j\} \subseteq A$ such that $B \subseteq \cup_{j=1}^{+\infty} A_j$ and $\sum_{j=1}^{+\infty} \rho(A_j) \leq \mu^*(B) + \varepsilon$. Then, by the additivity of $\rho$ on $A$,

$$\mu^*(B) + \varepsilon \geq \sum_{j=1}^{+\infty} \left( \rho(A_j \cap E) + \rho(A_j \cap \complement E) \right) \geq \mu^*(B \cap E) + \mu^*(B \cap \complement E),$$

since $\{A_j \cap E\}$ and $\{A_j \cap \complement E\}$ are coverings of $B \cap E$ and $B \cap \complement E$, resp., by elements of $A$. Since, $\varepsilon > 0$ was arbitrary, this proves (ii).

\[\square\]

**Lemma 3.6.** Let $\mathcal{X}, A, \rho$ and $\mu^*$ be as in Lemma 3.5. Then

(i) Given any $B \subseteq \mathcal{X}$ and $\varepsilon > 0$ there exists a collection $\{A_j\} \subseteq A$ be such that $B \subseteq \cup_{j=1}^{+\infty} A_j$ and $\mu^*\left( \cup_{j=1}^{+\infty} A_j \right) \leq \mu^*(B) + \varepsilon$.

Assume further that $\rho$ is $\sigma$-finite and let $\mathcal{M}(A)$ be the $\sigma$-algebra generated by $A$. The following properties hold true.

(ii) A set $E \subseteq \mathcal{X}$ is $\mu^*$-measurable if and only if there exists $A \in \mathcal{M}(A)$ such that $E \subseteq A$ and $\mu^*(A \setminus E) = 0$.

(iii) A set $E \subseteq \mathcal{X}$ is $\mu^*$-measurable if and only if there exists $B \in \mathcal{M}(A)$ such that $B \subseteq E$ and $\mu^*(E \setminus B) = 0$.

**Proof.** (i) If $\mu^*(B)$ we have nothing to prove. Suppose that $\mu^*(B)$ is finite. Then, given $\varepsilon > 0$, by definition of infimum, there exists $\{A_j\} \subseteq A$ such that $B \subseteq \cup_{j=1}^{+\infty} A_j$ and $\sum_{j=1}^{+\infty} \rho(A_j) \leq \mu^*(B) + \varepsilon$. Using Lemma 3.5 (i) we then have

$$\mu^*\left( \bigcup_{j=1}^{+\infty} A_j \right) \leq \sum_{j=1}^{+\infty} \mu^*(A_j) = \sum_{j=1}^{+\infty} \rho(A_j) \leq \mu^*(B) + \varepsilon,$$

as we wished to prove.

(ii) One direction is obvious. If $E \subseteq \mathcal{X}$ and there exists $B \in \mathcal{M}(A)$ such that $B \subseteq E$ and $\mu^*(E \setminus B) = 0$, then $E = B \cup (E \setminus B)$. It follows that $E$ is $\mu^*$-measurable since $B \in \mathcal{M}(A)$ and all sets of $\mu^*$-measure 0 are $\mu^*$-measurable.

Conversely, suppose $E \subseteq \mathcal{X}$ is $\mu^*$-measurable. Assume first that $\mu^*(E) < +\infty$. For each $k = 1, 2, \ldots$, we apply part (i) to $E$, with $\varepsilon = 1/k$. We find a collection of sets $\{A_j^{(k)}\}$ such that $A^{(k)} := \bigcup_{j=1}^{+\infty} A_j^{(k)} \supseteq E$ and $\mu^*(A^{(k)}) \leq \mu^*(E) + \frac{1}{k}$. Set

$$A = \bigcap_{k=1}^{+\infty} A^{(k)}.$$

Then $A \supseteq E$, so that $\mu^*(E) \leq \mu^*(A)$. Moreover, for each $k$,

$$\mu^*(A) = \mu^*\left( \bigcap_{k=1}^{+\infty} A^{(k)} \right) \leq \mu^*(A^{(k)}) \leq \mu^*(E) + \frac{1}{k},$$

so that $\mu^*(A) = \mu^*(E)$. Since $E$ is $\mu^*$-measurable,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \complement E) = \mu^*(E) + \mu^*(A \setminus E).$$
Since \( \mu^*(A) = \mu^*(E) < +\infty \), we can subtract it from both sides and obtain that \( \mu^*(A \setminus E) = 0 \). This proves (ii) when \( \mu^*(E) < +\infty \).

Next, suppose \( E \) is \( \mu^* \)-measurable and \( \mu^*(E) = +\infty \). Here we use the assumption that \( \rho \) is \( \sigma \)-finite, that is that \( \mathcal{X} = \bigcup_{j=1}^{+\infty} X_j \) with \( \rho(X_j) < +\infty \). We may assume that the \( X_j \)'s are disjoint. Let \( E_j = E \cap X_j \). By Lemma 3.5 (ii) we know that the \( X_j \)'s are \( \mu^* \)-measurable, so is \( E_j \) for each \( j \).

Then, for each \( j \) there exists \( A_j \in \mathcal{M}(\mathcal{A}) \) such that \( E_j \subseteq A_j \) and \( \mu^*(A_j \setminus E_j) = 0 \). Setting \( A = \bigcup_{j=1}^{+\infty} A_j \) we have that \( E \subseteq A \) and

\[
\mu^*(A \setminus E) = \mu^* \left( \bigcup_{j=1}^{+\infty} (A_j \setminus E_j) \right) \leq \sum_{j=1}^{+\infty} \mu^* (A_j \setminus E_j) = 0.
\]

This proves (ii).

Finally we prove (iii). We first assume that \( \mu^*(E) < +\infty \). We apply (ii) (which is now valid for all \( \mu^* \)-measurable sets) to \( F = \complement E \) and find \( A \in \mathcal{A}, A \supseteq \complement E \) and \( \mu^*(A \setminus \complement E) = 0 \). Setting \( B = \complement A \), we have \( B \in \mathcal{M}(\mathcal{A}), B \subseteq E \) and since \( E \) is \( \mu^* \)-measurable,

\[
\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap \complement E) = \mu^*(B) + \mu^*(B \cap \complement E).
\]

Now, \( \mu^*(B) \leq \mu^*(E) < +\infty \), so we can subtract it on both sides of the line of equations above. We then obtain

\[
\mu^*(B) + \mu^*(B \cap \complement E) = 0.
\]

The proof of the case \( \mu^*(E) \) is similar to the analogous case in (ii). This completes the proof. \( \square \)

We are finally ready to prove the result about the construction of a complete measure, that will be used in the next session to construct the Lebesgue measure on \( \mathbb{R} \) as a particular instance.

**Theorem 3.7.** Let \( \mathcal{X}, \mathcal{A}, \rho \) and \( \mu^* \) be as in Lemma 3.5. Further, assume that \( \rho \) is \( \sigma \)-finite. Let \( \mathcal{M} \) be the \( \sigma \)-algebra of the \( \mu^* \)-measurable sets and let \( \overline{\mu} = \mu^*|_{\mathcal{M}} \) be the complete measure given by Thm. 3.3.

Let \( \mathcal{M}(\mathcal{A}) \) be the \( \sigma \)-algebra generated by \( \mathcal{A} \) and let \( \nu \) be any measure on \( \mathcal{M}(\mathcal{A}) \) whose restriction to \( \mathcal{A} \) coincides with \( \rho \). Then \( \overline{\nu} \) is the completion of \( \nu \) and

\[
\overline{\mathcal{M}} = \mathcal{M}(\mathcal{A}) \cup \mathcal{N}, \tag{12}
\]

where \( \mathcal{N} = \{ N \subseteq \mathcal{X} : \text{there exists } F \in \mathcal{M}(\mathcal{A}), N \subseteq F, \nu(F) = 0 \} \).

**Proof.** (†) In order to show that \( (\overline{\nu}, \overline{\mathcal{M}}) \) is the completion of \( (\nu, \mathcal{M}(\mathcal{A})) \) we need to show that \( \overline{\nu}|_{\mathcal{M}(\mathcal{A})} = \nu \) and that (12) holds. We begin with the former one.

Let \( \mu^* \) be the outer measure given by (11) in Lemma 3.5. Let \( (\overline{\mathcal{M}}, \overline{\mu}) \) be the complete measure constructed in Thm. 3.3, starting from \( \mu^* \). By Lemma 3.5 we know that \( \mathcal{A} \subseteq \overline{\mathcal{M}} \). Hence \( \mathcal{M}(\mathcal{A}) \subseteq \overline{\mathcal{M}} \). Now, let \( E \in \mathcal{M}(\mathcal{A}) \) and let \( \{ A_j \} \subseteq \mathcal{A} \) be such that \( E \subseteq \bigcup_{j=1}^{+\infty} A_j \). Then

\[
\nu(E) \leq \sum_{j=1}^{+\infty} \nu(A_j) = \sum_{j=1}^{+\infty} \rho(A_j)
\]

so that

\[
\nu(E) \leq \inf \left\{ \sum_{j=1}^{+\infty} \rho(A_j) : E \subseteq \bigcup_{j=1}^{+\infty} A_j, A_j \in \mathcal{A} \right\} = \mu^*(E) = \overline{\mu}(E),
\]

where
since $E$ in particular is in $\overline{M}$. Thus, $\nu(E) \leq \overline{\mu}(E)$ for all $E \in \mathcal{M}(A)$. Moreover, setting $A = \bigcup_{j=1}^{+\infty} A_j$, since $\nu$ and $\overline{\mu}$ coincide on $\mathcal{A}$, we have that

$$\nu(A) = \lim_{n \to +\infty} \nu\left( \bigcup_{j=1}^{n} A_j \right) = \lim_{n \to +\infty} \overline{\mu}\left( \bigcup_{j=1}^{n} A_j \right) = \overline{\mu}(A).$$

Conversely, assume first that $\overline{\mu}(E) < +\infty$. Then, given $\varepsilon > 0$, there exists a collection $\{A_j\} \subseteq \mathcal{A}$ such that $E \subseteq \bigcup_{j=1}^{+\infty} A_j$ and $\overline{\mu}(A) \leq \overline{\mu}(E) + \varepsilon$, so that $\overline{\mu}(A \setminus E) < \varepsilon$. Then, using the first part too, we have that

$$\overline{\mu}(E) \leq \overline{\mu}(A) = \nu(A) = \nu(E) + \nu(A \setminus E) \leq \nu(E) + \overline{\mu}(A \setminus E) \leq \nu(E) + \varepsilon.$$  

Since $\varepsilon > 0$ was arbitrary, we have $\overline{\mu}(E) \leq \nu(E)$, hence $\overline{\mu}(E) = \nu(E)$, if $\overline{\mu}(E) < +\infty$ and $E \in \mathcal{M}(A)$. Finally, suppose $E \in \mathcal{M}(A)$ and $\overline{\mu}(E) = +\infty$. Here we assume that $\rho$ is $\sigma$-finite, that is that $\mathcal{X} = \bigcup_{j=1}^{+\infty} A_j$ with $\rho(A_j) < +\infty$. We may assume that the $A_j$’s are disjoint. Then

$$\overline{\mu}(E) = \sum_{j=1}^{+\infty} \overline{\mu}(E \cap A_j) = \sum_{j=1}^{+\infty} \nu(E \cap A_j) \leq \nu(E).$$

Hence, $\nu = \overline{\mu}$ on $\mathcal{M}(A)$.

The fact that (12) holds follows from Lemma 3.6 (ii). Indeed, Lemma 3.6 (ii) says that $E \subseteq \mathcal{X}$ is in $\overline{M}$ if and only if there exists $B \in \mathcal{M}(A)$ such that

$$E = B \cup (E \setminus B)$$

with $\overline{\mu}(E \setminus B) = 0$, where clearly $(E \setminus B) \in \overline{M}$. This proves (12) and therefore the theorem.  

3.2. The Lebesgue measure on $\mathbb{R}$. We now are now ready to introduce the main object of this course.

We consider the collection $\mathcal{A}$ of finite unions of disjoint left-open/right-closed intervals in $\mathbb{R}$, that is,

$$\mathcal{A} = \left\{ E \subseteq \mathbb{R} : E = \bigcup_{j=1}^{n} I_j, I_j \text{ disjoint}, I_j = (a_j, b_j] \text{ or } I_j = (a_j, +\infty), -\infty \leq a_j < b_j < +\infty \right\}.$$  

Lemma 3.8. Set $\rho(\emptyset) = 0$, $\rho(E) = +\infty$ if $E \in \mathcal{A}$ is unbounded, and if the intervals $\{(a_j, b_j]\}$, $j = 1, 2, \ldots, n$ are disjoint and $E = \bigcup_{j=1}^{n} (a_j, b_j]$, then

$$\rho(E) = \sum_{j=1}^{n} (b_j - a_j).$$

Then $\rho$ is a premeasure on the algebra $\mathcal{A}$.

Proof. It is clear that $E$ is well defined, that is, if $E = \bigcup_{j=1}^{n} (a_j, b_j] = \bigcup_{k=1}^{m} (c_k, d_k]$, then

$$\sum_{j=1}^{n} (b_j - a_j) = \sum_{k=1}^{m} (d_k - c_k).$$

It is also easy to see that $\rho$ is finitely additive. In fact, both assertions follow from the fact that an element of $\mathcal{A}$ can be written in a unique way as disjoint union of maximal disjoint left-open/right-closed intervals in $\mathbb{R}$. 
Thus, it remains to show that if \( E \) is countable union of disjoint left-open/right-closed intervals 
\( E = \bigcup_{j=1}^{+\infty} (a_j, b_j) \), and \( E \in \mathcal{A} \), then \( \rho(E) = \sum_{j=1}^{+\infty} \rho((a_j, b_j)) = \sum_{j=1}^{+\infty} (b_j - a_j) \). Since \( E \in \mathcal{A} \), it can be written as finite union of disjoint left-open/right-closed intervals or of the form \((a, +\infty)\), say \( I_j, j = 1, \ldots, n \). Using the finite additivity of \( \rho \), it suffices to consider the case when \( E \) is itself a left-open/right-closed interval \((a, b]\), or \((a, +\infty)\). Suppose then that \( I = \bigcup_{j=1}^{+\infty} I_j \), where \( I_j = (a_j, b_j] \). Notice that this includes also the case \( I = (a, +\infty) \). Since for each \( n \), 
\[ I = \left( \bigcup_{j=1}^{n} I_j \right) \cup \left( \bigcup_{j=n+1}^{+\infty} I_j \right) =: \left( \bigcup_{j=1}^{n} I_j \right) \cup J, \]
with \( J \in \mathcal{A} \), we have 
\[ \rho(I) = \rho \left( \bigcup_{j=1}^{n} I_j \right) + \rho(J) = \sum_{j=1}^{n} \rho(I_j) + \rho(J) \geq \sum_{j=1}^{n} \rho(I_j) \] 
for all \( n \). Hence, 
\[ \rho(I) \geq \sum_{j=1}^{+\infty} \rho(I_j). \]

Conversely, if \( \sum_{j=1}^{+\infty} \rho(I_j) = +\infty \) we have nothing to prove. Hence, assume that \( \sum_{j=1}^{+\infty} \rho(I_j) < +\infty \). Notice that this implies that \( I \) must be bounded. Let \( \varepsilon > 0 \) be given. Then, the sets \( \{(a_j, b_j + \varepsilon 2^{-j}) : j = 1, 2, \ldots \} \) form an open cover of \( I \). Therefore, there exists a finite subcollection \( \{(a_{j\ell}, b_{j\ell} + \varepsilon 2^{-j\ell}) : \ell = 1, 2, \ldots, m \} \) that still covers \( I \). Hence, \( I \subseteq \bigcup_{\ell=1}^{m} (a_{j\ell}, b_{j\ell} + \varepsilon 2^{-j\ell}] \). Using the monotonicity and the finite subadditivity (that follows from the finite additivity) of \( \rho \), we have 
\[ \rho(I) \leq \rho \left( \bigcup_{\ell=1}^{m} (a_{j\ell}, b_{j\ell} + \varepsilon 2^{-j\ell}] \right) \leq \sum_{\ell=1}^{m} \rho((a_{j\ell}, b_{j\ell} + \varepsilon 2^{-j\ell}] ) \]
\[ \leq \sum_{\ell=1}^{m} \rho(I_{j\ell}) + \varepsilon 2^{-j\ell} \leq \sum_{j=1}^{+\infty} (\rho(I_j) + \varepsilon 2^{-j}) \]
\[ \leq \sum_{j=1}^{+\infty} \rho(I_j) + \varepsilon, \]
for every \( \varepsilon > 0 \). Since \( \varepsilon > 0 \) was arbitrary, this proves the inequality \( \rho(I) \leq \sum_{j=1}^{+\infty} \rho(I_j) \), hence the lemma. \( \square \)

**Definition 3.9.** Let \( \mathcal{A} \) and \( \rho \) be as in Lemma 3.8, \( \mu^* \) the outer measure defined in (11) in Lemma 3.5. We define the Lebesgue measure space on \((\mathbb{R}, \mathcal{L}, m)\) where \( \mathcal{L} \) is the \( \sigma \)-algebra and as the complete measure \( m = \mu \) constructed in Thm. 3.7. The measure \( m \) is called the **Lebesgue measure** on \( \mathbb{R} \) and \( \mathcal{L} \) the \( \sigma \)-algebra of **Lebesgue measurable sets**.

**Remark 3.10.** We collect here some obvious but fundamental properties of the Lebesgue measure in \( \mathbb{R} \).

1. We begin by observing that the \( \sigma \)-algebra of Lebesgue measurable sets \( \mathcal{L} \), by Thm. 3.3 is the completion of the \( \sigma \)-algebra generated by the algebra \( \mathcal{A} \) of the finite unions of disjoint left-open/right-closed intervals in \( \mathbb{R} \). Then \( \mathcal{L} \) contains \( \mathcal{B}_{\mathbb{R}} \), the \( \sigma \)-algebra of Borel sets in \( \mathbb{R} \). More precisely, using Thm. 3.7 we have that, 
\[ \mathcal{L} = \mathcal{B}_{\mathbb{R}} \cup \mathcal{N}, \]
where \( \mathcal{N} = \{N \subseteq \mathbb{R} : \text{there exists } F \in \mathcal{B}_\mathbb{R}, N \subseteq F, m(F) = 0\} \).

(2) For each interval \( I \subseteq \mathbb{R} \), \( m(I) \) equals the length of \( I \) (possibly \( +\infty \)).

(3) Each open set \( A \subseteq \mathbb{R} \) has positive measure, or possibly \( = +\infty \). Indeed, it suffices to notice that \( A \) is at most countable union of disjoint open intervals \( I_n, n = 1, 2, \ldots \), since then \( m(A) \geq m(I) > 0 \). Then, given \( x \in A \), there exists an open interval \( I_x \subseteq A \) and containing \( x \). Let \( I_1 \) be the union of all such intervals. If \( I_1 = A \) we are done. Otherwise, there exists \( y \in A \setminus I_1 \), and an open interval \( I_y \) contained in \( A \) and disjoint from \( I_1 \). Let \( z \in I_y \cap \mathbb{Q} \). Define \( I_2 \) be the union of all open intervals containing \( z \) and contained in \( A \setminus I_1 \). This process is at most of countably many steps, and the conclusion follows.

(4) Each point \( \{x\} \) has measure 0, so countable sets all have measure 0. In particular, the function \( \chi_{[0,1] \cap \mathbb{Q}} \) is integrable and \( \int \chi_{[0,1] \cap \mathbb{Q}} \, dm = 0 \), and the function \( \chi_{[0,1] \cap (\mathbb{R} \setminus \mathbb{Q})} \) is also integrable and \( \int \chi_{[0,1] \cap (\mathbb{R} \setminus \mathbb{Q})} \, dm = \int \left( \chi_{[0,1]} - \chi_{[0,1] \cap (\mathbb{R} \setminus \mathbb{Q})} \right) \, dm = 1 \).

We now see other fundamental properties of the Lebesgue measure.

**Proposition 3.11.** Let \( E \in \mathcal{L} \). Then

(i) \( m(E) = \inf \{m(U) : E \subseteq U, U \text{ open}\} \),

and also

(ii) \( m(E) = \sup \{m(K) : K \subseteq E, K \text{ compact}\} \).

**Proof.** We preliminary observe that, if \( E \in \mathcal{L} \), then

\[
m(E) = \inf \left\{ \sum_{j=1}^{+\infty} (b_j - a_j) : E \subseteq \bigcup_{j=1}^{+\infty} (a_j, b_j) \right\}. \tag{13}
\]

Call \( \nu(E) \) the quantity on the right hand side above. By construction we have that

\[
m(E) = \inf \left\{ \sum_{j=1}^{+\infty} (b_j - a_j) : E \subseteq \bigcup_{j=1}^{+\infty} (a_j, b_j) \right\},
\]

so that clearly \( m(E) \leq \nu(E) \) (since in the case of \( m \) we take the infimum over a larger numerical set). On the other hand, given \( \varepsilon > 0 \), let \( \{(a_j, b_j)\} \) be a countable collection of left-open/right-closed intervals whose union covers \( E \) and such that \( m(E) + \varepsilon \geq \sum_{j=1}^{+\infty} (b_j - a_j) \). Then \( \{(a_j, b_j + \varepsilon 2^{-j})\} \) is a collection of open intervals whose union covers \( E \) and such that

\[
\sum_{j=1}^{+\infty} (b_j + \varepsilon 2^{-j} - a_j) = \sum_{j=1}^{+\infty} (b_j - a_j) + \varepsilon \leq m(E) + 2\varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, \( \nu(E) \leq m(E) \) and (13) follows.

(i) Since, if \( E \subseteq U \) we have \( m(E) \leq m(U) \), it is clear that

\[
m(E) \leq \inf \{m(U) : U \text{ open}, U \supseteq E\}.
\]

From (13) we know that given \( \varepsilon > 0 \) there exists a collection of open intervals \( \{I_j\} \) whose union covers \( E \) and such that

\[
m(E) + \varepsilon \geq \sum_{j=1}^{+\infty} m(I_j) \geq m\left( \bigcup_{j=1}^{+\infty} I_j \right) = m(U).
\]
This shows that \( m(E) \geq \inf\{m(U) : U \text{ open, } U \supseteq E\} \), and proves (i).

(ii) Suppose first that \( E \) is bounded. Notice that this implies that \( E \) is contained in some bounded interval \((a, b]\), so it has finite measure. If \( E \) is closed, then it is compact and the conclusion is obvious. Assume now that \( E \) is not closed. Then \( \overline{E} \setminus E \) is not empty, and part (i) gives that given \( \varepsilon > 0 \) there exists an open set \( U \supseteq (\overline{E} \setminus E) \) such that \( m(U) \leq m(\overline{E} \setminus E) + \varepsilon \).

Let \( K = E \setminus U = \overline{E} \setminus U \). Then \( K \subseteq E \) and it is compact, as they are both easy to check. Then,

\[
m(K) = m(E) - m(U) = m(E) - [m(U) - m(\overline{E} \setminus E)] \geq m(E) - \varepsilon.
\]

Thus (ii) holds in this case. Suppose now \( E \) is unbounded and define \( E_j = E \cap (j, j+1] \). We apply the argument above and, given \( \varepsilon > 0 \), there exist compact sets \( K_j, j \in \mathbb{Z} \) such that \( K_j \subseteq E_j \) and \( m(K_j) \leq m(E_j) + \varepsilon 2^{-|j|} \). Let \( H_n = \bigcup_{j=-n}^{n} K_j \), so that \( H_n \) is compact and \( H_n \subseteq \bigcup_{j=-n}^{n} E_j \subseteq E \).

Thus,

\[
m(H_n) = \sum_{j=-n}^{n} m(K_j) \geq \sum_{j=-n}^{n} (m(E_j) - \varepsilon 2^{-|j|}) \geq \sum_{j=-n}^{n} m(E_j) - 2\varepsilon.
\]

Since \( \sum_{j=-n}^{n} m(E_j) \rightarrow m(E) \) as \( n \rightarrow +\infty \), we have that

\[
m(H_n) \geq m(E) - 3\varepsilon,
\]

for \( n \) large enough. This gives (ii) also for unbounded sets, and we are done. \( \square \)

**Example 3.12.** Given the sets \( D = \mathbb{R} \setminus \mathbb{Q} \) and \( D_1 = D \cap (0, 1] \). Given \( M, \varepsilon > 0 \) determine \( H, K \) compacts, \( H \subseteq D, K \subseteq D_1 \) such that \( m(H) > M \) and \( m(K) > 1 - \varepsilon \).

Notice that the statement about \( D \) follows easily from the case of \( D_1 \) by setting \( H = \bigcup_{j=-n}^{n} (K + j) \) and \( n \) large enough so that \( m(H) = \sum_{j=-n}^{n} m(K) > (2n + 1)(1 - \varepsilon) > M \).

In the case of \( D_1 \), we observe that the proof of Thm. 3.11 (ii) provides the construction of such a set. Let \( E = D_1 \), so that \( \overline{E} = [0, 1] \) and \( \overline{E} \setminus E = \mathbb{Q} \cap [0, 1] \). Let \( U \) be an open set such that \( U \supseteq (\mathbb{Q} \cap [0, 1]) \) and \( m(U) \leq m((\mathbb{Q} \cap [0, 1])) + \varepsilon = \varepsilon \). In fact, given \( \varepsilon > 0 \), we may choose \( U = \bigcup_{j=1}^{+\infty} I_{q_j} \), where \( \{q_j\} \) is a denumeration of the rational numbers in \([0, 1]\) and \( I_{q_j} = (q_j - \varepsilon 2^{-(j+1)}, q_j + \varepsilon 2^{-(j+1)}) \). Then, we set \( K = E \setminus U = \overline{E} \setminus U \), that is,

\[
K = [0, 1] \setminus \bigcup_{j=1}^{+\infty} I_{q_j},
\]

which is clearly closed, bounded, hence compact, and

\[
m(K) \geq 1 - \sum_{j=1}^{+\infty} m(I_{q_j}) = 1 - \sum_{j=1}^{+\infty} \varepsilon 2^{-j} = 1 - \varepsilon. \quad \square
\]

**Proposition 3.13.** Let \( E \in \mathcal{L} \). Then \( E + y = \{x + y : x \in E\} \in \mathcal{L} \) for every \( y \in \mathbb{R} \) and \( m(E + y) = m(E) \). If \( r \in \mathbb{R} \) and \( rE = \{x + y : x \in E\} \), then \( rE \in \mathcal{L} \) and \( m(rE) = |r|m(E) \).
Proof. Since the collection of left-open/right-closed intervals in $\mathbb{R}$ is translation invariant, so is $\mathcal{B}_R$. Given $y, r \in \mathbb{R}$, we then define $m_y(E) = m(E + y)$ and $m^r(E) = m(rE)$. We consider the case of $m_y$ first. It is easy to see that $m_y$ is a measure on $\mathcal{B}_R$. Next, since the premeasure $\rho$ is invariant by translation, if $A \in \mathcal{A}$ then

$$m_y(A) = m(A + y) = \rho(A + y) = \rho(A),$$

so that $m_y|_\mathcal{A} = \rho$. Thus, we can apply the second part of Thm. 3.7 and see that $(m, \mathcal{L})$ is the completion of $(m_y, \mathcal{B}_R)$ and that $m = m_y$ on $\mathcal{B}_R$. In particular, if $E \in \mathcal{B}_R$ and $m(E) = 0$, then $m(E + y) = m_y(E) = 0$ which implies that the set of Lebesgue measure 0 are preserved by translation. Thus, also $m_y$ is complete and $m_y = m$, that is, $m$ is translation invariant.

The same proof shows that $m^r = rm$ when $r > 0$ is positive, since $\mathcal{A}$ is preserved by positive dilations, and $m^r = r\rho$ on $\mathcal{A}$. In this case we apply the second part of Thm. 3.7 to $m^r$, the premeasure $r\rho$ and the complete measure $rm$. We leave the elementary details to the reader.

To treat the case $r < 0$, it suffices to consider the case $r = -1$, that is, to reflection about the origin. In this case we need to start with the algebra $\mathcal{A}$ generated by left-closed/right-open intervals instead. Again, we leave the details to the reader. \qed

As an immediate corollary we obtain that the set $N$ constructed in Example 1.1 cannot be Lebesgue measurable. Thus, $\mathcal{L}$ is strictly contained in $\mathcal{P}(\mathbb{R})$.

The next result now follows easily from the previous result. We leave the proof as an exercise.

**Proposition 3.14.** Let $E \subseteq \mathbb{R}$. Then the following properties are equivalent.

(i) $E \in \mathcal{L}$;

(ii) there exists a $\mathcal{G}_\delta$-set $V$ and $N \in \mathcal{L}$ with $m(N) = 0$ such that $E = V \setminus N$;

(iii) there exists an $\mathcal{F}_\sigma$-set $W$ and $N' \in \mathcal{L}$ with $m(N') = 0$ such that $E = W \cup N'$.

We now compare the Lebesgue integral with the Riemann integral. We write $\mathcal{R}([a, b])$ to denote the space of Riemann integrable functions on the compact interval $[a, b]$ and we write $\mathcal{R} \int_a^b f(t) \, dt$ to denote the Riemann integral of $f \in \mathcal{R}([a, b])$. We recall that the definition of $f \in \mathcal{R}([a, b])$ requires $f$ to be bounded on $[a, b]$.

**Proposition 3.15.** The following properties hold true.

(i) If $f : [a, b] \to \mathbb{R}$ and $f \in \mathcal{R}([a, b])$, then $f$ is Lebesgue integrable on $[a, b]$ and

$$\int_{[a,b]} f \, dm = \mathcal{R} \int_a^b f(t) \, dt.$$

(ii) If $f : [a, +\infty) \to [0, +\infty)$, $f \in \mathcal{R}([a, b])$, for all $b > a$ and there exists the improper Riemann integral $\mathcal{R} \int_a^{+\infty} f(t) \, dt$, then $f$ is Lebesgue integrable on $[a, +\infty)$ and

$$\int_{[a,\infty)} f \, dm = \mathcal{R} \int_a^{+\infty} f(t) \, dt.$$

Analogously, if $f : [a, b) \to [0, +\infty)$, $f \in \mathcal{R}([a, c])$, for all $a < c < b$ and there exists the improper Riemann integral $\mathcal{R} \int_a^b f(t) \, dt$, then $f$ is Lebesgue integrable on $[a, b)$ and

$$\int_{[a,b)} f \, dm = \mathcal{R} \int_a^b f(t) \, dt.$$
(iii) If \( f : [a, b] \to \mathbb{R} \) is bounded, then \( f \in \mathcal{R}([a, b]) \) if and only if the set \( F \) of points of discontinuity of \( f \) is such that \( m(F) = 0 \).

Proof. (i) Given a partition \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) of \([a, b]\), let
\[
s(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad S(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),
\]
be the inferior and superior sums of \( f \), resp., w.r.t. the partition \( P \), where \( m_i = \inf_{x \in (x_{i-1}, x_i)} f(x) \) and \( M_i = \sup_{x \in (x_{i-1}, x_i)} f(x) \). Since \( f \in \mathcal{R}([a, b]) \), given \( n = 1, 2, \ldots \), there exist partitions \( P_n \) of \([a, b]\) such that \( P_{n+1} \) is a refinement of \( P_n \) and \( 0 \leq S(f, P_n) - s(f, P_n) \leq 1/n \).

For each \( n = 1, 2, \ldots \) fixed, set \( I_i(x_{i-1}, x_i) \), and define the simple functions
\[
s_n = \sum_{i=1}^{n} m_i \chi_{I_i}, \quad S_n = \sum_{i=1}^{n} M_i \chi_{I_i}.
\]
Then,
\[
0 \leq s_1 \leq s_2 \leq \cdots \leq f \leq \cdots \leq S_2 \leq S_1.
\]
Since the sequences \( \{s_n\} \) and \( \{S_n\} \) are monotone, let \( L(x) = \lim_{n \to +\infty} s_n(x) \) and \( U(x) = \lim_{n \to +\infty} S_n(x) \). Then, \( 0 \leq L(x) \leq f(x) \leq U(x) \) in \([a, b]\). Notice that \( L, U \) are bounded and measurable. Now, by the MCT and since \( f \in \mathcal{R}([a, b]) \),
\[
\int L \, dm = \lim_{n \to +\infty} \int s_n \, dm = \lim_{n \to +\infty} \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \lim_{n \to +\infty} s(f, P_n) = \mathcal{R} \int_{a}^{b} f(t) \, dt,
\]
and also,
\[
\int U \, dm = \lim_{n \to +\infty} \int S_n \, dm = \lim_{n \to +\infty} \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \lim_{n \to +\infty} S(f, P_n) = \mathcal{R} \int_{a}^{b} f(t) \, dt.
\]
Therefore, \( 0 \leq L \leq U \) and \( \int_{[a, b]} (U - L) \, dm = \int_{[a, b]} U \, dm - \int_{[a, b]} L \, dm = 0 \). This gives that \( U = L = f \) a.e. and
\[
\int_{[a, b]} f \, dm = \mathcal{R} \int_{a}^{b} f(t) \, dt,
\]
as we wished to prove.

Next we prove (iii). Let \( P = \bigcup_{n=1}^{+\infty} P_n \). Clearly \( P \) is countable, hence of measure 0. For \( x \in [a, b] \setminus P \), the following are equivalent:
- \( f \) is continuous at \( x \);
- \( U(x) = L(x) \).

Since \( x \in [a, b] \setminus P \), there exists a sequence of intervals \( I_n \) such that \( I_n \supseteq I_{n+1} \), \( \cap_{n=1}^{+\infty} I_n = \{x\} \). Indeed, given \( n \), \( x \) belongs to a unique interval \( I_n \) determined by the partition \( P_n \). \( P_{n+1} \) is a refinement of \( P_n \). Hence, \( I_n \supseteq I_{n+1} \), and \( \cap_{n=1}^{+\infty} I_n \) is non-empty, but its diameter is 0, since each partition \( P_n \) contains intervals of at most length 1/n. Then, \( L(x) = \lim_{n \to +\infty} \inf_{x \in I_n} f(x) \), \( U(x) = \lim_{n \to +\infty} \sup_{x \in I_n} f(x) \). It is now easy to convince oneself that the two conditions above are equivalent.

Next we observe that the following are equivalent:
- (a) \( f \) is continuous a.e.;
(b) \( L = U \) a.e.;
(c) \( f \in \mathcal{R}([a, b]). \)

Clearly (a) is equivalent to (b), and notice that in (i) we have proved that (c) implies (b). Suppose (b) does not hold, then \( 0 \leq \int L \, dm < \int U \, dm \), that is, \( U - L > \delta > 0 \) on a set of positive measure. This implies that difference \( \inf_{\mathcal{P}} S(f, \mathcal{P}) - \sup_{\mathcal{P}} s(f, \mathcal{P}) > 0 \), that is (c) does not hold. Hence, (b) implies (c), and we are done.

(ii) We only prove the first part, the proof of the second one being completely analogous. Since \( f \in \mathcal{R}([a, b]) \) for all \( b > a \), by (i) we have that \( \int_{[a,n]} f \, dm = \mathcal{R} \int_{a}^{n} f(t) \, dt \) for all \( n \) sufficiently large. Set \( f_n = \chi_{[a,n]} f \). Then \( 0 \leq f_1 \leq f_2 \leq \cdots \leq f \) and \( f_n \to f \) pointwise in \( [a, +\infty) \). By the MCT

\[
\int_{[a, +\infty]} f \, dm = \lim_{n \to +\infty} \int_{[a,n]} f \, dm = \lim_{n \to +\infty} \mathcal{R} \int_{a}^{n} f(t) \, dt = \mathcal{R} \int_{a}^{+\infty} f(t) \, dt.
\]

This proves (ii), hence the theorem.

\[\square\]

3.3. Examples.
Here and in what follows, \( dx = dm(x) \) denotes the Lebesgue measure on the real line. We also “import” the notation \( \int_{a}^{b} f(x) \, dx \) to denote the Lebesgue integral on the set \([a, b]\), that is, we write

\[
\int_{[a,b]} f \, dm = \int_{a}^{b} f(x) \, dx,
\]

since, by the previous result the two expressions are equal for \( f \) when both integral exist.

(1) Consider the functions \( f_n : [0, +\infty) \to [0, +\infty) \), \( f_n = \frac{1}{n} \chi_{[n,n+1)}. \) Show that \( f_n \to 0 \) uniformly in \([0, +\infty)\), but \( \int_{0}^{+\infty} f_n \, dx \neq \int_{0}^{+\infty} f \, dx \).

On the other hand, show that if \( f_n \to f \) uniformly on a compact interval \([a, b]\), \( f_n \in \mathcal{L}^1([a, b]) \), then \( \int_{a}^{b} f_n \, dx \to \int_{a}^{b} f \, dx \).

(2) We evaluate the following limits, if they exist:

\[
\lim_{k \to +\infty} \int_{0}^{k} \left( 1 - \frac{x}{k} \right)^k e^{x/2} \, dx, \quad \lim_{k \to +\infty} \int_{0}^{k} \left( 1 + \frac{x}{k} \right)^k e^{-2x} \, dx.
\]

(3) Let \( f_n : \mathbb{R} \to [0, +\infty) \) be given by \( f_n(x) = (1 + 4^n x^2)^{-1} \). Setting \( f = \sum_{n=1}^{+\infty} f_n \), evaluate \( \int f \, dx \).

We observe that for each \( n \), \( f_n \) is continuous on \( \mathbb{R} \), hence measurable. Since \( f_n \geq 0 \), \( f = \sum_{n=1}^{+\infty} f_n \) is well defined, non-negative, and measurable. We may apply Cor. 2.17 (2) to see that

\[
\sum_{n=1}^{+\infty} \int_{a}^{b} f_n \, dx = \int_{a}^{+\infty} \sum_{n=1}^{+\infty} f_n \, dx = \int f \, dx.
\]
We observe that each $f_n$ is integrable and then

$$
\int_{\mathbb{R}} f(x) \, dx = \sum_{n=1}^{+\infty} \int_{\mathbb{R}} f_n(x) \, dx
$$

$$
= 2 \sum_{n=1}^{+\infty} \int_{0}^{+\infty} \frac{1}{(1 + (2^n x)^2)} \, dx
$$

$$
= \sum_{n=1}^{+\infty} \frac{1}{2^{n-1}} \arctan(2^n x) \bigg|_{0}^{+\infty}
$$

$$
= \pi.
$$

(4) Let $f_n : (0, +\infty) \to [0, +\infty)$, $n = 1, 2, \ldots$ be given by

$$
f_n(x) = \frac{1}{xe^{n/x} \log(1 + nx)}.
$$

Show that $f_n \in L^1((0, +\infty))$ for all $n$ and that $\lim_{n \to +\infty} \int_{0}^{+\infty} f_n \, dx = 0$.

(5) Given the functions $f_n : [0, +\infty) \to \mathbb{R}$,

$$
f_n(x) = \frac{n^{4/3} x}{1 + n^{5/2} x^2},
$$

evaluate the limits

(a) $\lim_{n \to +\infty} \int_{1}^{+\infty} f_n(x) \, dx$,

(b) $\lim_{n \to +\infty} \int_{0}^{1} f_n(x) \, dx$,

justifying your answers.

It is easy to see that the $f_n$ are measurable and non-negative. (a) When $x \geq 1$, $f_n \to 0$ pointwise and

$$
0 \leq f_n(x) \leq \frac{1}{n^{7/6} x^2} \leq \frac{1}{x^2} \in L^1(1, +\infty).
$$

By the DCT it follows that $\int_{1}^{+\infty} f_n \, dx \to 0$ as $n \to +\infty$. (b) With the change of variables $nx = t$ we obtain that

$$
\int_{0}^{1} f_n(x) \, dx = \int_{0}^{+\infty} g_n(t) \, dt,
$$

where $g_n(t) = \frac{1}{n^{2/3} + t/\sqrt{t}} \chi_{(0,n)}(t)$. Then, $g_n \to 0$ pointwise on $(0, +\infty)$ and $|g_n(t)| \leq \frac{t}{1+t^2} \in L^1((0, +\infty))$. The conclusion now follows from the DCT.

(5) Let $f_k, g_k : [0, +\infty) \to \mathbb{R}$ be given respectively, by

$$
f_k(x) = \frac{1}{ \sqrt{k} } \chi_{(0,k)}(x) \frac{1}{ \sqrt{x} e^{(x-\frac{1}{2})^2} } \quad \text{and} \quad g_k(x) = \frac{1}{ k^2 } \chi_{[\frac{1}{k}, +\infty)}(x) \frac{1}{ x^3 } e^{-(x-k)^2}.
$$

Evaluate, if they exist, the limits,

(i) $\lim_{k \to +\infty} \int_{0}^{+\infty} f_k(x) \, dx$, \quad and (ii) $\lim_{k \to +\infty} \int_{0}^{+\infty} g_k(x) \, dx$. 

3.4. The Cantor set and its generalizations. In this section we introduce a noticeable example of a set in \([0, 1]\) that is uncountable and yet it has measure 0, while a variation on the same theme produces sets in \([0, 1]\) of measures arbitrarily close to 1.

**Definition 3.16.** Consider the compact interval \([0, 1]\). Let \(J_{1,0} = (1/3, 2/3)\) and \(C_1 = [0, 1] \setminus J_{1,0}\). Then \(C_1\) is union of two intervals, namely \(I_{1,0} = [0, 1/3]\) and \(I_{1,1} = [2/3, 1]\), each of length 1/3.

Next, we define \(C_2\) as the interval \(C_1\) taken away the two middle thirds of each \(I_{1,j}\), \(j = 0, 1\). Then \(C_2\) is union of \(2^2\) intervals, namely \(I_{2,0} = [0, 1/9]\), \(I_{2,1} = [2/9, 3/9]\), \(I_{2,2} = [6/9, 7/9]\), and \(I_{2,3} = [8/9, 9/9]\), each of length \(1/3^2\). We now iterate this process. Notice that at each step we double the number of intervals. Thus, we can write

\[
C_k = \bigcup_{j=0}^{2^k-1} I_{k,j}, \quad m(I_{k,j}) = 3^{-k} \text{ for all } j = 0, \ldots, 2^k - 1,
\]

and

\[
C_{k+1} = \bigcup_{j=0}^{2^k-1} (I_{k,j} \setminus J_{k,j}) =: \bigcup_{j=0}^{2^{k+1}-1} I_{k+1,j}, \quad \text{with } m(J_{k,j}) = 3^{-k}.
\]

We then construct sets \(C_k\), \(k = 1, 2, \ldots\) such that

- \(C_1 \supseteq C_2 \supseteq \cdots\);
- each \(C_k\) is compact so that \(\bigcap_{k=1}^{+\infty} C_k \neq \emptyset\);
- each \(C_k\) is union of \(2^k\) disjoint intervals, each of length \(3^{-k}\).

These properties are all easy to check. We finally set

\[
C = \bigcap_{k=1}^{+\infty} C_k.
\]

The set \(C\) is called the Cantor set.

**Proposition 3.17.** Let \(C\) be the Cantor set. Then, the following properties hold true.

(i) \(C\) is compact, nowhere dense and totally disconnected;

(ii) \(C\) has the cardinality of continuum;

(iii) \(C \in \mathcal{L}\) and \(m(C) = 0\).

We recall that a set is said to be nowhere dense if its closure has empty interior and a set is totally disconnected if its connected components are single points.

**Proof.** It is clear that \(C\) is compact and non-empty, as intersection of incapsulated compact sets. For any given \(\varepsilon > 0\) and an interval \(I \subseteq [0, 1]\), let \(k\) be such that \(2^{-k} < \varepsilon\). Since \(C \subseteq C_k\) for each \(k\), and \(C_k\) is union of disjoint compact intervals of length \(3^{-k}\), \(I\) cannot be contained in \(C_k\), hence in \(C\). Thus, \(C\) does not contain any interval. This implies that \(C\) is nowhere dense and totally disconnected. This proves (i).

To show (ii), it is easy to construct an injective correspondence between the set of sequences of 0 and 1 and \(C\). Given the sets \(I_{1,0}\) and \(I_{1,1}\) we associate the values 0 and 1, resp. Splitting \(I_{1,0}\) into \(I_{2,0}\) and \(I_{2,1}\) we associate to each of them the terms \{0, 0\} and \{0, 1\}, resp., and so on. At each step we have an interval \(I_{k,j}\) to which it is associated the finite sequence of length \(k\) of 0’s and 1’s, when we split it into two intervals, we associate to them the sequences of length \(k+1\) by adding a 0 to the sequence corresponding to interval on the left, and a 1 to the sequence
Corollary 3.19. There exist open sets 

\[ \alpha \] 

Hence, it suffices to select 

\[ C \] 

This proves the proposition.

(iii) Notice that 

\[ C = \lim_{k \to +\infty} m(C_k). \] 

The set \( C_{k+1} \) is obtained by removing \( 2^k \) intervals each of length \( 3^{-(k+1)} \) from \( C_k \). Then,

\[
m(C) = \lim_{k \to +\infty} m(C_{k+1}) = \lim_{k \to +\infty} \left( m(C_k) - \sum_{j=0}^{2^k-1} m(J_{k,j}) \right) = \lim_{k \to +\infty} \left( m(C_k) - \frac{2^k}{3^{k+1}} \right)
\]

\[
= \lim_{k \to +\infty} \left( m(C_{k-1}) - \frac{2^{k-1}}{3^k} - \frac{2^k}{3^{k+1}} \right) = \lim_{k \to +\infty} \left( 1 - \frac{1}{3} - \cdots - \frac{2^{k-1}}{3^k} - \frac{2^k}{3^{k+1}} \right)
\]

\[
= \lim_{k \to +\infty} 1 - \frac{1}{2} \sum_{j=1}^{k+1} \frac{2^j}{3^j}
\]

\[
= 1 - \frac{1}{2} \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 0.
\]

This proves the proposition. \( \square \)

Using a similar procedure, for any \( \varepsilon > 0 \) we can construct a nowhere dense, totally disconnected subset of \([0, 1]\), of measure greater than \( 1 - \varepsilon \).

**Proposition 3.18.** Let \( \varepsilon > 0 \). Then, there exists \( \alpha \), \( 0 < \alpha \leq 1/3 \) and a set \( C^{(\alpha)} \subseteq [0, 1] \), which is compact, nowhere dense, and totally disconnected set, such that \( m(C^{(\alpha)}) \geq 1 - \varepsilon \).

**Proof.** We imitate the construction of the Cantor set. Given \([0, 1]\), we remove the central interval of length \( \alpha \), that is we define \( C_1^{(\alpha)} = [0, 1] \setminus \left( \frac{1}{2} - \frac{\alpha}{2}, \frac{1}{2} + \frac{\alpha}{2} \right) \). From the resulting two intervals, we remove the central intervals, each of length \( \alpha^2 \), Then, at the step \( k + 1 \), we remove from \( C_k^{(\alpha)} \) the central intervals of length \( \alpha^{k+1} \), and we set \( C^{(\alpha)} = \bigcap_{k=1}^{+\infty} C_k^{(\alpha)} \). Notice that, if \( \alpha = \frac{1}{3} \), then \( C^{(\alpha)} = C \), the Cantor set.

The topological properties of \( C^{(\alpha)} \) are proved as in the case of \( C \). Finally, observing that the set \( C_{k+1}^{(\alpha)} \) is obtained by removing \( 2^k \) intervals each of length \( \alpha^{k+1} \) from \( C_k^{(\alpha)} \), we have

\[
m(C^{(\alpha)}) = \lim_{k \to +\infty} m(C_{k+1}^{(\alpha)}) = \lim_{k \to +\infty} \left( m(C_k^{(\alpha)}) - 2^k \alpha^{k+1} \right)
\]

\[
= \lim_{k \to +\infty} 1 - \frac{1}{2} \sum_{j=1}^{k+1} (2\alpha)^j
\]

\[
= 1 - \frac{1}{2} \cdot \frac{2\alpha}{1 - 2\alpha}
\]

\[
= 1 - 3\alpha.
\]

Hence, it suffices to select \( \alpha \) so that \( \frac{1 - 3\alpha}{1 - 2\alpha} > 1 - \varepsilon \). \( \square \)

**Corollary 3.19.** There exist open sets \( U \subseteq \mathbb{R} \) such that \( m(U \setminus U) > 0 \).
Proof. Notice that, since $U$ is open, $\overline{U} \setminus U = \partial U$, the boundary of $U$.

We observe that for any $0 < \alpha \leq 1/3$, $C^{(\alpha)}$ is a perfect set, that is, it coincides with its derived set, the set of its accumulation points. Since $C^{(\alpha)}$ is closed, it suffices to notice that each point of $C^{(\alpha)}$ is of accumulation for $C^{(\alpha)}$ itself. Thus, it follows that $C^{(\alpha)} = \partial C^{(\alpha)}$.

It suffices to consider $U = ^c C^{(\alpha)}$, with $0 < \alpha < 1/3$, so that $m(C^{(\alpha)}) > 0$. Now, $\partial U = \partial( ^c U) = \partial C^{(\alpha)} = C^{(\alpha)}$, so that $m(\partial U) > 0$. \hfill \Box

3.5. Integrals depending on a parameter. The following result is quite often very useful. We state it for a generic measure space $(X, M, \mu)$, although we will essentially use it in the case of the Lebesgue measure $(\mathbb{R}, \mathcal{L}, m)$.

Theorem 3.20. Let $(X, M, \mu)$ be a measure space, $f : X \times [a, b] \to \mathbb{R}$ (or $\mathbb{C}$), such that $f(\cdot, t) \in L^1(\mu)$ for every $t \in [a, b]$ and let

$$F(t) = \int_X f(x, t) \, d\mu(x).$$

(1) Suppose that there exists $g \in L^1(\mu)$ such that $|f(x, t)| \leq g(x)$ for all $t \in [a, b]$ and suppose that $\lim_{t \to t_0} f(x, t) = f(x, t_0)$. Then

$$\lim_{t \to t_0} F(t) = F(t_0).$$

In particular, if $f(x, \cdot)$ is continuous for every $x \in X$, then $F$ is continuous.

(2) Suppose that $\partial_t f$ exists and that there exists $g \in L^1(\mu)$ such that $|\partial_tf(x, t)| \leq g(x)$ for all $(x, t) \in X \times [a, b]$. Then $F$ is differentiable and

$$F'(t) = \int_X \partial_t f(x, t) \, d\mu(x).$$

We remark that in the applications this result is often used in its local form, that is, in the nbhb of a given point $t_0$ of the parameter. It is in fact sufficient just take $[a, b]$ to be a sufficiently small nbhb of $t_0$.

Proof. (1) Let $\{t_n\} \subseteq [a, b]$ be any sequence converging to $t_0$ and set $f_n(x) = f(x, t_n)$. Then $f_n \to f(x) := f(x, t_0)$, $|f_n| \leq g \in L^1(\mu)$, and we can apply the DCT to $\{f_n\}$ and obtain

$$\lim_{n \to +\infty} F(t_n) = \lim_{n \to +\infty} \int_X f_n(x) \, d\mu = \int_X f(x, t_0) \, d\mu = F(x, t_0).$$

Since the sequence $t_n \to t_0$ was arbitrary, the conclusion follows.

(2) Again, let $\{t_n\} \subseteq [a, b]$ be any sequence converging to $t_0$. Set

$$h_n(x) = \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0}.$$ 

Then, $\lim_{n \to +\infty} h_n(x) = \partial_t f(x, t_0)$. Hence, $\partial_t f(\cdot, t_0)$ is measurable. Moreover, using the mean value theorem we see that

$$|h_n(x)| \leq \sup_{t \in [a, b]} |\partial_t f(x, t)| \leq g(x).$$
Therefore, we can apply the DCT to \( \{h_n\} \), using again the fact that the sequence \( t_n \to t_0 \) was arbitrary, and obtain that
\[
F'(t_0) = \lim_{n \to +\infty} \frac{F(t_n) - F(t_0)}{t_n - t_0} = \lim_{n \to +\infty} \int_X \frac{f(x, t_n) - f(x, t_0)}{t_n - t_0} d\mu(x) = \lim_{n \to +\infty} \int_X h_n(x) d\mu
\]
\[
= \int_X \partial_t f(x, t_0) d\mu,
\]
as we wished to show.

Example 3.21. The Gamma function. For \( z \in \mathbb{C}, \Re z > 0 \) we set
\[
\Gamma(z) = \int_{0}^{+\infty} t^{z-1} e^{-t} dt.
\]
(We recall that \( t^{z-1} = e^{(z-1) \log t} \).)

Then:
1. For all \( a \in \mathbb{C}, \Re z > 0, a \Gamma(z) = \Gamma(z + 1) \);
2. \( \Gamma(1) = 1 \);
3. For all non-negative integer \( n \), \( \Gamma(n + 1) = n! \);
4. \( \Gamma \in C^\infty((0, +\infty)) \).

Furthermore, identifying \( z = x + iy \in \mathbb{C} \) with \( (x, y) \in \mathbb{R}^2 \), the function \( \Gamma \) is \( C^\infty \) in \( \{(x, y) : x > 0\} \). We prove the above statements in the case \( z \) real, that is, \( z = x > 0 \). The case of the complex \( z \) is proven in exactly the same way, however we do not need in the present class.

We begin by observing that, for every fixed \( x > 0, t^{x-1}e^{-t} \in L^1((0, +\infty)) \). For, \( t^{x-1}e^{-t} \leq t^{-1} \) for \( 0 < t < 1 \), while \( t^{x-1}e^{-t} \) is easily seen to be integrable on \( (1, +\infty) \). Then, integrating by parts we see that
\[
\Gamma(x) = \int_{0}^{+\infty} t^{x-1}e^{-t} dt = \lim_{\varepsilon \to 0^+, b \to +\infty} \int_{\varepsilon}^{b} t^{x-1}e^{-t} dt
\]
\[
= \lim_{\varepsilon \to 0^+, b \to +\infty} \frac{1}{x} \left( t^x e^{-t} \right]_{\varepsilon}^{b} + \frac{1}{x} \int_{\varepsilon}^{b} t^x e^{-t} dt
\]
\[
= 0 + \frac{1}{x} \int_{0}^{+\infty} t^x e^{-t} dt
\]
\[
= \frac{\Gamma(x + 1)}{x}.
\]

This shows (1), while (2) it is obvious. To show (3), assume by induction that \( \Gamma(n) = (n - 1)! \).

By (1) we now have that \( \Gamma(n + 1) = n\Gamma(n) = n(n - 1)! = n! \). It easily follows from Thm. 3.20 (2), and we leave the details as an exercise.

\[\text{\footnote{In fact, } \Gamma \text{ is differentiable in the complex sense, that is, } \Gamma \text{ is \textit{holomorphic}. We will not stress this, however fundamental, aspect at the present time.}}\]
3.6. More on $L^1(m)$. In this section we prove some more results about the Lebesgue space $L^1(m)$, that by Thm. 2.31, we recall, is a Banach space. We recall, if $\mathcal{X}$ is a topological space, that the support of a continuous function $g$ is the closure of the set $\{x \in \mathcal{X} : g(x) \neq 0\}$. We denote by $C_c(\mathcal{X})$ the space of continuous functions with compact support in $\mathcal{X}$.

We begin with an elementary lemma, which holds true on any $\sigma$-finite measure space.

**Lemma 3.22.** Let $(\mathcal{X}, \mathcal{M}, \mu)$ be $\sigma$-finite measure space and let $f \in L^1(\mu)$. For any $\varepsilon > 0$ there exists a set $A$ such that $\int_A |f|d\mu < \varepsilon$ and $f$ is bounded on $A$.

**Proof.** For sake of simplicity, we prove only in the case of $(\mathbb{R}, \mathcal{L}, m)$. We invite the reader to prove it in the general case.

Define

$$B_n = \{x : |x| \leq n \text{ and } |f(x)| \leq n\},$$

so that $B_1 \subseteq B_2 \subseteq \cdots$, and $\bigcup_{n=\infty}^\infty B_n = \mathbb{R}$. Define the sequence $f_n = \chi_{B_n}f$. Then, $f_n \to f$ pointwise, $|f_n| \leq |f| \in L^1(m)$, so that $f_n \to f$ in $L^1(m)$ as $n \to +\infty$, by the DTC. It suffices to choose $n_\varepsilon$ so that $\int_{B_2} |f|d\mu = \|f - f_{n_\varepsilon}\|_{L^1(m)} < \varepsilon$ and set $A = B_{n_\varepsilon}$. The conclusion now follows. □

**Proposition 3.23.** (Absolute continuity of the measure) Let $f \in L^1(\mathbb{R})$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $m(F) < \delta$, then

$$\int_F |f|d\mu < \varepsilon.$$

**Proof.** Given $\varepsilon > 0$, let $A$ be a set such that $\int_A |f|d\mu < \varepsilon/2$ and $f$ is bounded on $A$, say $M = \sup_{x \in \mathbb{R}}|f(x)|$. Now, chose $\delta = \varepsilon/(2M)$. Then, if $F$ is a measurable set with $m(F) < \delta$ we have

$$\int_F |f(x)|dx \leq \int_{A \cap F} |f(x)|dx + \int_{A^c \cap F} |f(x)|dx$$

$$\leq \int_A |f(x)|dx + \int_{A^c \cap F} |f(x)|dx$$

$$< \frac{\varepsilon}{2} + Mm(F) < \frac{\varepsilon}{2} + M\delta$$

$$= \varepsilon. \quad \Box$$

We now recall

**Lemma 3.24.** (Urysohn’s Lemma)$^5$ Let $\mathcal{X}$ be a locally compact Hausdorff space, $K$ compact and $U$ open, $K \subseteq U$. Then there exists $g$ continuous such that $g = 1$ on $K$ and $g = 0$ on $^cU$.

In particular, if $U$ is taken such that $\overline{U}$ is compact, then the function $g$ is continuous and has compact support.

**Theorem 3.25.** (Lusin’s Theorem) (1) Let $f$ be a measurable function on $\mathbb{R}$. Suppose that there exists $A \subseteq \mathbb{R}$ such that $m(A) < +\infty$ and $f = 0$ on $^cA$. Then, for any $\varepsilon > 0$, there exists $g \in C_c(\mathbb{R})$ such that

$$m(\{x \in \mathbb{R} : f(x) \neq g(x)\}) < \varepsilon.$$

If $f$ is bounded, $g$ can be chosen so that $|g| \leq |f|$. (2) The space $C_c(\mathbb{R})$ is dense in $L^1(m)$.

$^5$See e.g. [Rudin, Real and Complex Analysis, 3rd Ed., McGraw–Hill Editor]
Proof. Assume first that \( f \) is bounded, say \(|f| \leq 1\), and that \( A \) is compact. Then, by Thm. 2.9 (2), there exists a sequence of simple functions \( \{ \varphi_n \} \) such that \(|\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f|\), \( \varphi_n \to f \) pointwise and uniformly on every set on which \( f \) is bounded – hence on all of \( \mathbb{R} \) in this case. Set \( \psi_1 = \varphi_1 \) and \( \psi_n = \varphi_n - \varphi_{n-1} \) for \( n \geq 2 \). Then, \( \sum_{k=1}^{n} \psi_k = \varphi_n \) so that \( \sum_{n=1}^{\infty} \psi_n = f \), pointwise and uniformly since \( f \) is bounded. Observe that, with the notation of Thm. 2.9, assuming \( f \) real-valued for simplicity,

\[
\psi_n = \varphi_n - \varphi_{n-1} = (s_n^+ - s_{n-1}^+) - (s_n^- - s_{n-1}^-)
\]

We claim that \( \psi_n = 2^{-n} \chi_{T_n} \), for a certain set \( T_n \). Indeed, assume for a moment that \( 0 \leq f \leq 1 \), and recall the construction of \( s_n \):

\[
s_n(x) = \sum_{k=0}^{2^n-1} k2^{-n} \chi_{E_{n,k}}(x),
\]

where

\[
E_{n,k} = f^{-1}((k2^{-n}, (k+1)2^{-n}]),
\]

and similarly for \( s_{n-1} \). (Notice that \( 0 \leq k \leq 2^n - 1 \) only, since \( f \leq 1 \).) The function \( s_{n+1} \) was constructed by splitting the sets \( E_{n,k} = E_{n+1,2k} \cup E_{n+1,2k+1} \) as in (3), we have that \( s_{n+1} = s_n \) on \( E_{n+1,2k} \), while \( s_{n+1} - s_n = 2^{-(n+1)} \) on \( E_{n+1,2k+1} \), for each \( k = 0, \ldots, 2^n - 1 \). This establishes the claim for \( s_n^+ - s_{n-1}^+ \), hence for \( \psi_n \).

Next, fix an open set \( U \) such that \( A \subseteq U \) and \( \overline{U} \) is compact. Given \( \varepsilon > 0 \) there exist sets \( K_n, U_n \), where \( K_n \) is compact, \( U_n \) is open, and such that

\[
K_n \subseteq T_n \subseteq U_n \subseteq U \quad \text{and} \quad m(U_n \setminus K_n) < \varepsilon 2^{-n}.
\]

By Urysohn’s Lemma there exist continuous functions \( h_n \) such that \( 0 \leq h_n \leq 1 \), \( h_n = 1 \) on \( K_n \), and \( h_n = 0 \) on \( \complement U_n \). Define

\[
g(x) = \sum_{n=1}^{+\infty} 2^{-n} h_n(x).
\]

Then, the series converges uniformly, \( g \) is continuous, and it is 0 outside \( U \), which is is bounded, so that \( \text{supp}(g) \) is compact. Moreover, observe that \( h_n = 1 \) on \( K_n \), so that \( 2^{-n} h_n = \psi_n \) on \( K_n \), and that \( h_n = 0 = \psi_n \) on \( \complement U_n \). Hence,

\[
\{ x : g(x) \neq f(x) \} \subseteq \bigcup_{n=1}^{+\infty} U_n \setminus K_n
\]

and

\[
m\{ x : g(x) \neq f(x) \} \leq \sum_{n=1}^{+\infty} m(U_n \setminus K_n) < \varepsilon.
\]

This proves (1) when \(|f| \leq 1\) and \( A \) is compact.

Now, let \( A \) be any set such that \( m(A) < +\infty \). Then, given any \( \varepsilon > 0 \), there exist \( U, K \), with \( K \) compact, \( U \) open \( K \subseteq A \subseteq U \), and such that \( m(U \setminus K) < \varepsilon/2 \). Then, by the argument above, we can find a continuous function with support in \( K \) such that

\[
m\{ x \in K : g(x) \neq f(x) \} < \varepsilon/2.
\]
Since also $f$ vanishes outside $U$

$$m(\{x : g(x) \neq f(x)\}) = m(\{x \in U : g(x) \neq f(x)\})$$

$$\leq m(\{x \in K : g(x) \neq f(x)\}) + m(U \setminus K)$$

$$< \varepsilon .$$

Thus, the conclusion holds true when $f$ is bounded.

Finally, we remove the assumption that $f$ is bounded. For each $n$, let $B_n = \{x : |f(x)| \leq n\}$. Since $f$ is real valued, $\bigcup_{n=1}^{+\infty} B_n = \mathbb{R}$, so that $\mathcal{C}B_1 \supseteq \mathcal{C}B_2 \supseteq \cdots$, and $\bigcap_{n=1}^{+\infty} \mathcal{C}B_n = \emptyset$. Thus, $m(\mathcal{C}B_n) \to 0$ as $n \to +\infty$. Then, given $\varepsilon > 0$ we can find an integer $n$ such that $m(\mathcal{C}B_n) \leq \varepsilon/2$ and then $g$ continuous with compact support contained in $B_n$ such that $m(\{x : g(x) \neq \chi_{B_n}(y)\}) < \varepsilon/2$. Therefore,

$$m(\{x : g(x) \neq f(x)\}) \leq m(\{x \in B_n : g(x) \neq f(x)\}) + m(\mathcal{C}B_n)$$

$$< \varepsilon .$$

This proves (1).

In order to prove (2), given $f \in L^1(m)$, given any $\varepsilon > 0$, by Lemma 3.22 there exists $A \subseteq \mathbb{R}$ such that $m(A) < +\infty$ and $\int_A |f| \ dm < \varepsilon/2$ and $f$ is bounded on $A$, say $|f| \leq M$ on $A$. Part (1) now gives that there exists $g$ continuous and with support in $A$ such that $m(A_1) < \varepsilon/4M$, where $A_1 = \{x \in A : g(x) \neq f(x)\}$, and $|g| \leq |f|$. Then

$$\|f - g\|_{L^1(m)} \int_{A_1} |f - g| \ dm \int_{A \setminus A_1} |f - g| \ dm + \int_{A} |f - g| \ dm$$

$$\leq \varepsilon/2 + \varepsilon/2 .$$

The proof is now complete. □

4. Product measure spaces and the Lebesgue integral in $\mathbb{R}^n$

Our current goal is to define the Lebesgue measure on the higher-dimensional euclidean space $\mathbb{R}^n$, and to reduce the computations to integrals in lower dimensions. In order to do this, we first present the theory of integration on product measure spaces and the fundamental theorems of Tonelli and Fubini on the equality of iterated integrals.

4.1. Product measure spaces.

**Definition 4.1.** Let $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{Y}, \mathcal{N})$ be measurable spaces. A subset of $\mathcal{X} \times \mathcal{Y}$ of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$ is called a measurable rectangle. We define the product $\sigma$-algebra in $\mathcal{X} \times \mathcal{Y}$ as the $\sigma$-algebra generated by the collection of measurable rectangles and we denote it by $\mathcal{M} \times \mathcal{N}$.

Next, given two measure spaces $(\mathcal{X}, \mathcal{M}, \mu)$ and $(\mathcal{Y}, \mathcal{N}, \nu)$ we wish to construct a measure on $\mathcal{M} \times \mathcal{N}$ such that the measure of any measurable rectangle $A \times B$ equals $\mu(A)\nu(B)$.

Throughout this section, we denote by $\mathcal{A}$ the collection of finite unions of disjoint measurable rectangles in $\mathcal{X} \times \mathcal{Y}$.

**Lemma 4.2.** The following properties hold:
(i) \( \mathcal{A} \) is an algebra;
(ii) if \( \mathcal{M}(\mathcal{A}) \) denotes the \( \sigma \)-algebra generated by \( \mathcal{A} \), then \( \mathcal{M}(\mathcal{A}) = \mathcal{M} \times \mathcal{N} \).

**Proof.** (†) In order to prove that \( \mathcal{A} \) is an algebra we need to show that it is closed under the complement and finite unions of sets in \( \mathcal{A} \). Given two measurable rectangles \( A_1 \times B_1 \) and \( A_2 \times B_2 \), we preliminary observe that \((A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)\) is also a measurable rectangle.

Now, given a measurable rectangle \( A \times B \) we have that
\[
^c(A \times B) = (\mathcal{X} \times ^cB) \cup (^cA \times B),
\]
that is, \( ^c(A \times B) \) is finite union of disjoint measurable rectangles, hence in \( \mathcal{A} \). Next, let \( A_1 \times B_1 \) and \( A_2 \times B_2 \) be measurable rectangles \( \mathcal{A} \) not necessarily disjoint (otherwise we have nothing to prove). By the first part, \( \mathcal{A} \ni ^c(A_2 \times B_2) = \bigcup_{j=1}^n (E_j \times F_j) \), where \( E_j \times F_j \) are disjoint measurable rectangles. Then,
\[
(A_1 \times B_1) \setminus (A_2 \times B_2) = (A_1 \times B_1) \cap \left( (E_1 \times F_1) \cup \cdots \cup (E_n \times F_n) \right)
= \left( (A_1 \cap E_1) \times (B_1 \cap F_1) \right) \cup \cdots \cup \left( (A_1 \cap E_n) \times (B_1 \cap F_n) \right),
\]
hence a finite union of disjoint measurable rectangles, which belongs to \( \mathcal{A} \). Therefore,
\[
(A_1 \times B_1) \cup (A_2 \times B_2) = \left( (A_1 \times B_1) \cap ^c(A_2 \times B_2) \right) \cup (A_2 \times B_2) \in \mathcal{A}.
\]
By induction, we obtain that
\[
(A_1 \times B_1) \cup \cdots \cup (A_n \times B_n) \in \mathcal{A},
\]
that is, \( \mathcal{A} \) is closed under finite unions. Finally, let \( A \times B \in \mathcal{A} \), that is, \( A \times B = \bigcup_{j=1}^n (E_j \times F_j) \), where \( E_j \times F_j \) are disjoint measurable rectangles. Then,
\[
^c(A \times B) = \left( (E_1 \times F_1) \cup \cdots \cup (E_n \times F_n) \right)
= (E_1 \times F_1) \cap \cdots \cap (E_n \times F_n)
= \left( (\mathcal{X} \times ^cF_1) \cup (^cE_1 \times F_1) \right) \cap \cdots \cap \left( (\mathcal{X} \times ^cF_n) \cup (^cE_n \times F_n) \right)
= \bigcup_{j=1}^n \left\{ (A_j \times B_j) : A_j, A_j' = \mathcal{X} \text{ or } ^c\mathcal{X}, B_j, B_j' = F_j \right\}.
\]
Hence, \( ^c(A \times B) \) is union of intersections of disjoint measurable rectangles, hence it is union of disjoint measurable rectangles. This shows that \( \mathcal{A} \) is an algebra, i.e. (i).

In order to prove (ii), observe that \( \mathcal{M} \times \mathcal{N} \) contains the measurable rectangles, hence their finite unions. Thus, \( \mathcal{M} \times \mathcal{N} \) contains \( \mathcal{A} \) and therefore the \( \sigma \)-algebra generated by it, that is, \( \mathcal{M}(\mathcal{A}) \). Conversely, \( \mathcal{M}(\mathcal{A}) \) contains all the measurable rectangles, hence the \( \sigma \)-algebra generated by them, that is, \( \mathcal{M} \times \mathcal{N} \). This completes the proof. \( \square \)

Now, let \( (\mathcal{X}, \mathcal{M}, \mu) \) and \( (\mathcal{Y}, \mathcal{N}, \nu) \) be measure spaces and let \( \mathcal{A} \) be the algebra of finite unions of disjoint measurable rectangles. If \( E \subseteq \mathcal{X} \times \mathcal{Y} \) and \( E = \bigcup_{j=1}^n A_j \times B_j \) is an element of \( \mathcal{A} \) (that is, \( A_j \times B_j \) are disjoint measurable rectangles, \( j = 1, \ldots, n \)), we set
\[
\rho(E) = \sum_{j=1}^n \mu(A_j)\nu(B_j).
\]
Lemma 4.3. With the hypotheses as above, \( \rho \) is a premeasure on \( A \).

Proof. (†) First of all, we need to show that \( \rho \) is well defined, that is, for any \( \mathcal{E} \in \mathcal{A} \), the value of \( \rho(E) \) is independent of the decomposition of \( \mathcal{E} \) as finite unions of disjoint measurable rectangles. Observe that, if \( A \times B \) is a measurable rectangle and \( A \times B = \bigcup_{j=1}^{m} A'_j \times B'_j \), where \( A'_j \times B'_j \) are disjoint measurable rectangles, we have

\[
\chi_A(x)\chi_B(y) = \sum_{j=1}^{m} \chi_{A'_j \times B'_j}(x,y) = \sum_{j=1}^{m} \chi_{A'_j}(x)\chi_{B'_j}(y).
\]

We now integrate w.r.t. \( x \) on both sides of the equalities above and see that

\[
\mu(A)\chi_B(y) = \int \chi_A(x)\chi_B(y) \, d\mu(x) = \sum_{j=1}^{m} \mu(A'_j)\chi_{B'_j}(y).
\]

We proceed by integrating w.r.t. \( y \) and obtain

\[
\rho(A \times B) = \mu(A)\nu(B) = \int \mu(A)\chi_B(y) \, d\nu(y) = \int \sum_{j=1}^{m} \mu(A'_j)\chi_{B'_j}(y) \, d\nu(y)
\]

\[
= \sum_{j=1}^{m} \mu(A'_j)\nu(B'_j). \tag{15}
\]

This shows that \( \rho(A \times B) = \sum_{j=1}^{m} \mu(A'_j)\nu(B'_j) \) for any decomposition of \( A \times B \) as finite unions of disjoint measurable rectangles. This easily implies that \( \rho \) is well defined. Now, \( \rho \) is finitely additive by construction.

We need to show \( \rho \) is countably additive, that is, if \( \{A_j \times B_j\} \) is a collection of disjoint measurable rectangles whose union is in \( \mathcal{A} \), then

\[
\rho\left(\bigcup_{j=1}^{+\infty} (A_j \times B_j)\right) = \sum_{j=1}^{+\infty} \rho(A_j \times B_j). \tag{16}
\]

By the finite additivity of \( \rho \), we first show that it suffices to consider the case in which \( \bigcup_{j=1}^{+\infty} (A_j \times B_j) \) is a single measurable rectangle. The argument is analogous to the one in the proof of Lemma 3.8, but we repeat it for sake of completeness. If \( A \times B = \bigcup_{k=1}^{m} (A^{(k)} \times B^{(k)}) \), with \( A^{(k)} \times B^{(k)} \) disjoint measurable rectangles, it is possible to find subcollections \( \{(A^{(k)}_j \times B^{(k)}_j)\}, k = 1, 2, \ldots, n, \) of \( \{(A_j \times B_j)\} \) such that, for each \( k = 1, 2, \ldots, n, \)

\[
A^{(k)} \times B^{(k)} = \bigcup_{j=1}^{+\infty} (A^{(k)}_j \times B^{(k)}_j).
\]

If we know that for each \( k, \rho(A^{(k)} \times B^{(k)}) = \sum_{j=1}^{+\infty} \rho(A^{(k)}_j \times B^{(k)}_j), \) the conclusion then follows from the finite additivity of \( \rho \).

Thus, let us show (16) when \( \bigcup_{j=1}^{+\infty} (A_j \times B_j) = (A \times B) \). We proceed as in the argument leading to (15). We have that

\[
\chi_A(x)\chi_B(y) = \sum_{j=1}^{+\infty} \chi_{A_j}(x)\chi_{B_j}(y).
\]

Integrating first w.r.t. \( x \) we have

\[
\mu(A)\chi_B(y) = \int \chi_A(x)\chi_B(y) \, d\mu(x) = \sum_{j=1}^{+\infty} \mu(A_j)\chi_{B_j}(y).
\]
Next we integrate w.r.t. $y$ and obtain

$$\rho\left(\bigcup_{j=1}^{+\infty}(A_j \times B_j)\right) = \rho(A \times B) = \mu(A)\nu(B)$$

$$= \int \mu(A)\chi_B(y) \, d\nu(y) = \int \sum_{j=1}^{+\infty} \mu(A_j)\chi_{B_j}(y) \, d\nu(y)$$

$$= \sum_{j=1}^{m} \mu(A_j)\nu(B_j).$$

This shows that $\rho$ is a premeasure and we are done. \(\square\)

**Definition 4.4.** Let $(\mathcal{X}, \mathcal{M}, \mu)$, $(\mathcal{Y}, \mathcal{N}, \nu)$ be measure spaces, $\mathcal{M} \times \mathcal{N}$ the product $\sigma$-algebra.

We define an outer measure $(\mu \times \nu)^*$ on $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$ as

$$(\mu \times \nu)^*(E) = \inf \left\{ \sum_{j=1}^{+\infty} \rho(A_j) : E \subseteq \bigcup_{j=1}^{+\infty} A_j, A_j \in \mathcal{A} \text{ for all } j \right\}.$$

Lemma 3.5 shows that indeed $(\mu \times \nu)^*$ is an outer measure, that $\mathcal{A}$ is contained in the $\sigma$-algebra of $(\mu \times \nu)^*$-measurable sets, and $(\mu \times \nu)^*$ restricted to $\mathcal{A}$ coincides with $\rho$. As a consequence, we also obtain that $\mathcal{M} \times \mathcal{N}$ is contained in the $\sigma$-algebra of $(\mu \times \nu)^*$-measurable sets.

Furthermore, if we also assume that $(\mathcal{X}, \mathcal{M}, \mu)$ and $(\mathcal{Y}, \mathcal{N}, \nu)$ are $\sigma$-finite, Thm. 3.7 shows that there exists a measure defined on the $\sigma$-algebra of $(\mu \times \nu)^*$-measurable sets. Thus, we define the product measure $\mu \times \nu$ on the product $\sigma$-algebra $\mathcal{M} \times \mathcal{N}$

$$(\mu \times \nu)(E) = (\mu \times \nu)^*(E), \quad E \in \mathcal{M} \times \mathcal{N}.$$  

Finally, $\mu \times \nu$ is the unique measure on $\mathcal{M} \times \mathcal{N}$ such that $(\mu \times \nu)|_{\mathcal{A}} = \rho$.

We observe that the $\sigma$-algebra of $(\mu \times \nu)^*$-measurable sets equals $\mathcal{M} \times \mathcal{N}$ union the subsets of sets of $(\mu \times \nu)^*$-measure 0.

4.2. **Integration on product measure spaces.** Having constructed the product measure space $(\mathcal{X} \times \mathcal{Y}, \mathcal{M} \times \mathcal{N}, \mu \times \nu)$ we now have all the results about the integration theory in an abstract measure space at our disposal. However, we wish to relate the calculus of integrals on $\mathcal{X} \times \mathcal{Y}$ to the integrals on $\mathcal{X}$ and on $\mathcal{Y}$. This is the goal of the current section.

Thus, let the measure spaces $(\mathcal{X}, \mathcal{M}, \mu)$ and $(\mathcal{Y}, \mathcal{N}, \nu)$ be given. Given a set $E \in \mathcal{M} \times \mathcal{N}$ we define its $x$-section as

$$E_x = \{y \in \mathcal{Y} : (x, y) \in E\}$$

and its $y$-section as

$$E^y = \{x \in \mathcal{X} : (x, y) \in E\}.$$  

If $f$ is a function on $\mathcal{X} \times \mathcal{Y}$ we define the $x$-section as the function $f_x(y) = f(x, y)$ and the $y$-section as $f^y(x) = f(x, y)$. We now have

**Proposition 4.5.** With the notation above, the following hold true:

(i) if $E \in \mathcal{M} \times \mathcal{N}$ then, for every $x \in \mathcal{X}$, $E_x \in \mathcal{N}$ and, for every $y \in \mathcal{Y}$, $E^y \in \mathcal{M}$;

(ii) if $f$ is a function on $\mathcal{X} \times \mathcal{Y}$ that is $\mathcal{M} \times \mathcal{N}$-measurable, for every $x \in \mathcal{X}$, $f_x$ is $\mathcal{N}$-measurable and, for every $y \in \mathcal{Y}$, $f^y$ is $\mathcal{M}$-measurable.
Proof. This is easy. Let \( \mathcal{R} = \{ E \in \mathcal{M} \times \mathcal{N} : E_x \in \mathcal{N} \text{ and } y \in \mathcal{Y} \text{ for all } x \in \mathcal{X}, y \in \mathcal{Y} \} \).

Then, \( \mathcal{R} \) contains all measurable rectangles \( A \times B \), since

\[
(A \times B)_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases} \quad \text{and} \quad (A \times B)^y = \begin{cases} A & \text{if } y \in B \\ \emptyset & \text{if } y \notin B \end{cases}.
\]

(17)

Next, notice that \( \mathcal{R} \) is a \( \sigma \)-algebra, since

\[
\left( \bigcup_{j=1}^{+\infty} E_j \right)_x = \bigcup_{j=1}^{+\infty} E_{j_x} \quad \text{and} \quad \left( \bigcup_{j=1}^{+\infty} E_j \right)^y = \bigcup_{j=1}^{+\infty} E_{j^y},
\]

and also

\[
(^cE)_x = ^c(E_x) \quad \text{and} \quad (^cE)^y = ^c(E^y).
\]

Therefore, \( \mathcal{R} \) contains the \( \sigma \)-algebra generated by the measurable rectangles, hence the \( \sigma \)-algebra generated by \( \mathcal{A} \), which equals \( \mathcal{M} \times \mathcal{N} \). Hence, \( \mathcal{R} = \mathcal{M} \times \mathcal{N} \), and this shows (i).

Next, we need the following notion.

**Definition 4.6.** A collection \( \mathcal{C} \) of subsets of a set \( \mathcal{X} \) is called a *monotone class* if it is closed with respect to countable increasing unions and countable decreasing intersections.

Clearly, a \( \sigma \)-algebra is a monotone class, and in particular \( \mathcal{P}(\mathcal{X}) \) is a monotone class. It is easy to see that intersections of monotone classes is still a monotone class. Hence, given any \( \mathcal{E} \subseteq \mathcal{P}(\mathcal{X}) \) there exists the smallest monotone class containing \( \mathcal{E} \); namely the intersection of all monotone classes containing \( \mathcal{E} \), family that is not empty since it contains \( \mathcal{P}(\mathcal{X}) \). We denote by \( \mathcal{C}(\mathcal{E}) \) such monotone class and we call it the monotone class generated by \( \mathcal{E} \).

**Theorem 4.7. (Monotone Class Lemma)** Let \( \mathcal{A} \) be an algebra of subsets of \( \mathcal{X} \). Then, \( \mathcal{M}(\mathcal{A}) = \mathcal{C}(\mathcal{A}) \), that is, the monotone class generated by \( \mathcal{A} \) coincides with the \( \sigma \)-algebra generated by \( \mathcal{A} \).

**Proof.** (†) See [F], Monotone Class Lemma 2.35. \( \square \)

We use the Monotone Class Lemma to prove the following result, which is fundamental for the proof of the Tonelli–Fubini theorem that will follow.

**Proposition 4.8.** Let \( (\mathcal{X}, \mathcal{M}, \mu), (\mathcal{Y}, \mathcal{N}, \nu) \) be \( \sigma \)-finite measure spaces. Let \( E \in \mathcal{M} \times \mathcal{N} \). Then, for every \( x \in \mathcal{X} \), the function \( y \mapsto \mu(E^y) \) is \( \mathcal{N} \)-measurable and for every \( y \in \mathcal{Y} \), the function \( x \mapsto \nu(E_x) \) is \( \mathcal{M} \)-measurable. Moreover,

\[
(\mu \times \nu)(E) = \int_{\mathcal{X}} \nu(E_x) \, d\mu(x) = \int_{\mathcal{Y}} \mu(E^y) \, d\nu(y).
\]

**Proof.** We first assume that \( \mu \) and \( \nu \) are finite measures, that is, \( \mu(\mathcal{X}), \nu(\mathcal{Y}) < +\infty \). Let \( \mathcal{C} \) be the collection of all sets \( E \in \mathcal{M} \times \mathcal{N} \) for which the conclusions in the statement are true. We claim that all measurable rectangles \( A \times B \) belong to \( \mathcal{C} \). Indeed, by (17) it follows that

\[
\mu((A \times B)^y) = \begin{cases} \mu(A) & \text{if } y \in B \\ 0 & \text{if } y \notin B \end{cases} = \mu(A) \chi_B(y),
\]
so that the function \( y \mapsto \mu((A \times B)^y) \) is \( \nu \)-measurable. Analogously,

\[
\nu((A \times B)_x) = \begin{cases} 
\nu(B) & \text{if } x \in A \\
0 & \text{if } x \notin A
\end{cases} = \nu(B)\chi_A(x),
\]

so that the function \( x \mapsto \nu((A \times B)_x) \) is \( \mu \)-measurable. Moreover,

\[
(\mu \times \nu)(A \times B) = \mu(A)\nu(B) = \int_Y \mu(A)\chi_B(y) \, d\nu(y) = \int_Y \mu((A \times B)^y) \, d\nu(y),
\]

and also

\[
(\mu \times \nu)(A \times B) = \mu(A)\nu(B) = \int_X \nu(B)\chi_A(x) \, d\mu(x) = \int_X \nu((A \times B)_x) \, d\mu(x).
\]

This proves the claim. By finite additivity, also finite unions of disjoint measurable rectangles also verify the conclusions in the statement, so that \( C \) contains the algebra \( A \) of finite unions of disjoint measurable rectangles.

If we show that \( C \) is a monotone class, then \( C \) would contain the monotone class generated by the algebra \( A \), that, by the Monotone Class Lemma, coincides with \( M \times N \). This would imply the result, under the assumption that both \( \mu \) and \( \nu \) are finite measures. In order to show that \( C \) is a monotone class, we need to show that it closed under countable increasing unions and countable decreasing intersections of sets. Suppose then that \( \{E_n\} \subseteq C \) and \( E_1 \subseteq E_2 \subseteq \cdots \). Set \( E = \cup_{n=1}^{+\infty} E_n \). Then \( \{\mu(E_n^y)\} \) is an increasing sequence of non-negative, \( N \)-measurable functions converging to \( \mu(E^y) \). Thus, \( \mu(E^y) \) is also \( N \)-measurable, and by the MCT we have that

\[
\int_Y \mu(E^y) \, d\nu = \lim_{n \to +\infty } \int_Y \mu(E_n^y) \, d\nu = \lim_{n \to +\infty } (\mu \times \nu)(E_n) = (\mu \times \nu)(E).
\]

By switching the roles of \( \mu \) and \( \nu \) and arguing in the same way, we also obtain that

\[
\int_X \nu(E_x) \, d\mu = (\mu \times \nu)(E).
\]

Thus, \( C \) is closed under countable increasing unions.

On the other hand, suppose that \( \{E_n\} \subseteq C \) and \( E_1 \supseteq E_2 \supseteq \cdots \). Set \( E = \cap_{n=1}^{+\infty} E_n \). Here we are going to use the assumption that \( \mu \) and \( \nu \) are finite measures. We have that \( \{\mu(E_n^y)\} \) is a decreasing sequence of non-negative, \( N \)-measurable functions converging to \( \mu(E^y) \). Thus, \( \mu(E^y) \) is also \( N \)-measurable. Moreover,

\[
0 \leq \mu(E_n^y) \leq \mu(\mathcal{X}) \in L^1(\nu),
\]

since \( \mu(\mathcal{X}) \), \( \nu(\mathcal{Y}) < +\infty \). Thus, we can apply the DCT and obtain

\[
\int_Y \mu(E^y) \, d\nu = \lim_{n \to +\infty } \int_Y \mu(E_n^y) \, d\nu = \lim_{n \to +\infty } (\mu \times \nu)(E_n) = (\mu \times \nu)(E).
\]

Again, by switching the roles of \( \mu \) and \( \nu \) and arguing in the same way, we have

\[
\int_X \nu(E_x) \, d\mu = (\mu \times \nu)(E).
\]

Thus, \( C \) is closed under countable decreasing intersections and \( C \) is a monotone class. This proves the assertion in the case \( \mu, \nu \) are finite measures.
Suppose now that \((\mathcal{X}, \mathcal{M}, \mu), (\mathcal{Y}, \mathcal{N}, \nu)\) are \(\sigma\)-finite measure spaces. Let \(\{X_j\} \subseteq \mathcal{M}, \{Y_j\} \subseteq \mathcal{N}\) be increasing sequences of sets such that \(\bigcup_{j=1}^{+\infty} X_j = \mathcal{X}, \bigcup_{j=1}^{+\infty} Y_j = \mathcal{Y}\). Let \(E\) be any set in \(\mathcal{M} \times \mathcal{N}\). The previous argument applies to the set \(E_j = E \cap (X_j \times Y_j)\) to give that for every \(j\), the functions \(y \mapsto \mu((E_j)^{\nu})\) and \(x \mapsto \nu((E_j)^{\mu})\) are \(\mathcal{N}\)-measurable and \(\mathcal{M}\)-measurable, resp., for every \(y \in Y_j, x \in X_j\), resp. Therefore, the functions \(y \mapsto \mu((E_j)^{\nu})\chi_{Y_j}(y)\) and \(x \mapsto \nu((E_j)^{\mu})\chi_{X_j}(x)\) are \(\mathcal{N}\)-measurable and \(\mathcal{M}\)-measurable, resp., for every \(y \in \mathcal{Y}, x \in \mathcal{X}\), resp. Moreover,

\[
(\mu \times \nu)(E_j) = \int_{X_j} \nu((E_j)^{\nu}) \, d\mu(x) = \int_{\mathcal{X}} \nu((E_j)^{\nu}) \chi_{X_j}(x) \, d\mu(x)
\]

\[
= \int_{Y_j} \mu((E_j)^{\mu}) \, d\nu(y) = \int_{\mathcal{Y}} \mu((E_j)^{\mu}) \chi_{Y_j}(y) \, d\nu(y).
\]

Finally, we apply the MCT to the increasing sequences \(\{\nu((E_j)^{\nu})\chi_{X_j}\}\) and \(\{\mu((E_j)^{\mu})\chi_{Y_j}\}\), that converge to \(\nu(E_x)\) and \(\mu(E_y)\), resp., and obtain that

\[
(\mu \times \nu)(E) = \lim_{j \to +\infty} (\mu \times \nu)(E_j)
\]

\[
= \lim_{j \to +\infty} \int_{\mathcal{X}} \nu((E_j)^{\nu}) \chi_{X_j}(x) \, d\mu = \int_{\mathcal{X}} \nu(E_x) \, d\mu
\]

\[
= \lim_{j \to +\infty} \int_{\mathcal{Y}} \mu((E_j)^{\mu}) \chi_{Y_j}(y) \, d\nu = \int_{\mathcal{Y}} \mu(E_y) \, d\nu.
\]

This completes the proof. \(\square\)

**Theorem 4.9. (The Tonelli–Fubini Theorem.)** Let \((\mathcal{X}, \mathcal{M}, \mu), (\mathcal{Y}, \mathcal{N}, \nu)\) be \(\sigma\)-finite measure spaces.

1. (Tonelli) Let \(f\) be a non-negative \(\mathcal{M} \times \mathcal{N}\)-measurable function on \(\mathcal{X} \times \mathcal{Y}\). Then, the functions \(g(y) = \int_{\mathcal{X}} f(y \times \mu(x)) \, d\mu(x)\) and \(h(x) = \int_{\mathcal{Y}} f(x \times \nu(y)) \, d\nu(y)\) are \(\mathcal{N}\)-measurable and \(\mathcal{M}\)-measurable, resp., and

\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d(\mu \times \nu)(x, y) = \int_{\mathcal{Y}} g(y) \, d\nu(y) = \int_{\mathcal{X}} h(x) \, d\mu(x). \tag{18}
\]

2. (Fubini) Let \(f \in L^1(\mathcal{X} \times \mathcal{Y}, (\mu \times \nu)(x, y))\). Then, the function \(g(y) = \int_{\mathcal{X}} f(y \times \mu(x)) \, d\mu(x)\) belongs to \(L^1(\mathcal{Y}, \nu)\) for a.e. \(x\) and the function \(h(x) = \int_{\mathcal{Y}} f(x \times \nu(y)) \, d\nu(y)\) belongs to \(L^1(\mathcal{X}, \mu)\) for a.e. \(y\) and equalities (18) hold.

Notice that (18) can also be written in terms of *iterated integrals*

\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d(\mu \times \nu)(x, y) = \int_{\mathcal{Y}} \left( \int_{\mathcal{X}} f(x, y) \, d\mu(x) \right) \, d\nu(y) = \int_{\mathcal{X}} \left( \int_{\mathcal{Y}} f(x, y) \, d\nu(y) \right) \, d\mu(x).
\]

It is customary to use the notations

\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d(\mu \times \nu)(x, y) = \int_{\mathcal{X}} \int_{\mathcal{Y}} f(x, y) \, d\nu(y) \, d\mu(x) = \iint f \, d\mu d\nu.
\]

**Proof.** (1) Observe that Tonelli’s theorem reduces to Prop. 4.8 in the case of characteristic functions. By finite additivity of the integrals and measures, the result follows for non-negative simple functions. Let \(f \geq 0\) be \((\mathcal{M} \times \mathcal{N})\)-measurable. By Thm. 2.9 (1) there exists an increasing
sequence \( \{s_n\} \) of non-negative simple functions, such that \( s_n \to f \) pointwise. Then, we can apply the MCT to obtain

\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d(\mu \times \nu)(x, y) = \lim_{n \to +\infty} \int_{\mathcal{X} \times \mathcal{Y}} s_n(x, y) \, d(\mu \times \nu)(x, y)
\]

and also

\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d(\mu \times \nu)(x, y) = \lim_{n \to +\infty} \int_{\mathcal{X} \times \mathcal{Y}} s_n(x, y) \, d(\mu \times \nu)(x, y)
\]

This proves (1).

In order to prove Fubini’s theorem, we proceed in an analogous fashion. For \( f \in L^1(\mu \times \nu) \), we Thm. 2.9 (2) to find a sequence \( \{\varphi_n\} \) of simple functions, such that \( 0 \leq |\varphi_1| \leq |\varphi_n| \leq \cdots |f|, \varphi_n \to f \) pointwise. Then, we argue as in (19) and (20), using the DCT, instead of the MCT. This completes the proof of Tonelli–Fubini’s theorem.

**Example 4.10.** The hypotheses in the Tonelli–Fubini theorem cannot be relaxed, as the following examples show.

1. Let \( f : (0, 1) \times (-1, 1) \to \mathbb{R} \) be given by \( f(x, y) = y/x \). Then, clearly

\[
\int_0^1 \left( \int_{-1}^1 \frac{y}{x} \, dy \right) \, dx = \int_0^1 0 \, dx = 0,
\]

while,

\[
\int_{-1}^1 \left( \int_0^1 \frac{y}{x} \, dx \right) \, dy = \int_{-1}^1 y \left( \int_0^1 \frac{1}{x} \, dx \right) \, dy = \int_{-1}^1 g(y) \, dy
\]

where \( g(y) = +\infty \) if \( 0 < y < 1 \) and \( g(y) = -\infty \) if \( -1 < y < 0 \). This last integral does not exist and the iterated integrals are not equal.

2. Let \( T_1 = \{(x, y) : 0 < x < 1, 0 < y < x\} \), \( T_2 = \{(x, y) : 0 < y < 1, 0 < x < y\} \), and define \( f : (0, 1) \times (0, 1) \to \mathbb{R} \) be given \( f(x, y) = \frac{1}{x^2} \chi_{T_1}(x, y) - \frac{1}{y^2} \chi_{T_2}(x, y) \). Then,

\[
\int_0^1 \left( \int_0^1 f(x, y) \, dy \right) \, dx = \int_0^1 \left( \int_0^x \frac{1}{x^2} \, dy - \int_x^1 \frac{1}{y^2} \, dy \right) \, dx
\]

\[
= \int_0^1 \left( \frac{1}{x} + \left( \frac{1}{y} \right) \right) \, dy = \int_0^1 \left( \frac{1}{x} + 1 - \frac{1}{x} \right) \, dy
\]

\[
= 1.
\]
On the other hand,
\[
\int_0^1 \left( \int_0^1 f(x, y) \, dx \right) \, dy = \int_0^1 \left( - \int_0^y \frac{1}{y^2} \, dx + \int_y^1 \frac{1}{x^2} \, dx \right) \, dy \\
= \int_0^1 \left( - \frac{1}{y} + \left( - \frac{1}{x^1} \right) \right) \, dy = \int_0^1 \left( - \frac{1}{y} - 1 + \frac{1}{y} \right) \, dy \\
= -1.
\]

Thus, again the iterated integrals are not equal.

**Remark 4.11.** The measure \((\mu \times \nu)\) is not complete, even if \((\mu, \mathcal{M})\) and\((\nu, \mathcal{N})\) are complete measures. As an example consider \((\mathcal{X}, \mu, \mathcal{M}) = (\mathbb{R}, \nu, \mathcal{N}) = (\mathbb{R}, m, \mathcal{L})\), the Lebesgue measure on \(\mathbb{R}\). Let \(A\) be any set of measure 0 and \(B\) the example of a non-measurable set constructed in Example 1.1. Then \((A \times B) \notin \mathcal{L} \times \mathcal{L}\), but it is contained in a set of \((m \times m)\)-measure 0.

This is not a significant drawback, since we can always complete the measure \((\mu \times \nu)\). However, the Tonelli–Fubini theorem takes a slightly more complicated form in the case of complete measures.

**Theorem 4.12. (The Tonelli–Fubini Theorem for complete measures)** Let \((\mathcal{X}, \mathcal{M}, \mu), (\mathcal{Y}, \mathcal{N}, \nu)\) be complete \(\sigma\)-finite measure spaces. Let \((\mathcal{X} \times \mathcal{Y}, \mathcal{T}, \tau)\) be the completion of the product measure space \((\mathcal{X} \times \mathcal{Y}, \mathcal{M} \times \mathcal{N}, \mu \times \nu)\) and \(f\) be \(\mathcal{T}\)-measurable.

1. **(Tonelli)** If \(f \geq 0\) then \(f_x\) is \(\mathcal{N}\)-measurable for \(\mu\)-a.e. \(x \in \mathcal{X}\) and \(f^y\) is \(\mathcal{M}\)-measurable for \(\nu\)-a.e. \(y \in \mathcal{Y}\) and
\[
\int_{\mathcal{X} \times \mathcal{Y}} f(x, y) \, d\tau(x, y) = \int_{\mathcal{Y}} \left( \int_{\mathcal{X}} f(x, y) \, d\mu(x) \right) \, d\nu(y) = \int_{\mathcal{X}} \left( \int_{\mathcal{Y}} f(x, y) \, d\nu(y) \right) \, d\mu(x) \quad (21)
\]

2. **(Fubini)** If \(f \in L^1(\tau)\) then \(f_x\) is \(\mathcal{N}\)-measurable for \(\mu\)-a.e. \(x \in \mathcal{X}\) and \(f^y\) is \(\mathcal{M}\)-measurable for \(\nu\)-a.e. \(y \in \mathcal{Y}\). Moreover \(f_x \in L^1(\nu)\) for \(\mu\)-a.e. \(x \in \mathcal{X}\), \(f^y \in L^1(\mu)\) for \(\nu\)-a.e. \(y \in \mathcal{Y}\) and \((21)\) holds.

**Proof.** (†) We sketch the proof, leaving the details to the reader.

Let \(f\) be \(\mathcal{T}\)-measurable. Since \((\mathcal{T}, \tau)\) is the completion of \((\mathcal{M} \times \mathcal{N}, \mu \times \nu)\), by Prop. 2.12 it follows that there exists an \((\mathcal{M} \times \mathcal{N})\)-measurable function \(g\) such that \(g = f - \tau\)-a.e. Let \(h = f - g\). We can apply Tonelli–Fubini theorem to \(g\) in either case \(f \in L^+(\tau)\) or \(f \in L^1(\tau)\) and obtain that
\[
\int_{\mathcal{X} \times \mathcal{Y}} g(x, y) \, d\tau(x, y) = \int_{\mathcal{X} \times \mathcal{Y}} g(x, y) \, d(\mu \times \nu)(x, y) \\
= \int_{\mathcal{Y}} \left( \int_{\mathcal{X}} g(x, y) \, d\mu(x) \right) \, d\nu(y) = \int_{\mathcal{X}} \left( \int_{\mathcal{Y}} g(x, y) \, d\nu(y) \right) \, d\mu(x).
\]

Hence, it suffices to show that if \(h\) is \(\mathcal{T}\)-measurable and \(h = 0\) \(\tau\)-a.e., then \(h_x, h^y\) are in \(L^1(\nu)\) and in \(L^1(\mu)\), hence measurable, resp., and
\[
\int_{\mathcal{Y}} h_x \, d\nu = \int_{\mathcal{X}} h^y \, d\mu = 0 \quad (22)
\]
for \( \mu \)-a.e. \( x \), and \( \nu \)-a.e. \( y \), resp. Indeed, using the observation at the end of Section 2, (22) would imply that

\[
\int_{X \times Y} (f(x,y) - g(x,y)) \, d\tau(x,y) = \int_{X \times Y} h(x,y) \, d\tau(x,y) = \int_{X \times Y} h(x,y) \, d(\mu \times \nu)(x,y)
\]

\[
= \int_Y \left( \int_X h(x,y) \, d\mu(x) \right) \, d\nu(y) = \int_X \left( \int_Y h(x,y) \, d\nu(y) \right) \, d\mu(x) = 0.
\]

Thus, we prove (22). Since \( h = 0 \) \( \tau \)-a.e., there exists \( E \in \mathcal{M} \times \mathcal{N} \) such that \( (\mu \times \nu)(E) = 0 \) and \( h = 0 \) on \( ^cE \). We then have

\[
\int_{X \times Y} h(x,y) \, d\tau = \int_{X \times Y} h(x,y) \, d(\mu \times \nu) = \int_E h(x,y) \, d(\mu \times \nu).
\]

Since \( (\mu \times \nu)(E) = 0 \), we have

\[
0 = \int_Y \mu(E^y) \, d\nu = \int_X \nu(E_x) \, d\mu
\]

which imply that

\[
\mu(E^y) = 0 \quad \nu \text{-a.e.} \quad \text{and} \quad \nu(E_x) = 0 \quad \mu \text{-a.e.} \quad .
\]

(23)

Then, there exists a sequence of simple functions \( \tau \)-measurable \( \{\varphi_n\} \) such that \( 0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |h| \) and \( \varphi_n \to h \). Notice that \( \varphi_{n_x} \to h_x \) and \( \varphi_{n^y} \to h^y \), for all \( x, y \).

Observe also that, clearly, \( \varphi_n = 0 \) on \( ^cE \), so that \( \varphi_{n_x} = 0 \) on \( ^c(E_x) \) and \( \varphi_{n^y} = 0 \) on \( ^c(E^y) \), for all \( x, y \). Thus, by (23), \( \varphi_{n_x} = \chi_{E_x} \mu \text{-a.e.} \) and \( \varphi_{n^y} = \chi_{E^y} \nu \text{-a.e.} \). Since \( \chi_{E_x} \) and \( \chi_{E^y} \) are \( \mathcal{N} \) and \( \mathcal{M} \)-measurable, resp., and \( \mu \) and \( \nu \) are complete, by Prop. 4.8 it follows that also \( \varphi_{n_x} \) and \( \varphi_{n^y} \) are \( \mathcal{N} \) and \( \mathcal{M} \)-measurable, resp. This easily implies that \( h_x \in L^1(\nu) \) and \( h^y \in L^1(\mu) \) for \( \mu \)-a.e. \( x \) and \( \nu \)-a.e. \( y \) and thus (22), and hence the theorem, follow. \( \Box \)

### 4.3. The Lebesgue integral in \( \mathbb{R}^n \)

Clearly, the results of the previous section on product measure spaces can be extended to the case of cartesian products of any finite number of measure spaces. We assume the validity of all these results, without formally state any of them.

We define the Lebesgue measure in \( m_n \) on \( \mathbb{R}^n \) as the completion of \( m \times \cdots \times m \) (\( n \) copies of \( m \)) on \( \mathcal{B}_\mathbb{R} \times \cdots \times \mathcal{B}_\mathbb{R} \), or, equivalently, on \( \mathcal{L} \times \cdots \times \mathcal{L} \). We denote by \( \mathcal{L}_n \) the \( \sigma \)-algebra domain of \( m_n \), and call these sets Lebesgue measurable sets in \( \mathbb{R}^n \).

In this section we prove some properties of \( m_n \) that extend similar properties valid in the case \( n = 1 \). If \( R = R_1 \times \cdots \times R_n \subseteq \mathbb{R}^n \) is a rectangle, we call \( R_j \) its sides, \( j = 1, \ldots, n \).

**Theorem 4.13.** Let \( E \in \mathcal{L}_n \). Then, the following properties hold true.

(i) \( m_n(E) = \inf \{ m_n(U) : E \subseteq U, U \text{ open} \} = \sup \{ m_n(K) : K \subseteq E, K \text{ compact} \} \);

(ii) \( E = A_1 \setminus N_1 \), where \( A_1 \) is a \( G_\delta \)-set and \( m_n(N_1) = 0 \), and also \( E = A_2 \cup N_2 \), where \( A_2 \) is an \( F_\sigma \)-set and \( m_n(N_2) = 0 \);

(iii) if \( m_n(E) < +\infty \), then for every \( \varepsilon > 0 \) there exists a finite collection of rectangles \( \{R_j\} \), \( j = 1, \ldots, N \), whose sides are intervals and such that \( m_n(E \bigtriangleup \bigcup_{j=1}^N R_j) < \varepsilon \).
Proof. (†) (i) By construction of the product measure, given \( E \in \mathcal{L} \), and \( \varepsilon > 0 \), there exists a collection \( \{ R_j \} \) of rectangles such that \( E \subseteq \bigcup_{j=1}^{+\infty} R_j \) and \( \sum_{j=1}^{+\infty} m_n(R_j) \leq m_n(E) + \varepsilon \). In order to prove the first equality, notice that we may assume that \( m_n(E) < +\infty \), since otherwise we have nothing to prove. Then, \( \sum_{j=1}^{+\infty} m_n(R_j) \) is a convergent series, so that there exists \( M > 0 \) such that \( m_n(R_j) \leq M \) for all \( j \).

For each \( j \) fixed, \( R_j = R_j^{(1)} \times \cdots \times R_j^{(n)} \), with \( R_j^{(k)} \in \mathcal{L} \), \( k = 1, \ldots, n \). By Prop. 3.11, for each \( j \) fixed, we can find open sets \( U_j^{(k)} \) such that \( R_j^{(k)} \subseteq U_j^{(k)} \) and \( m(U_j^{(k)}) < m(R_j^{(k)}) + 2^{-j}\varepsilon/(NM^{n-1}) \), where \( N \) is a sufficiently large fixed positive integer. Then

\[
R \subseteq U := \bigcup_{j=1}^{+\infty} \left( U_j^{(1)} \times \cdots \times U_j^{(n)} \right).
\]

Notice that the \( U_j \) are open rectangles. Hence \( U \) is open, and

\[
m_n(U) \leq \sum_{j=1}^{+\infty} m_n(U_j) = \sum_{j=1}^{+\infty} n \prod_{k=1}^{n} m(U_j^{(k)}) \leq \sum_{j=1}^{+\infty} \prod_{k=1}^{n} \left( m(R_j^{(k)}) + 2^{-j}\varepsilon/(NM^{n-1}) \right)
\]

\[
\leq \sum_{j=1}^{+\infty} (m_n(R_j) + \varepsilon 2^{-j})
\]

\[
\leq m_n(E) + 2\varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, the first equality in (i) is proved. In order to prove the second equality, we proceed exactly as in the proof of (ii) in Prop. 3.11, replacing the intervals \((j, j + 1]\) with the cubes \( Q_j = (0, 1)^n + j \), where \( j \in \mathbb{Z}^n \).

The proofs of (ii), which exactly the same as the ones in Prop. 3.14, is again left as an exercise. (iii) Let \( m_n(E) < +\infty \), and let \( \varepsilon > 0 \) be given. We can find open rectangles \( \{ U_j \} \) such that \( \sum_{j=1}^{+\infty} m_n(U_j) \leq m_n(E) + \varepsilon/2 \). Hence, \( m_n(U_j) < +\infty \) for all \( j \). For each \( j \), the sides of \( U_j \) are countable unions of disjoint open intervals, say \( U_j^{(\ell)} = \bigcup_{k=1}^{+\infty} I_j^{(\ell)} \), \( \ell = 1, \ldots, n \). Thus, for each \( j \) fixed,

\[
U_j = \left( \bigcup_{k_1=1}^{+\infty} I_{j,k_1}^{(1)} \right) \times \cdots \times \left( \bigcup_{k_n=1}^{+\infty} I_{j,k_n}^{(n)} \right) = \bigcup_{k_1,\ldots,k_n=1}^{+\infty} I_{j,k_1}^{(1)} \times \cdots \times I_{j,k_n}^{(n)}.
\]

Hence, for each \( j \), we can find a finite subunion \( V_j \) of the \( I_{j,k_1}^{(1)} \times \cdots \times I_{j,k_n}^{(n)} \)'s such that

\[
m_n(U_j) \leq m_n(V_j) + 2^{-j}\varepsilon/2.
\]

Notice that the sides of the \( V_j \)'s are finite unions of intervals. Then, if \( N \) is large enough so that \( \sum_{j=N+1}^{+\infty} m_n(U_j) < \varepsilon/2 \) we have

\[
m_n\left( E \setminus \bigcup_{j=1}^{N} V_j \right) \leq m_n\left( \bigcup_{j=1}^{N} \left( U_j \setminus V_j \right) \cup \bigcup_{j=N+1}^{+\infty} U_j \right)
\]

\[
\leq m_n\left( \bigcup_{j=1}^{N} \left( U_j \setminus V_j \right) \right) + m_n\left( \bigcup_{j=N+1}^{+\infty} U_j \right)
\]

\[
< \varepsilon.
\]
On the other hand,

\[ m_n\left(\bigcup_{j=1}^N V_j \setminus E\right) \leq m_n\left(\bigcup_{j=1}^{+\infty} U_j \setminus E\right) \leq \sum_{j=1}^{+\infty} m_n(U_j) - m_n(E) < \varepsilon. \]

This proves (iii), and we are done. \( \square \)

Next, we prove that \( m_n \) is also translation invariant.

**Proposition 4.14.** Let \( E \in \mathcal{L}_n \), and \( f \in L^+(m_n) \) or \( E \in L^1(m_n) \). For \( a \in \mathbb{R} \) define \( E + a = \{ x + a \; x \in E \} \in \mathcal{L}_n \) and \( f_a = f(\cdot + a) \). Then, \( m_n(E + a) = m_n(E) \) and \( \int f_a \, dm_n = \int f \, dm_n \).

**Proof.** (†) By composition of translations along each axis, it suffices to consider the case of translations along one axis, that is, \( a = (0, \ldots, a_j, 0, \ldots, 0) \). Then, if \( E = E_1 \times \cdots \times E_n \) is a rectangle, then \( m_n(E + a) = m(E_1) \cdots m(E_j + a_j) \cdots m(E_n) = m_n(E) \), by the 1-dimensional result Prop. 3.13. Then, the result is true for the elements of algebra \( \mathcal{A} \) of finite unions of disjoints measurable rectangles. By the construction of the product measure, the invariance now follows. Notice also that in particular the class of sets of measure zero is invariant under translations.

Next, let \( f \) be Lebesgue measurable and \( f_a = f(\cdot + a) \). Let \( B \) be a Borel set in \( \mathbb{R} \) (or in \( \mathbb{C} \), if \( f \) is complex-valued). Then, \( f^{-1}(B) \) is a Lebesgue measurable set, hence \( f^{-1}(B) = E \cup N \), where \( E \) is a Borel set in \( \mathbb{R}^n \) and \( N \) is contained in a Borel set of measure 0. Then, denoting by \( \tau_a \) the translation by \( a \in \mathbb{R}^n \), so that \( f(\cdot + a) = f \circ \tau_a \),

\[ (f_a)^{-1}(B) = \tau_a^{-1}(f^{-1}(B)) = \tau_a^{-1}(E \cup N) = (E - a) + (N - a), \]

that is, \( (f_a)^{-1}(B) \) is union of a Borel set and of a set of measure 0. Hence, it is Lebesgue measurable, and \( f_a \) is measurable.

Finally, the identity \( \int f_a \, dm_n = \int f \, dm_n \) holds true when \( f = \chi_E \), and \( E \) is a measurable set. Hence, it holds true for simple functions, and by the MCT and the DCT it holds true for \( f \in L^+(m_n) \) and \( f \in L^1(m_n) \), resp. \( \square \)

We now wish to show how the Lebesgue measure behaves under change of variables. We begin with a linear transformation. Let \( T \) be an invertible linear transformation and we write \( T \in GL(n, \mathbb{R}) \). This means that \( T : \mathbb{R}^n \to \mathbb{R}^n \) is a linear bijection, and this is the case if and only if \( \det T \neq 0 \). We denote by \( \langle \cdot, \cdot \rangle \) the inner product in \( \mathbb{R}^n \). Then, we identify \( T \) with the matrix \( (T_{jk})_{j,k=1,\ldots,n} \), where \( T_{jk} = \langle Te_j, e_k \rangle \), and \( e_1, \ldots, e_n \) is the canonical basis of \( \mathbb{R}^n \).

It is well known that every \( T \in GL(n, \mathbb{R}) \) can be decomposed as composition of elementary transformations. Namely,

- translations in each direction by a scalar multiple of another component of the variable: \( T(x_1, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, x_j + cx_k, \ldots, x_n) \), with \( j = 1, \ldots, n \) and \( c \in \mathbb{R} \);
- dilations in each direction: \( T(x_1, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, cx_j, \ldots, x_n) \), with \( c \in \mathbb{R} \setminus \{0\} \), \( j = 1, \ldots, n \);
- interchanges of coordinates: \( T(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_n) = (x_1, \ldots, x_k, \ldots, x_j, \ldots, x_n) \), \( 1 \leq j < k \leq n \).

**Theorem 4.15.** (Linear change of variables formula) Let \( T \in GL(n, \mathbb{R}) \). Then the following properties hold.
(1) If \( f \) is \( \mathcal{L}_n \)-measurable, then \( f \circ T \) is also \( \mathcal{L}_n \)-measurable. If \( f \in L^+(m_n) \) or \( f \in L^1(m_n) \) then
\[
\int f(x) \, dm_n = |\det T| \int (f \circ T)(x) \, dm_n.
\]

(2) If \( E \in \mathcal{L}_n \), then \( T(E) \in \mathcal{L}_n \) and \( m_n(T(E)) = |\det T| m_n(E) \).

Proof. If \( f \) is Borel measurable, then \( f \circ T \) is Borel measurable, since \( T \) is continuous. Moreover, since \( f \circ (T \circ S) = (f \circ T) \circ S \), and \( \det(T \circ S) = (\det T)(\det S) \), it suffices to prove the result for the elementary matrices as above. We may apply the Tonelli–Fubini theorem, according to whether \( f \in L^+(m_n) \) or \( f \in L^1(m_n) \). We can integrate in the variable \( x_j \) first and notice that
\[
\int_{\mathbb{R}} f(x_1, \ldots, x_j + cx_k, \ldots, x_n) \, dx_j = \int_{\mathbb{R}} f(x_1, \ldots, x_j, \ldots, x_n) \, dx_j
\]
and also that
\[
\int_{\mathbb{R}} f(x_1, \ldots, cx_j, \ldots, x_n) \, dx_j = |c| \int_{\mathbb{R}} f(x_1, \ldots, x_j, \ldots, x_n) \, dx_j,
\]
using the 1-dimensional results. Analogously, integrating in the variables \((x_j, x_k)\) first
\[
\int_{\mathbb{R}^2} f(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_n) \, dx_j \, dx_k = \int_{\mathbb{R}^2} f(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_n) \, dx_j \, dx_k.
\]
This proves (1) for \( f \) Borel measurable. If \( E \) is a Borel set, then (1) applied to \( \chi_E \) gives (2) in this case. In particular, \( T \) and \( T^{-1} \) preserve the Borel sets of measure 0. This easily imply that every \( T \in \text{GL}(n, \mathbb{R}) \) preserves the Lebesgue sets of measure 0.

Finally, we are in the position to prove (1) for a general Lebesgue measurable function \( f \). Indeed, there exists \( g \) which is Borel measurable such that \( g = f \) a.e., that is, \( f = g \) on a Borel set \( E \) and \( m_n(cE) = 0 \). Then, \( f \circ T = g \circ T \) on the Borel set \( T^{-1}(E), m_n(c(T^{-1}E)) = 0 \), i.e. \( f \circ T = g \circ T \) a.e., and we have
\[
\int (f \circ T)(x) \, dx = \int (g \circ T)(x) \, dx = |\det T| \int g(x) \, dx = |\det T| \int f(x) \, dx.
\]
This proves the theorem.

We extend the previous result to \( C^1 \)-diffeomorphisms of \( \mathbb{R}^n \). We recall that a map \( \Phi : \mathbb{R}^n \to \mathbb{R}^n \) is called a \( C^1 \)-diffeomorphisms if \( \Phi \) is \( C^1 \), is injective, and its Jacobian matrix \((\text{Jac } \Phi)(x)\) is invertible at every point \( x \in \mathbb{R}^n \). We recall also that the Jacobian matrix is the matrix of the partial derivatives of the components of \( \Phi \), that is, denoting by \( \nabla f \) the gradient of a scalar-valued function \( f \),
\[
\Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_n \end{pmatrix} \quad \text{and} \quad \text{Jac } \Phi = \begin{pmatrix} \nabla \Phi_1 \\ \vdots \\ \nabla \Phi_n \end{pmatrix} = \begin{pmatrix} \partial_{x_1} \Phi_1 & \cdots & \partial_{x_n} \Phi_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} \Phi_n & \cdots & \partial_{x_n} \Phi_n \end{pmatrix}.
\]

We also recall that, if \( \Phi \) is a \( C^1 \)-diffeomorphism, then \( \Phi^{-1} \) is well defined, and the inverse function theorem gives that also \( \Phi^{-1} \) is a \( C^1 \)-diffeomorphism and it holds that
\[
\text{Jac } (\Phi^{-1})(\Phi(x)) = (\text{Jac } \Phi(x))^{-1},
\]
for all \( x \in \mathbb{R}^n \).

We now have the general change of variables formula.
Theorem 4.16. (Change of variables formula for $C^1$-diffeomorphisms) Let $\Omega$ be an open set in $\mathbb{R}^n$ and let $\Phi : \Omega \to \mathbb{R}^n$ be a $C^1$-diffeomorphism. Then the following properties hold.

1. If $f$ is $L_n$-measurable on $\Phi(\Omega)$, then $f \circ \Phi$ is $L_n$-measurable on $\Omega$. If $f \in L^+(m_n)$ or $f \in L^1(m_n)$ then
   \[ \int_{\Phi(\Omega)} f(x) \, dx = \int_{\Omega} (f \circ \Phi)(x) |\text{det}(\text{Jac} \, \Phi(x))| \, dx. \]

2. If $E \in L_n$, then $\Phi(E) \in L_n$ and
   \[ m_n(\Phi(E)) = \int_E |\text{det}(\text{Jac} \, \Phi(x))| \, dx. \]

Proof. (†) See [F], Theorem 2.47.

We conclude this section with the extension of a classical result that relates the area under the graph of a (positive) function and the integral of the function itself.

Proposition 4.17. Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be measurable, where $\Omega$ is a measurable set. Let $G_f$ denote the region between the graph of $f$ and its domain:

\[ G_f = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R} : 0 \leq y \leq f(x) \}. \]

Then $G_f$ is measurable and $m(G_f) = \int_\Omega f \, dm_n$.

Proof. (†) We leave the proof as an exercise. In order to prove measurability, notice that the map $(x, y) \mapsto f(x) - y$ is the composition of $(x, y) \mapsto (f(x), y)$ and $(z, y) \mapsto z - y$.

4.4. Polar coordinates in $\mathbb{R}^n$. We wish to extend the familiar formulas for integration in polar coordinates in $\mathbb{R}^2$ and $\mathbb{R}^3$, that were defined using the Riemann integral in one and two variables, and that now we recall.

With the polar coordinates $\Phi : \mathbb{R}^2 \setminus \{(0, 0)\} \to (0, +\infty) \times [0, 2\pi)$, $\Phi(x, y) = (r \cos \theta, r \sin \theta)$, we had

\[ \int_{\mathbb{R}^2} f(x, y) \, dxdy = \int_0^{+\infty} \int_0^{2\pi} f(r \cos \theta, r \sin \theta) \, d\theta \, dr. \]

In the case of $\mathbb{R}^3$, we used what are called spherical coordinates $\Phi : \mathbb{R}^3 \setminus \{(0, 0, 0)\} \to (0, +\infty) \times [0, \pi] \times [0, 2\pi)$, $\Phi(x, y, z) = (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi)$, and had the formula

\[ \int_{\mathbb{R}^3} f(x, y, z) \, dxdydz = \int_0^{+\infty} \int_0^\pi \int_0^{2\pi} f(r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) \, d\theta \, r^2 \sin \phi \, d\phi dr. \]

Again, these formulas were given using the theory of the Riemann integral. We wish to put them in the context of abstract integration, with respect to well-defined measures.

We observe that, in both cases, we can define the length of an arc on unit circle $S_1$ and the surface area of a “rectangle” on the unit sphere $S_2$ as follows. The area of the sector $\Omega_{\theta_0} = \{ (x, y) = (r \cos \theta, r \sin \theta) : 0 < r < R, \alpha - \frac{\theta_0}{2} < \theta < \alpha + \frac{\theta_0}{2} \}$ of angle $\theta_0$ is $m(\Omega_{\theta_0}) = \frac{1}{2} R^2 \theta_0$. This equation can be used to define the length of an arc $A_{\theta_0}$, of angle $\theta_0$, by solving for $\theta_0$: setting

\[ \sigma(A_{\theta_0}) = \frac{m(\Omega_{\theta_0})}{\frac{1}{2} R^2} = \theta_0. \]
Analogous reasoning can be made in the case of the “rectangle” on the unit sphere $S_2$. Let $\Omega_{(\phi_1, \phi_2, \theta_0)} = \{(x, y, z) = (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi) : 0 < r < R, \phi_1 < \phi < \phi_2, \alpha - \frac{\theta_0}{2} < \theta < \alpha + \frac{\theta_0}{2}\}$. An elementary calculation shows that 

$$m(\Omega_{(\phi_1, \phi_2, \theta_0)}) = \theta_0\left(\cos \phi_1 - \cos \phi_2\right).$$

Again, we can define the surface area of the corresponding rectangle $R_{(\phi_1, \phi_2, \theta_0)}$ on the unit sphere, as

$$\sigma(R_{(\phi_1, \phi_2, \theta_0)}) = \frac{m(\Omega_{(\phi_1, \phi_2, \theta_0)})}{\frac{1}{3}r^3}.$$

Let us go back to the case of $\mathbb{R}^n$. We denote by $S_{n-1}$ the unit sphere in $\mathbb{R}^n$: $S_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. For $x \in \mathbb{R}^n \setminus \{0\}$ we write 

$$x = rx', \quad \text{where} \quad r = |x|, \ x' = \frac{x}{|x|}.$$ 

These are called the polar coordinates in $\mathbb{R}^n$. The corresponding map $\Phi : \mathbb{R}^n \setminus \{0\} \to (0, +\infty) \times S_{n-1}$ is then $\Phi(x) = (r, x')$. It is a continuous bijection, and its inverse is also continuous, being given by $\Phi^{-1}(r, x') = rx'$. We denote by $(m_n)_*$ the measure on $(0, +\infty) \times S_{n-1}$ induced by $m_n$ by the formula

$$(m_n)_*(A) = m_n(\Phi^{-1}(A)),$$

where $A$ is a Borel set in $(0, +\infty) \times S_{n-1}$. We denote by $\rho = \rho_n$ the measure on $(0, +\infty)$ given by, for $E \subseteq (0, +\infty)$,

$$\rho(E) = \int_E r^{n-1} \, dr.$$ 

**Definition 4.18.** Let $E$ be a Borel set in $S_{n-1}$. For $a > 0$ define 

$$E_a = \Phi^{-1}((0, a] \times E) = \{x \in \mathbb{R}^n : x = rx', 0 < r \leq a, x' \in E\},$$

and set 

$$\sigma(E) = n \cdot m_n(E_1).$$

It is clear that $\sigma$ is a measure on the $\sigma$-algebra of Borel sets in $S_{n-1}$, since the map $E \mapsto E_1$ takes Borel sets in $S_{n-1}$ to Borel sets in $\mathbb{R}^n$ and commutes with unions, intersections and complements.

We wish to show that the measures $(m_n)_*$ and $(\rho \times \sigma)$ coincides on the Borel subsets of $(0, +\infty) \times S_{n-1}$.

**Theorem 4.19.** With the above definition, $\sigma$ is the unique measure on the unit sphere $S_{n-1}$ in $\mathbb{R}^n$ such that $(\rho \times \sigma) = (m_n)_*$. Moreover, if $f \in L^1(m_n)$, we have

$$\int f(x) \, dm_n(x) = \int_0^{+\infty} \int_{S_{n-1}} f(rx') \, d\sigma(x') \, r^{n-1} \, dr. \quad (24)$$

**Proof.** We have just observed that $\sigma$ is indeed a measure on the $\sigma$-algebra of Borel sets in $S_{n-1}$. Moreover, writing $T_a(x) = ax$ for $a > 0$ and $x \in \mathbb{R}^n$,

$$m_n(E_a) = m_n(T_a(E_1)) = a^n m_n(E_1) = \frac{a^n}{n} \sigma(E),$$
using Thm. 4.15 (2) and the definition of $\sigma$. Therefore, for $0 < a < b$,
\[
(m_n)_\ast((a,b] \times E) = m_n(E_b \setminus E_a) = m_n(E_b) - m_n(E_a) \\
= \frac{b^n}{n} \sigma(E) - \frac{a^n}{n} \sigma(E) = \frac{b^n - a^n}{n} \sigma(E) = \sigma(E) \int_a^b r^{n-1} dr \\
= (\rho \times \sigma)((a,b] \times E).
\]
Therefore, the measures $(m_n)_\ast$ and $(\rho \times \sigma)$ coincides on the sets of the form $(a,b] \times E$, where $E$ is a Borel in $S_{n-1}$. It is easy to see that the $\sigma$-algebra generated by such sets is indeed the $\sigma$-algebra of Borel sets of $(0, +\infty) \times S_{n-1}$, so that $(m_n)_\ast = (\rho \times \sigma)$.

Formula (24) holds true for simple functions and passing to the limit, using the MCT if $f \in L^1(m_n)$ or the DCT if $f \in L^1(m_n)$, we obtain the result.

Recall that a function $f$ on $\mathbb{R}^n$ is said to be radial if it depends on $|x|$, that is, if there exists $g$ defined on $(0, +\infty)$ such that $f(x) = g(|x|)$, for all $x$.

**Corollary 4.20.** (1) If $f \in L^1(m_n)$ or if $f \in L^1(m_n)$ is radial, and $f(x) = g(|x|)$, for all $x$, then
\[
\int f(x) \, dm_n(x) = \omega_n \int_0^{+\infty} g(|x|) r^{n-1} dr
\]
where $\omega_n = \sigma(S_{n-1})$ denotes the surface area of $S_{n-1}$.

(2) If $B_r = \{x : |x| < r\}$ denotes the ball centered at the origin and radius $r > 0$, and $f_a(x) = |x|^{-a}$, we have
   (i) $f_a \in L^1(B_r)$ if $a < n$, while if $g(x) \geq c|x|^{-n}$, then $g \notin L^1(B_r)$;
   (ii) $f_a \in L^1(cB_r)$ if $a > n$, while if $g(x) \geq c|x|^{-n}$, then $g \notin L^1(cB_r)$.

**Proof.** The part (1) follows at once from the previous result, while part (2) follows from (1). \qed

We conclude this part with the evaluation of some fundamental explicit integrals.

**Proposition 4.21.** We have:
   (i) for $a > 0$, $\int e^{-a|x|^2} \, dx = \left(\frac{\pi}{a}\right)^{n/2}$;
   (ii) $\sigma(S_{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$;
   (iii) $m_n(B_1) = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}$. 
Proof. (i) Let \( I_n = \int_{\mathbb{R}^n} e^{-a|x|^2} \, dx \). We observe that, by Cor. 4.20 (1),
\[
I_2 = 2\pi \int_0^{+\infty} e^{-r^2} r \, dr = -\left(\frac{\pi}{a}\right) e^{-ax^2}\bigg|_0^{+\infty} = \frac{\pi}{a}.
\]
Next, we observe that, since \( e^{-a|x|^2} = \prod_{k=1}^n e^{-ax_k^2} \), using Tonelli’s theorem we have
\[
I_n = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left( \prod_{k=1}^n e^{-ax_k^2} \right) \, dx_1 \cdots dx_n = (I_1)^n.
\]
In particular, \( I_1 = \sqrt{I_2} = \left(\frac{\pi}{a}\right)^{1/2} \) and \( I_n = \left(\frac{\pi}{a}\right)^{n/2} \), as we wished to show.

(ii) We now observe that
\[
\pi^{n/2} = \int e^{-|x|^2} \, dx = \sigma(S_{n-1}) \int_0^{+\infty} e^{-r^2} r^{n-1} \, dr
\]
\[
= \frac{\sigma(S_{n-1})}{2} \int_0^{+\infty} e^{-s} s^{n-1} \, ds
\]
\[
= \frac{\sigma(S_{n-1})}{2} \Gamma(n/2).
\]
This gives (ii).

Finally, we know that \( m_n(B_1) = \sigma(S_{n-1})/n \), so that
\[
m_n(B_1) = \frac{\pi^{n/2}}{\frac{n}{2} \Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)},
\]
by the properties of the Gamma function. \( \square \)

5. Hausdorff measures

5.1. A quick review of submanifolds in \( \mathbb{R}^n \).

Definition 5.1. A subset \( M \) of \( \mathbb{R}^n \) is called a \( C^1 \)-submanifold, of dimension \( k \), \( 1 \leq k \leq n - 1 \), if for every \( x \in M \) there exist a nbgh \( U \) of \( x \), an open set \( V \subseteq \mathbb{R}^k \) and \( \varphi : V \to U \) such that

- \( \varphi \) is injective;
- \( \varphi(V) = M \cap U \);
- the rank of the Jacobian of \( \varphi \) is maximal at every point \( x \in V \), that is, \( \text{rank} \left( \text{Jac} \varphi(x) \right) = k \) for all \( x \in V \).

The map \( \varphi \), or the pair \( (\varphi, V) \), is called a local parametrization for \( M \cap U \), or local chart.

We also assume that \( M \) can be covered by at most finitely many of such sets \( M \cap U \).

Remark 5.2. Using the implicit function theorem, it is easy to see that this definition is equivalent to requiring the following: For each \( x \in M \) there exist a nbgh \( U \) of \( x \) and a \( C^1 \)-function \( F : U \to \mathbb{R}^{n-k} \) such that

- \( U \cap M = \{ x \in \mathbb{R}^n : F(x) = 0 \} \);
- the rank of the Jacobian of \( F \) is maximal at every point \( x \in U \), that is, \( \text{rank} \left( \text{Jac}(F)(x) \right) = n - k \) for all \( x \in U \).
Indeed, we may apply the implicit function theorem to $F$ at the point $x$ where $F(x) = 0$, separating the variable $x$ as $x = (x', x'')$, with $x' \in \mathbb{R}^k$ and $x'' \in \mathbb{R}^{n-k}$ in such a way that $\det \text{Jac}_x(F)(x', x'') \neq 0$ in a nbgh $U$ of $x$. Then, there exist a nbgh $V' \subseteq \mathbb{R}^k$ of $x'$, a nbgh $V''$ of $x''$ in $\mathbb{R}^{n-k}$ and a unique $C^1$ function $\varphi : V' \to V''$ such that $F(x', \varphi(x')) = 0$. Then, the set $M \cap U = \{(x', x'') \in U : F(x', x'') = 0\} = \varphi(V')$, and $\varphi$ satisfies the hypotheses as in Def. 5.1.

**Remark 5.3.** We quickly review the familiar cases when $n = 2, 3$, and $k = 1, 2$, that is, the concepts of surfaces and surface integrals in $\mathbb{R}^3$, in analogy with the theory of (piecewise) $C^1$ curves and line integrals.

Let $A$ be a connected open set in $\mathbb{R}^2$ and $D$ a set such that $A \subseteq D \subseteq \overline{A}$, so that, in particular $\overline{D} = A$. Let $\varphi : D \to \mathbb{R}^3$, injective, $\varphi \in C^1(D)$ and such that $\text{rank}(\text{Jac}(\varphi)) = 2$ on $D$. Setting $\Sigma = \varphi(D)$, the pair $(\Sigma, \varphi)$ is called a surface in $\mathbb{R}^3$ and $\varphi$ its parametrization. Notice that we allow $\varphi$ also to be defined on a portion or the whole of the boundary of $A$. We will in particular use this observation in the case of the sphere in $\mathbb{R}^3$.

We observe that $\text{rank}(\text{Jac}(\varphi)) = 2$ if and only if the matrix $(\varphi_u \varphi_v)$ has rank 2, if and only if $\varphi_u \wedge \varphi_v \neq 0$, where we denote by $(u, v)$ the coordinates in $D \subseteq \mathbb{R}^2$,

$$
\varphi_u = \begin{pmatrix} \partial_u \varphi_1 \\ \partial_u \varphi_2 \\ \partial_u \varphi_3 \end{pmatrix} \quad \text{and} \quad \varphi_v = \begin{pmatrix} \partial_v \varphi_1 \\ \partial_v \varphi_2 \\ \partial_v \varphi_3 \end{pmatrix}.
$$

Then, the surface area of $\Sigma$ is defined as

$$\text{Area}(\Sigma) = \int_D \|\varphi_u \wedge \varphi_v\| \, dudv,
$$

and the surface integral of a continuous function defined on $M$ as

$$
\int_{\Sigma} f \, d\sigma = \int_D f(\varphi(u, v)) \|\varphi_u \wedge \varphi_v\| \, dudv.
$$

The reader should check that the unit sphere in $\mathbb{R}^3$, parametrized by the spherical coordinates $\varphi : [0, \pi] \times [0, 2\pi], \varphi(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$, is an example of surface in $\mathbb{R}^3$.

### 5.2 Hausdorff measures

Although we will work in $\mathbb{R}^n$, we begin with a few definitions and facts that have their natural setting in the class of metric spaces.

**Definition 5.4.** Let $(\mathcal{X}, d)$ be a metric space, with distance $d$. Let $\mu^*$ be an outer measure on $\mathcal{X}$. We say that $\mu^*$ is a *metric outer measure* if for any $A, B \subseteq \mathcal{X}$ with $d(A, B) > 0$ we have that

$$
\mu^*(A \cup B) = \mu^*(A) + \mu^*(B).
$$

We recall that $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$. It is immediate to check that $d(A, B) > 0$ implies that $A \cap B = \emptyset$, while $d(A, B) = d(\overline{A}, \overline{B})$ so that $d(A, B) = 0$ does not imply that $A \cap B = \emptyset$.

**Proposition 5.5.** Let $\mu^*$ be a metric outer measure in a metric space $(\mathcal{X}, d)$. Then, the Borel sets are $\mu^*$-measurable.

**Proof.** (†) In order to check that the Borel $\sigma$-algebra $\mathcal{B}_\mathcal{X}$ is contained in the $\sigma$-algebra of $\mu^*$-measurable sets, it suffices to check that every closed $F$ is $\mu^*$-measurable. In turn, by (10), it suffices to check that

$$
\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \cap ^{c}F) \quad \text{for all } A \text{ with } \mu^*(A) < +\infty.
$$
For the complete proof, we refer the reader to [F] Prop. 11.16.

We now introduce the concept of Hausdorff measure. For $\alpha \geq 0$ we set

$$\gamma_\alpha = \frac{\pi^{\alpha/2}}{2^\alpha \Gamma\left(\frac{\alpha}{2} + 1\right)}.$$  

(25)

Recall that in Prop. 4.21 (iii) we computed the Lebesgue volume of the unit ball in $\mathbb{R}^n$. Notice that we have that $\gamma_n = \frac{m_n(B_1)}{2^n}$.

**Definition 5.6.** Let $(\mathcal{X}, d)$ be a second-countable metric space, $\delta > 0$ and $\alpha \geq 0$. For $A \subseteq \mathcal{X}$ set

$$H_\alpha^{(\delta)}(A) = \gamma_\alpha \inf \left\{ \sum_{j=1}^{+\infty} (\text{diam } B_j)^\alpha : A \subseteq \bigcup_{j=1}^{+\infty} B_j, \text{diam } B_j \leq \delta \right\}.$$  

(26)

Notice that the the quantity $H_\alpha^{(\delta)}(A)$ is increasing as $\delta$ decreases, since we are taking the infimum over a smaller set of coverings. Thus, we define

$$H_\alpha(A) = \lim_{\delta \to 0^+} H_\alpha^{(\delta)}(A).$$  

(27)

We call $H_\alpha(A)$ the $\alpha$-dimensional Hausdorff outer measure of $A$.

**Remarks 5.7.** A few remarks are in order.

1. In fact, it is not necessary to restrict the definition of Hausdorff measure to second-countable metric spaces. This assumption guarantees that every open cover has a countable subcover. On a general metric space, we need to allow the open covers also to be uncountable.

2. We will soon show, see Prop. 5.8 that $H_\alpha$ is indeed a metric outer measure, and by Prop. 5.4 it will then follow that $H_\alpha|_{\mathcal{B}_X}$ is a measure; see Cor. 5.9.

3. In the definition (26) the $B_j$’s are generic subsets of $\mathcal{X}$. It is equivalent to require that the $B_j$’s are closed, since $\text{diam } \overline{B_j} = \text{diam } B_j$, or open, since one replaces $B_j$ by an open set of diameter $\leq \text{diam } B_j + \epsilon 2^{-j}$. In the case of $\mathcal{X} = \mathbb{R}^n$, it is possible to show that it is equivalent to take the $B_j$’s to be open, or closed, (euclidean) balls. However, this fact is not elementary and we refer to [KP], Cor. 4.3.9.

**Proposition 5.8.** $H_\alpha$ is a metric outer measure.

*Proof.* Prop. 3.1 gives that $H_\alpha^{(\delta)}$ is an outer measure for each $\delta > 0$. Then, clearly $H_\alpha(\emptyset) = \lim_{\delta \to 0^+} H_\alpha^{(\delta)}(\emptyset) = 0$, if $A \subseteq B$ are subsets in $\mathcal{X}$, then,

$$H_\alpha(A) = \lim_{\delta \to 0^+} H_\alpha^{(\delta)}(A) \leq \lim_{\delta \to 0^+} H_\alpha^{(\delta)}(B) = H_\alpha(B).$$

Finally, if $A = \bigcup_{j=1}^{+\infty} A_j$, then, using the MCT we have

$$H_\alpha(A) = \lim_{\delta \to 0^+} H_\alpha^{(\delta)}(A) \leq \lim_{\delta \to 0^+} \sum_{j=1}^{+\infty} H_\alpha^{(\delta)}(A_j) = \sum_{j=1}^{+\infty} \lim_{\delta \to 0^+} H_\alpha^{(\delta)}(A_j) = \sum_{j=1}^{+\infty} H_\alpha(A_j).$$

---

$^6$We recall that a topological space is called second-countable if its topology has a countable base. In particular, every cover by open sets as a countable subcollection that is still a cover, see Theorem 16.9 in S. Willard, *General Topology*, (1970) Addison-Wesley Publishing.
Now we show that $\mathcal{H}_\alpha$ is a metric outer measure. Let $A, B \subseteq \mathcal{X}$, $d(A, B) > 0$. Let $\delta$ be such that $0 < \delta < d(A, B)$, and let $\{E_j\}$ be a covering of $A \cup B$ with $\text{diam} \, E_j < \delta$ for each $j$. Then, each $E_j$ can intersect only one among $A$ and $B$, that is, for every $j$, if $E_j \cap A \neq \emptyset$, then $E_j \cap B = \emptyset$, and vice versa, if $E_j \cap B \neq \emptyset$, then $E_j \cap A = \emptyset$. Then, we split the covering $\{E_j\}$ into two disjoint collections $\{E_j^{(A)}\}$ and $\{E_j^{(B)}\}$ such that $A \subseteq \bigcup_{j=1}^{+\infty} E_j^{(A)}$ and $B \subseteq \bigcup_{j=1}^{+\infty} E_j^{(B)}$. We then have,

$$H_\alpha^{(\delta)}(A) + H_\alpha^{(\delta)}(B) \leq \sum_{j=1}^{+\infty} H_\alpha^{(\delta)}(E_j^{(A)}) + \sum_{j=1}^{+\infty} H_\alpha^{(\delta)}(E_j^{(A)}) = \sum_{j=1}^{+\infty} H_\alpha^{(\delta)}(E_j).$$

This inequality holds for all coverings $\{E_j\}$ of $A \cup B$, with diameters $\leq \delta$. Thus, by taking the infimum over such coverings on the right hand side, we see that

$$H_\alpha^{(\delta)}(A) + H_\alpha^{(\delta)}(B) \leq H_\alpha^{(\delta)}(A \cup B),$$

and then we pass to the limit as $\delta \to 0^+$ to obtain

$$H_\alpha(A) + H_\alpha(B) \leq H_\alpha(A \cup B),$$

as we wished to show.

\begin{proof}
Corollary 5.9. If $(\mathcal{X}, d)$ is a second-countable metric space, then $(\mathcal{X}, \mathcal{H}_\alpha, \mathcal{B}_\mathcal{X})$ is a measure space.

Proof. The previous proposition shows that $\mathcal{H}_\alpha$ is a outer measure, which is also metric, in the sense of Def. 5.4. Prop. 5.5 gives that Borel sets $\mathcal{B}_\mathcal{X}$ in $\mathcal{X}$ are $\mathcal{H}_\alpha$-measurable, so $\mathcal{H}_\alpha|_{\mathcal{B}_\mathcal{X}}$ is a measure, by Carathéodory’s Theorem 3.3; hence the conclusion.

The proposition that follows will imply that we can define the Hausdorff dimension of any set $A \subseteq \mathcal{X}$.

Proposition 5.10. Let $A \subseteq \mathcal{X}$. Then, the following properties hold true:

(i) if $H_\alpha(A) < +\infty$ for some $\alpha$, then $H_\beta(A) = 0$ for all $\beta > \alpha$;

(ii) if $H_\alpha(A) > 0$ for some $\alpha$, then $H_\gamma(A) = +\infty$ for all $\gamma < \alpha$;

(iii) $\sup \{ \gamma : H_\gamma(A) = +\infty \} = \inf \{ \beta : H_\beta(A) = 0 \} := \alpha$.

Such value $\alpha$ is called the Hausdorff dimension of $A$.

Proof. (i) Suppose $H_\alpha(A) < +\infty$. Let $\varepsilon_0 > 0$ be fixed. For any $\delta > 0$, there exists a covering $\{B_j\}$ of $A$ with $\text{diam} \, B_j \leq \delta$ such that $\gamma_\alpha \sum_{j=1}^{+\infty} (\text{diam} \, B_j)^\alpha \leq H_\alpha^{(\delta)}(A) + \varepsilon_0$. Since $H_\alpha^{(\delta)}(A) \leq H_\alpha(A)$ for every $\delta > 0$, for such covering of $A$ we have

$$\gamma_\alpha \sum_{j=1}^{+\infty} (\text{diam} \, B_j)^\alpha \leq H_\alpha(A) + \varepsilon_0.$$

Now, if $\beta > \alpha$ we have

$$\sum_{j=1}^{+\infty} (\text{diam} \, B_j)^\beta \leq \delta^{\beta - \alpha} \sum_{j=1}^{+\infty} (\text{diam} \, B_j)^\alpha \leq \frac{1}{\gamma_\alpha} \delta^{\beta - \alpha} (H_\alpha(A) + \varepsilon_0),$$

so that

$$H_\beta^{(\delta)}(A) \leq \frac{\gamma_\beta}{\gamma_\alpha} \delta^{\beta - \alpha} (H_\alpha(A) + \varepsilon_0).$$
Letting $\delta \to 0^+$ we see that $H_\beta(A) = 0$. This proves (i).

In order to prove (ii), we simply notice that, seeking a contradiction, if $H_\gamma(A) < +\infty$ for some $\gamma < \alpha$, then part (i) implies that $H_\alpha(A) = 0$, a contradiction. Then (ii) follows.

Finally, given $A \subseteq \mathcal{X}$, parts (i) and (ii) show that the value $\alpha$ is well defined, and we are done. \hfill \Box

5.3. Hausdorff measures in $\mathbb{R}^n$. We now restrict ourselves to the case of $\mathbb{R}^n$, endowed with the standard euclidean distance. From now on therefore, $H_\alpha$ will denote the $\alpha$-Hausdorff measure on the $\sigma$-algebra of Borel sets in $\mathbb{R}^n$. Our goal is to show that when $\alpha = k = 0, 1, \ldots, n$, then the Hausdorff measure $H_k$ provides the right notion of measure on a $k$-dimensional submanifold in $\mathbb{R}^n$.

**Theorem 5.11.** On the $\sigma$-algebra of Borel sets in $\mathbb{R}^n$, $H_n = m_n$.

**Proof.** (†) We prove that there exist constants $C_n \leq C'_n$ such that

$$H_n \leq C_n m_n \leq C'_n H_n.$$  \hfill (28)

To show that it is possible to choose $C_n = C'_n = \gamma_n$ requires a couple of results that fall beyond the scope of this course. For the proof we refer to [L] Teorema 2.1. \hfill \Box

We now give a few examples of the Hausdorff dimension of some given sets.

**Example 5.12.** (1) Let $A = \{p\}$ be the set consisting a single point in $\mathbb{R}^n$. Then $H_0(\{p\}) = 1$, and therefore the Hausdorff dimension of $\{p\}$ is 0.

Indeed, for any covering $\{B_j\}$ of $\{p\}$ by sets of diameter $\leq \delta$, since $\gamma_0 = 1$,

$$\gamma_0 \sum_j (\text{diam } B_j)^0 = \text{cardinality}(\{B_j\}) \geq 1.$$

Thus, $H_0^\delta(\{p\}) \geq 1$ for all $\delta > 0$, which implies $H_0(\{p\}) \geq 1$. Conversely, a single ball $B(p, \delta/2)$ covers $\{p\}$ so that, $H_0^\delta(\{p\}) \leq 1$ and therefore also $H_0(\{p\}) \leq 1$.

(2) Using the fact that $H_0$ is a measure, it follows at once from (1) that if $A \subseteq \mathbb{R}^n$ is finite or countable, then $H_0(A) = \text{cardinality}(A)$.

(3) Is $A \subseteq \mathbb{R}^n$ is a Borel measurable set such that $m_n(A) > 0$, then, its Hausdorff dimension equals $n$. The conclusion follows at once from Prop. 5.10.

(4) Let $C$ be the Cantor set in $\mathbb{R}$. Then, the Hausdorff dimension of $C$ is $\log 2/\log 3$. For a proof, see [L].

We now wish to define an intrinsic notion of measure on a $k$-dimensional submanifold $M$ of $\mathbb{R}^n$. We begin with a lemma.

**Lemma 5.13.** Let $k$ be an integer, $1 \leq k \leq n$ and let $T : \mathbb{R}^k \to \mathbb{R}^n$ be an injective linear transformation and $T^*$ its adjoint. Then, setting $\mathcal{J}(T) = \sqrt{\det(T^*T)}$, for any Borel set $A \subseteq \mathbb{R}^k$,

$$H_k(T(A)) = \mathcal{J}(T) H_k(A).$$

**Proof.** We begin by observing that $T^*T : \mathbb{R}^k \to \mathbb{R}^k$ is positive semi-definitive, since $\langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = ||Tv||^2$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^k$ and $\| \cdot \|$ the norm in $\mathbb{R}^k$.

If $k = n$, then $T \in \text{GL}(n, \mathbb{R})$, and $\det(T^*T) = (\det T)^2$, so that $\mathcal{J}(T) = |\det T|$. Hence, the result follows from Thm.‘s 4.15 and 5.11.
Assume now $k < n$. Then, there exists a rotation $R$ such that
\[ R(T(R^k)) = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_{k+1} = \cdots = x_n = 0 \} = \mathbb{R}^k \times \{0\}. \]
Set $S = RT$. Then, identifying $\mathbb{R}^k \times \{0\}$ with $\mathbb{R}^k$ itself, $S$ becomes a linear transformation of $\mathbb{R}^k$ into itself. Using the result for transformation in the same dimension $k$, we have that
\[ H_k(S(A)) = J(S)H_k(A). \]
Moreover, $S^*S = (RT)^*RT = T^*R^*RT = T^*T$, so that $J(S) = J(T)$ and, using the fact that $H_k$ is rotational invariant, we have
\[ H_k(T(A)) = H_k(RT(A)) = H_k(S(A)) = J(S)H_k(A) = J(T)H_k(A). \]
This completes the proof.

We now generalize the previous result to $C^1$-injections, as in Def. 5.1.

**Theorem 5.14.** Let $V$ be an open set in $\mathbb{R}^k$, $k \geq 1$, $\varphi : V \to \mathbb{R}^n$ a $C^1$, injective map such that \( \text{Jac} \varphi(x) \) has rank $k$ at every point $x \in V$. Let $A \subseteq V$ be any Borel set, then $\varphi(A)$ is Borel measurable in $\mathbb{R}^n$ and
\[ H_k(\varphi(A)) = \int_A J(\text{Jac} \varphi(x)) \, dH_k(x). \quad (29) \]
Moreover, if $f$ is a Borel measurable function on $\varphi(V) = M$ such that, either $f \in L^+(M, H_k)$ or $f \in L^1(M, H_k)$, then
\[ \int_M f(y) \, dH_k(y) = \int_V f(\varphi(x)) J(\text{Jac} \varphi(x)) \, dH_k(x). \quad (30) \]
Notice that the integrals on the right hand sides of 29 and (30) are evaluated on Borel sets of $\mathbb{R}^k$, so the measures $dH_k$ equal $dm_k$ in both cases.

**Proof.** (†) See [F], Lemma 11.22 and Thm. 11.25.

**Example 5.15.** Let $\gamma$ be a simple regular curve in $\mathbb{R}^n$, that for the moment we take without its end points. This means that we are given an injective, $C^1$ map, $\varphi : (a, b) \to \mathbb{R}^n$, such that $|\varphi'(t)| \neq 0$, for all $t \in (a, b)$. Then, the length of $\gamma$ was defined as
\[ \ell(\gamma) = \int_a^b |\gamma'(t)| \, dt, \]
and the line integral of a continuous function defined on $\gamma$ as
\[ \int_{\gamma} f \, ds = \int_a^b f(\gamma(t))|\gamma'(t)| \, dt. \]

We set $M = \varphi((a, b))$ and apply Thm. 5.14. Observe that $\text{Jac} \varphi = \begin{pmatrix} \varphi'_1 \\ \vdots \\ \varphi'_n \end{pmatrix}$ so that
\[ J(\text{Jac} \varphi(t))^2 = \begin{pmatrix} \varphi'_1(t) & \cdots & \varphi'_n(t) \end{pmatrix} \begin{pmatrix} \varphi'_1(t) \\ \vdots \\ \varphi'_n(t) \end{pmatrix} = |\varphi'(t)|^2. \]
Therefore, by (29) we have that
\[ H_1(M) = H_1(\varphi((a, b))) = \int_{(a, b)} \mathcal{J}(\text{Jac } \varphi(t)) \, dH_1(t) = \int_a^b |\varphi'(t)| \, dt = \ell(M). \]

Similarly, if \( f \) is a continuous function non-negative on \( M \), then it is Borel measurable and by (30) we have
\[ \int_M f(y) \, dH_1(y) = \int_{(a, b)} f(\varphi(t)) \mathcal{J}(\text{Jac } \varphi(t)) \, dH_1(t) = \int_a^b f(\varphi(t))|\varphi'(t)| \, dt. \]

Therefore, the integral w.r.t. the 1-dimensional Hausdorff measure generalizes both the notion of length of a regular curve and the one of line integral of continuous function.

It is also easy to extend the above argument to the case of a piecewise \( C^1 \), simple curve, and including the endpoints. We have seen in Example 5.12 (1) that \( H_0 \) coincides with the counting measure in \( \mathbb{R}^n \), so that \( H_0(\{p_1, \ldots, p_N\}) = N \), for any finite collection of points. This also implies that \( H_\alpha(\{p_1, \ldots, p_N\}) = 0 \) for any \( \alpha > 0 \). Then, if \( \varphi : (a, b) \cup \{\varphi(a), \varphi(b)\} \rightarrow \mathbb{R}^3 \), then
\[ H_1(M) = H_1(\varphi((a, b))) + H_1(\varphi(a)) + H_1(\varphi(b)) = H_1(\varphi((a, b))), \]
and then the argument proceeds as before. Analogously for the integral of \( f \):
\[ \int_M f(y) \, dH_1(y) = \int_{(a, b)} f(\varphi(t)) \mathcal{J}(\text{Jac } \varphi(t)) \, dH_1(t) + 0 = \int_a^b f(\varphi(t))|\varphi'(t)| \, dt. \]

**Example 5.16.** Let \( M \) be a surface in \( \mathbb{R}^3 \) and assume that it is parametrized by a single \( C^1 \), injective map, \( \varphi : V \rightarrow \mathbb{R}^3 \), where \( V \) is an open set in \( \mathbb{R}^2 \), and \( \text{Jac } \varphi(y) \) has rank 2 for all \( y \in V \). Then, adopting the notation in Example 5.3, we have that
\[ \text{Area}(M) = \int_V \|\varphi_u \wedge \varphi_v\| \, dudv, \]
and the surface integral of a continuous function defined on \( M \) as
\[ \int_M f \, d\sigma = \int_V f(\varphi(u, v)) \|\varphi_u \wedge \varphi_v\| \, dudv. \]

Again, we have want to apply Thm. 5.14. Observe that \( \text{Jac } \varphi = \begin{pmatrix} \nabla \varphi_1 \\ \nabla \varphi_2 \\ \nabla \varphi_3 \end{pmatrix} = (\varphi_u \quad \varphi_v) \) so that
\[ \mathcal{J}(\text{Jac } \varphi(t)) = \begin{pmatrix} t \varphi_u \\ t \varphi_v \end{pmatrix} = \begin{pmatrix} |\varphi_u|^2 & \varphi_u \cdot \varphi_v \\ \varphi_u \cdot \varphi_v & |\varphi_v|^2 \end{pmatrix}. \]

Then, it is elementary to check the equality
\[ \mathcal{J}(\text{Jac } \varphi(t)) = \|\varphi_u \wedge \varphi_v\|. \]

Therefore, by (29) we have that
\[ H_2(M) = H_2(\varphi(V)) = \int_V \mathcal{J}(\text{Jac } \varphi(y)) \, dH_2(y) = \int_V \|\varphi_u \wedge \varphi_v\| \, dudv = \text{Area}(M). \]
Similarly, if \( f \) is a continuous function non-negative on \( M \), then it is Borel measurable and by (30) we have
\[
\int_M f(y) \, dH_2(y) = \int_V f(\varphi(u, v)) \, J(\text{Jac } \varphi(u, v)) \, dH_2(u, v) = \int_V f(\varphi(u, v)) \|\varphi_u \land \varphi_v\| \, dudv.
\]
Therefore, the integral w.r.t. the 2-dimensional Hausdorff measure generalizes both the notion of area of a regular surface and the one of surface integral of a continuous function.

**Example 5.17.** Let \( M \) be the graph of a \( C^1 \) function \( f : V \subseteq \mathbb{R}^k \rightarrow \mathbb{R} \), that is, \( M = \{y \in \mathbb{R}^{k+1} : y = (u, f(u)), \ u \in V\} \). Then, \( \varphi : V \rightarrow \mathbb{R}^{k+1} \) given by \( \varphi(u) = (u, f(u)) \) is clearly injective, \( C^1 \) and its Jacobian has maximal rank 1. Then,
\[
(Surface \ area)(M) = \int_V \sqrt{1 + |\nabla f(u)|^2} \, du,
\]
and the surface integral of a continuous function defined on \( M \) as
\[
\int_M f \, d\sigma = \int_V f(\varphi(u)) \sqrt{1 + |\nabla f(u)|^2} \, du.
\]
Again, we have want to apply Thm. 5.14. Observe that \( \text{Jac } \varphi = (1 \ \nabla f) \) so that
\[
J(\text{Jac } \varphi(t))^2 = 1 + |\nabla f(u)|^2,
\]
as it is elementary to check.

Therefore, by (29) we have that
\[
H_k(M) = H_k(\varphi(V)) = \int_V J(\text{Jac } \varphi(y)) \, dH_2(y) = \int_V \sqrt{1 + |\nabla f(u)|^2} \, du = (Surface \ area)(M).
\]

Similarly, if \( g \) is a continuous function non-negative on \( M \), then it is Borel measurable and by (30) we have
\[
\int_M g(y) \, dH_2(y) = \int_V g(\varphi(u)) \, J(\text{Jac } \varphi(u)) \, dH_k(u) = \int_V g(\varphi(u)) \sqrt{1 + |\nabla f(u)|^2} \, du.
\]
Again, the integral w.r.t. the \( k \)-dimensional Hausdorff measure generalizes both the notion of surface area of a regular surface and the one of surface integral of a continuous function.
6. The fundamental theorem of calculus and the theorems of Gauss and Green

In this final part we generalize the fundamental theorem of calculus and the notion and technique of integration by parts to the case of domains in higher dimensions.

One of the main issues that appear while passing from 1 to higher dimensions is the shape of open, and in particular, of connected open sets. It is well known that on the real line the connected open sets are just the open intervals, and that every open set can be written as at most countable union of disjoint open intervals. The situation is considerably more complicated in higher dimensions, and we begin by addressing this issue.

6.1. The fundamental theorem of calculus and Gauss theorem. In order to simplify our presentation, but also because in practice it suffices to consider this case, we restrict our attention to connected open sets.

Definition 6.1. A bounded, connected open set \( \Omega \subseteq \mathbb{R}^n \), \( n \geq 2 \), is called a \( C^k \)-regular domain if:

(i) \( \Omega = \overline{\Omega} \);

(ii) the boundary \( \partial \Omega \) of \( \Omega \) is a \( C^k \), \( (n-1) \)-dimensional submanifold in \( \mathbb{R}^n \).

Notice that condition (i) rules out domains such as \( A = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 < 4 \right\} \setminus \left\{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1, y \geq 0 \right\} \).

It is easy to see that \( A \) is bounded, open, connected, and that \( \overline{A} = B(0,2) \), so that \( \overline{A} = B(0,2) \) which is strictly larger than \( A \).

Our first goal is to show that \( C^k \)-regular domains can be analytically described in a fairly simple way.

Lemma 6.2. Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded set given by

\[
\Omega = \left\{ x \in \mathbb{R}^n : \psi(x) < 0 \right\},
\]

where \( \psi : \mathbb{R}^n \to \mathbb{R} \) is a \( C^k \)-function such that and \( \nabla \psi(x) \neq 0 \) if \( \psi(x) = 0 \). Then \( C^k \)-regular domain.

Such a function \( \psi \) is called a defining function for \( \Omega \). As an example we consider the case of the ball of radius 1 in \( \mathbb{R}^n \), \( B = B(0,1) \), when, clearly

\[
B = \left\{ x \in \mathbb{R}^n : 1 - |x|^2 < 0 \right\}
\]

that is, \( \psi(x) = 1 - |x|^2 \) is a defining function for \( B \).

Proof. (†) Clearly, \( \Omega \) is open and it is bounded by assumption. It is it is connected as continuous image of a connected set.

We claim that

\[
\partial \Omega = \overline{\partial \Omega} = \left\{ x \in \mathbb{R}^n : \psi(x) = 0 \right\}.
\]

If is clear that both \( \partial \Omega, \partial \overline{\Omega} \subseteq \{ x \in \mathbb{R}^n : \psi(x) = 0 \} \). Thus, it suffices to show that if \( \psi(x_0) = 0 \), then \( x_0 \) belongs to both \( \partial \Omega \) and \( \partial \overline{\Omega} \). By the mean value property, for each \( x \in \mathbb{R}^n \), there exists \( \xi_x \in s_{x,x_0} \), the segment joining \( x \) to \( x_0 \), such that

\[
\psi(x) - \psi(x_0) = \nabla \psi(\xi_x) \cdot (x - x_0).
\]
As customary, we denote by $\nu(x_0)$ the unit vector $\nabla \psi(x_0)/|\nabla \psi(x_0)|$ (which is normal to the level curves of $\psi$), and observe that is well defined since $\nabla \psi(x_0) \neq 0$. If we take $x$ on the straight line passing through $x_0$ with direction $\nu(x_0)$, that is, $x$ is of the form $x = x_0 + t\nu(x_0)$, with $t \in \mathbb{R}$. Plugging into (32) we obtain,

$$\psi(x) = t(\nabla \psi(\xi_x), \nu(x_0)).$$

Next, since $\langle \nabla \psi(\xi_x), \nu(x_0) \rangle \rightarrow \langle \nabla \psi(x_0), \nu(x_0) \rangle = |\nabla \psi(x_0)| > 0$ as $x \rightarrow x_0$, it follows that there exists $\delta > 0$ such that if $x \in B(x_0, \delta)$ (that is, $|x - x_0| < \delta$), then $\psi(x) > 0$ if $t > 0$ and $\psi(x) < 0$ if $t < 0$. These conditions can be expressed as

$$\psi(x + t\nu(x_0)) > 0 \quad \text{if} \quad 0 < t < \delta,$$

and

$$\psi(x + t\nu(x_0)) < 0 \quad \text{if} \quad -\delta < t < 0.$$

Therefore,

$$x_0 + t\nu(x_0) \in \Omega \quad \text{and} \quad x_0 + t\nu(x_0) \notin \overline{\Omega} \quad \text{if} \quad -\delta < t < 0.$$

This clearly implies that $x_0 \in \partial \Omega \cap \partial \overline{\Omega}$, and proves the claim (31).

Now, using (31), we see that condition (i) in Def. 6.1 follows since

$$\Omega \subseteq \overline{\Omega} = \overline{\Omega} \setminus \partial \overline{\Omega} \subseteq \{x : \psi(x) \leq 0\} \setminus \partial \overline{\Omega} = \{x : \psi(x) < 0\} = \Omega.$$

Finally, using (31) again, we have that $\partial \Omega = \{x : \psi(x) = 0\}$ and the conclusion follows from Remark 5.2. \qed

We now wish to show that every $C^k$-regular domains can be described as in Lemma 6.2, thus proving the converse implication. We need a few preliminary notation and definitions.

**Definition 6.3.** Given $x \in \mathbb{R}^n$ and $k \in \{1, \ldots, n\}$, we denote by $\hat{x}_k$ the element of $\mathbb{R}^{n-1}$ obtained from $x = (x_1, \ldots, x_n)$ by removing the coordinate $x_k$. Thus, e.g. $x = (x_1, \hat{x}_1) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and also $x = (\hat{x}_n, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.  

We say that a domain $D \subseteq \mathbb{R}^n$ is normal w.r.t. to $x_k$, or also $k$-normal, if there exist an open set $V \subseteq \mathbb{R}^{n-1}$, $b > 0$ and a function $g : V \rightarrow (0, b)$ (or $g : V \rightarrow (-b, 0)$, resp.) such that

$$D = \{(\hat{x}_k, x_k) \in V \times (0, b) : 0 < x_k < g(\hat{x}_k)\}$$

or

$$D = \{(\hat{x}_k, x_k) \in V \times (-b, 0) : g(\hat{x}_k) < x_k < 0\},$$

resp.

In other words, $D$ is $k$-normal if it is the portion of space lying between the domain $V$ in the hyperplane $x_k = 0$ and the graph of the function $g$, that may lie above or below the hyperplane $x_k = 0$.

Next, we recall the definition of *partition of unity*. Below we denote by $\text{supp} \, g$ the support of a continuous function $g$.

**Definition 6.4.** Given a compact subset $K$ of $\mathbb{R}^n$ and a finite open cover $\{B_j\}$ of $K$, $j = 1, \ldots, m$, we say that a collection $\{p_j\}$ of functions, $j = 1, \ldots, m$, is a $C^\infty$ partition of unity subordinated to the covering $\{p_j\}$ if

(i) for each $j$, $p_j \in C^\infty(\mathbb{R}^n)$ and $\text{supp} \, p_j \subseteq B_j$;
(ii) $0 \leq p_j \leq 1$, $j = 1, \ldots, m$, and $\sum_{j=1}^{m} p_j(x) = 1$ for all $x \in K$.

**Theorem 6.5.** Given a compact set $K$ and a finite open cover $\{B_j\}$ of $K$, there exists a partition of unity subordinated to the cover $\{B_j\}$.

*Proof.* (†) This is a standard result and it appears in many analysis and geometry books. For a proof we refer to [F] or [L]. □

Before our first main result of this section, we need a further lemma.

**Lemma 6.6.** Let $\Omega$ be a $C^k$-regular domain, $k \geq 1$. Let $A$ be open, $A \supset \overline{\Omega}$ and $h : A \to \mathbb{R}$ be a $C^1$-function, whose support is contained in $\Omega$. Then, for every $j = 1, \ldots, n$,

\[
\int_{\Omega} \partial_{x_j} h(x) \, dx = 0.
\]

*Proof.* We use Fubini’s theorem and the 1-dimensional result. Observe that supp$(\partial_{x_j} h)$ $\subseteq$ supp $h$, and moreover supp$(\partial_{x_j} h)$ $\subseteq$ $\Omega$ implies that also supp$(\partial_{x_j} h) \cap \partial \Omega = \emptyset$. Moreover, if $R > 0$ is large enough so that $\overline{\Omega} \subseteq [-R,R]^n$, which is possible since $\Omega$ is bounded, then $h$ and $\nabla h$ equal 0 on $\subseteq [-R,R]^n \setminus \Omega$. Therefore,

\[
\int_{\Omega} \partial_{x_j} h(x) \, dx = \int_{[-R,R]^n} \partial_{x_j} h(x) \, dx = \int_{[-R,R]^{n-1}} \left( \int_{-R}^{R} \partial_{x_j} h(\hat{x}_j, x_j) \, dx_j \right) \hat{x}_j = 0,
\]

since $h(\hat{x}_j, R) = h(\hat{x}_j, -R) = 0$. □

**Lemma 6.7.** Let $D$ be a $k$-normal domain, say

\[
D = \{(\hat{x}_k, x_k) \in V \times (0, b) : 0 < x_k < g(\hat{x}_k)\},
\]

where $g : V \to (0, b)$. Let $f : D \to \mathbb{R}$ be a $C^1$-function, whose support is contained in the set $\{(\hat{x}_k, x_k) \in V \times (0, b)\}$. Then, for every $j = 1, \ldots, n$,

\[
\int_{\Omega} \partial_{x_j} f \, dH_n = \int_{\mathcal{G}(g)} f \nu_j \, dH_{n-1}, \tag{33}
\]

where $\mathcal{G}(g) = \{x \in \mathbb{R}^n : x = (\hat{x}_k, x_k), x_k = g(\hat{x}_k)\}$.

Notice that the measure $dH_n$ on the left hand side is just the Lebesgue measure, while the measure $dH_{n-1}$ on the right hand side is the induced surface measure, and the integration is restricted to the portion of the boundary of $D$ where the function $f$ does not vanish.

*Proof.* We need to distinguish two cases: $k = j$ and $k \neq j$. We begin with the former one, $k = j$. We have

\[
\int_{\Omega} \partial_{x_j} f \, dH_n = \int_{V} \left( \int_{0}^{g(\hat{x}_k)} \partial_{x_j} f(\hat{x}_j, x_j) \, dx_j \right) \, d\hat{x}_j = \int_{V} f(\hat{x}_j, g(\hat{x}_j)) - f(\hat{x}_j, 0) \, d\hat{x}_j = \int_{V} f(\hat{x}_j, g(\hat{x}_j)) \, d\hat{x}_j, \tag{34}
\]
since we are taking \( j = k \) and the support of \( f \) is contained in \( \{(\hat{x}_k, x_k) \in V \times (0, b)\} \), so that \( f(\hat{x}_k, 0) = 0 \).

Consider now the integral on the right hand side of (33). Notice that \( D = \{(\hat{x}_k, x_k) \in V \times (0, b) : \psi(x) := x_k - g(\hat{x}_k) < 0\} \), so that the normal at the points of \( \mathcal{G}(g) \) is

\[
\nu = \frac{\nabla \psi}{|\nabla \psi|}.
\] (35)

Again since \( j = k \), we have

\[
\nu_j = \frac{1}{\sqrt{1 + |\nabla g(\hat{x}_j)|^2}} \quad x \in \mathcal{G}(g).
\]

Since \( \mathcal{G}(g) \) is the the graph of the function \( g \), by Example 5.17 the integral over \( \mathcal{G}(g) \) w.r.t. \( dH_{n-1} \) is given by integration against the density \( \sqrt{1 + |\nabla g(\hat{x}_j)|^2} \). Precisely,

\[
\int_{\mathcal{G}(g)} f \nu_j dH_{n-1} = \int_V f(\hat{x}_j, g(\hat{x}_j)) \frac{1}{\sqrt{1 + |\nabla g(\hat{x}_j)|^2}} \sqrt{1 + |\nabla g(\hat{x}_j)|^2} d\hat{x}_j
\]

\[
= \int_V f(\hat{x}_j, g(\hat{x}_j)) d\hat{x}_j.
\]

Substituting this equality into the right hand side of (34) we obtain the result in the case \( j = k \).

Next, suppose \( j \neq k \). Using the fundamental theorem of calculus in one variable we have,

\[
\int_{\Omega} \partial_{x_j} f dH_n = \int_V \left( \int_0^{g(\hat{x}_k)} \partial_{x_j} f(\hat{x}_k, x_k) \, dx_k \right) d\hat{x}_k
\]

\[
= \int_V \frac{\partial}{\partial x_j} \left( \int_0^{g(\hat{x}_k)} f(\hat{x}_k, x_k) \, dx_k \right) d\hat{x}_k - \int_V f(\hat{x}_k, x_k) \partial_{x_j} g(\hat{x}_k) \, d\hat{x}_k.
\]

We now use Lemma 6.6 applied to the function

\[
h(\hat{x}_k) = \begin{cases} f(\hat{x}_k, x_k) \, dx_k & \hat{x}_k \in V \\
0 & \hat{x}_k \not\in V.
\end{cases}
\]

Then, \( h \) is a \( C^1 \) function, with support contained in the regular domain \( V \subseteq \mathbb{R}^{n-1} \), so that Lemma 6.6 gives that

\[
\int_V \frac{\partial}{\partial x_j} \left( \int_0^{g(\hat{x}_k)} f(\hat{x}_k, x_k) \, dx_k \right) d\hat{x}_k = 0.
\] (36)

Therefore,

\[
\int_{\Omega} \partial_{x_j} f dH_n = - \int_V f(\hat{x}_k, x_k) \partial_{x_j} g(\hat{x}_k) \, d\hat{x}_k.
\]

Again, consider the integral on the right hand side of (33). Since \( \nu \) is given by equation (35) and \( j \neq k \) we have,

\[
\nu_j = - \frac{\partial_{x_j} g}{\sqrt{1 + |\nabla g|^2}}.
\]
Hence,

\[
\int_{\mathcal{G}(g)} f\nu_j dH_{n-1} = \int_{\mathcal{V}} f(\hat{x}_j, g(\hat{x}_k)) \frac{-\partial_{x_j} g(\hat{x}_k)}{\sqrt{1 + |\nabla g(\hat{x}_k)|^2}} \sqrt{1 + |\nabla g(\hat{x}_k)|^2} \, d\hat{x}_k
\]

\[
= - \int_{\mathcal{V}} f(\hat{x}_k, g(\hat{x}_k)) \partial_{x_j} g(\hat{x}_k) \, d\hat{x}_j.
\]

Substituting this equality into the right hand side of (36) we obtain the result in the case \( j \neq k \). This completes the proof. \( \square \)

We now come to one of the main results of this part.

**Theorem 6.8.** Let \( \Omega \subseteq \mathbb{R}^n \) be a \( C^1 \)-regular domain, \( A \supseteq \overline{\Omega} \) an open set, \( f \in C^1(A) \). Then, for every \( j \in \{1, \ldots, n\} \) we have

\[
\int_\Omega \partial_{x_j} f \, dH_n = \int_{\partial\Omega} f\nu_j \, dH_{n-1}.
\]

**Remark 6.9.** We remark that the above identity (37) is often written as

\[
\int_\Omega \partial_{x_j} f(x) \, dx = \int_{\partial\Omega} f(x')\nu_j(x') \, d\sigma(x'),
\]

thus writing the Lebesgue measure on the left hand side with its standard notation \( dx \) and indicating the induced surface measure on the right hand side as \( d\sigma \).

**Proof.** By the implicit function theorem, see also Remark 5.2, for every \( x_0 \in \partial\Omega \), there exists a nbhb \( U_{x_0} \) of \( x_0 \) such that \( \Omega \cap U_{x_0} \) is a \( k \)-normal domain, for some \( k \in \{1, \ldots, n\} \), and \( \partial\Omega \cap U_{x_0} \) can be expressed as the graph of a \( C^k \)-function. We can use the compactness of \( \overline{\Omega} \) we can select a finite such cover \( \{B_\ell\}, \ell = 1, \ldots, N \), and \( B_\ell \cap \Omega \) is a \( k \)-normal domain, for some \( k \). Moreover, there exists an open set \( B_0 \) such that

\[
B_0 \supseteq \Omega \setminus \bigcup_{\ell=1}^N B_\ell, \quad \text{and} \quad \overline{B}_0 \subseteq \Omega.
\]

Then, \( \{B_\ell : \ell = 0, 1, \ldots, N\} \) is an open cover of \( \overline{\Omega} \), and by Prop. 6.5 there exists a \( C^\infty \) partition of unity subordinated to \( \{B_\ell\}, \ell = 0, \ldots, N \); call it \( \{p_\ell\}, \ell = 0, \ldots, N \). In particular we have

\[
\text{supp } p_\ell \subseteq B_\ell, \quad \ell = 0, \ldots, N, \quad \text{and} \quad \sum_{\ell=0}^N p_\ell(x) = 1 \quad \text{on } \overline{\Omega}.
\]
Therefore, using the fact that $p_0$ has compact support in $\Omega$ together with Lemma 6.6, and then Lemma 6.7, we have

\[ \int_{\Omega} \partial_{x_j} f(x) \, dx = \int_{\Omega} \frac{\partial}{\partial x_j} \left( \sum_{\ell=0}^{N} f(x)p_\ell(x) \right) \, dx = \sum_{\ell=0}^{N} \int_{\Omega} \frac{\partial}{\partial x_j} (f(x)p_\ell(x)) \, dx \]

\[ = \sum_{\ell=1}^{N} \int_{\Omega} \frac{\partial}{\partial x_j} (f(x)p_\ell(x)) \, dx = \sum_{\ell=1}^{N} \int_{\Omega \cap B_\ell} \partial_{x_j} (f(x)p_\ell(x)) \, dx \]

\[ = \sum_{\ell=1}^{N} \int_{\partial \Omega \cap B_\ell} f(x')p_\ell(x') \nu_j(x') \, dH_{n-1}(x') = \sum_{\ell=1}^{N} \int_{\partial \Omega} f(x')p_\ell(x') \nu_j(x') \, dH_{n-1}(x') \]

\[ = \int_{\partial \Omega} f(x') \nu_j(x') \, dH_{n-1}(x') , \]

since $\text{supp } p_\ell \subseteq B_\ell$ and $\sum_{\ell=1}^{N} p_\ell(x') = 1$ on $\partial \Omega$. This completes the proof. \[ \square \]

**Corollary 6.10. (Integration by parts in higher dimension)** Let $\Omega \subseteq \mathbb{R}^n$ be a $C^1$-regular domain, $A \supseteq \Omega$ an open set, $f, g \in C^1(A)$. Then, for every $j \in \{1, \ldots, n\}$ we have

\[ \int_{\Omega} g \partial_{x_j} f \, dH_n = \int_{\partial \Omega} g f \nu_j \, dH_{n-1} - \int_{\Omega} f \partial_{x_j} g \, dH_n . \quad (38) \]

**Proof.** The proof is now immediate. By the previous theorem we have

\[ \int_{\Omega} g \partial_{x_j} f \, dH_n + \int_{\Omega} f \partial_{x_j} g \, dH_n = \int_{\Omega} \frac{\partial}{\partial x_j} (fg) \, dH_n \]

\[ = \int_{\partial \Omega} gf \nu_j \, dH_{n-1} , \]

and the conclusion follows. \[ \square \]

Recall that if $F = (F_1, \ldots, F_n)$ is a $C^1$ vector field on an open set $A \subseteq \mathbb{R}^n$, then the divergence of $F$ is defined as

\[ \text{div } F = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j} . \]

**Corollary 6.11. (Divergence Theorem, or Gauss Theorem)** Let $\Omega \subseteq \mathbb{R}^n$ be a $C^1$-regular domain, $A \supseteq \Omega$ an open set, $F \in C^1(A, \mathbb{R}^n)$ a $C^1$ vector field on $A$. Then,

\[ \int_{\Omega} \text{div } F \, dH_n = \int_{\partial \Omega} \langle F, \nu \rangle \, dH_{n-1} . \quad (39) \]

Notice that the divergence theorem says that the outward flux of a vector field through a closed surface equals the integral of the divergence over the region inside the surface w.r.t. the volume measure.
Proof. This is also immediate consequence of Thm. 6.8:

$$\int_\Omega \text{div} F \, dH_n = \sum_{j=1}^n \int_\Omega \frac{\partial F_j}{\partial x_j} \, dH_n$$

$$= \sum_{j=1}^n \int_{\partial \Omega} F_j \nu_{x_j} \, dH_{n-1}$$

$$= \int_{\partial \Omega} \langle F, \nu \rangle \, dH_{n-1}. \quad \square$$

6.2. Extension to domains with lower regularities. (†)

As we wish to extend the previous result to more general domain, such the cube for instance, whose boundary is piecewise regular. We begin by making this idea into a precise definition.

Definition 6.12. We say that a bounded domain $\Omega \subseteq \mathbb{R}^n$, with $n \geq 2$, is a piecewise $C^k$-regular domain if

(i) $\Omega = \tilde{\Omega}$;

(ii) there exists an integer $N$ such the boundary $\partial \Omega = \Sigma_1 \cup \cdots \Sigma_N \cup S$ such that

(ii-1) for each $j = 1, \ldots, N$, $\Sigma_j$ is a relatively open subset of $\partial \Omega$;

(ii-2) for each $j = 1, \ldots, N$ there exists a $C^k$, $(n-1)$-dimensional submanifold $M_j$ such that $\Sigma_j \subseteq$;

(ii-3) the subset $S$ is compac and it is contained in an $(n-2)$-submanifold;

(ii-4) $\Sigma_j \cap \Sigma_k \subseteq S$, for $j \neq k$.

In this case, we call $\Sigma_1 \cup \cdots \Sigma_N$ the regular part of the boundary, and we denote it by $\partial_{\text{reg}} \Omega$.

Example 6.13. The most typical example of such domain is a cube, or rectangle in $\mathbb{R}^n$. Let us limit ourselves to the the cases of $n = 2, 3$.

1. Let $R = (a, b) \times (c, d)$ be an open rectangle in $\mathbb{R}^2$. Then the $\partial R = \Sigma_1 \cup \ldots \Sigma_4 \cup S$, where $\Sigma_1, \ldots, \Sigma_4$ are the four open sides, say $\Sigma_1 = \{(x, c) : a < x < b\}$, $\Sigma_2 = \{(b, y) : c < y < d\}$, etc., while $S$ is the union of the four vertices $S = \{(a, c), (b, c), (b, d), (a, d)\}$. It is elementary to check that $R$ is a piecewise $C^1$-regular domain.

2. The same reasoning applies to the case of a rectangle in $\mathbb{R}^3$,

$$R = (a, b) \times (c, d) \times (\alpha, \beta) = \{(x, y, z) \in \mathbb{R}^3 : a < x < b, c < y < d, \alpha < z < \beta\}.$$  

Now

$$\partial R = \Sigma_1 \cup \cdots \Sigma_8 \cup S$$

where for $j = 1, \ldots, 8$, $\Sigma_j$ is one of the open faces, and $S$ is the union of the open edge, that are 1-dimensional manifolds, and of the vertices, that have Hausdorff dimension 0.

With the above definitions, it is easy to extend the results of the previous section to the case of piecewise $C^k$-regular domains. We observe that, if $\Omega$ is a piecewise $C^k$-regular domain, then on the regular part of its boundary $\partial_{\text{reg}} \Omega$, the outward unit normal vector $\nu$ is well defined. Hence we have the following results.

This part is only required for the written part of the exam.
Theorem 6.14. Let $\Omega \subseteq \mathbb{R}^n$ be a piecewise $C^1$-regular domain, $A \supseteq \overline{\Omega}$ an open set, $f \in C^1(A)$. Then, for every $j \in \{1, \ldots, n\}$ we have
\[
\int_{\Omega} \partial_x j f \, dH_n = \int_{\partial_{\text{reg}} \Omega} f \nu_j \, dH_{n-1}.
\] (40)

Corollary 6.15. (Integration by parts in higher dimension) Let $\Omega \subseteq \mathbb{R}^n$ be a piecewise $C^1$-regular domain, $A \supseteq \overline{\Omega}$ an open set, $f, g \in C^1(A)$. Then, for every $j \in \{1, \ldots, n\}$ we have
\[
\int_{\Omega} g \partial_x j f \, dH_n = \int_{\partial_{\text{reg}} \Omega} gf \nu_j \, dH_{n-1} - \int_{\Omega} f \partial_x j g \, dH_n.
\] (41)

Corollary 6.16. (Divergence Theorem, or Gauss Theorem) Let $\Omega \subseteq \mathbb{R}^n$ be a piecewise $C^1$-regular domain, $A \supseteq \overline{\Omega}$ an open set, $\mathbf{F} \in C^1(A, \mathbb{R}^n)$ a $C^1$ vector field on $A$. Then,
\[
\int_{\Omega} \text{div} \mathbf{F} \, dH_n = \int_{\partial_{\text{reg}} \Omega} \langle \mathbf{F}, \nu \rangle \, dH_{n-1}.
\] (42)

6.3. Green’s Theorem and identities. We begin with the classical Green’s formula in $\mathbb{R}^2$.

Theorem 6.17. (Green’s Theorem) Let $\Omega \subseteq \mathbb{R}^2$ be a $C^1$-regular domain, $A \supseteq \overline{\Omega}$ an open set, $\omega = f \, dx + g \, dy$ a 1-form in $\mathbb{R}^2$ with coefficients in $C^1(A)$. Then,
\[
\int_{\partial \Omega} f \, dx + g \, dy = \int_{\Omega} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dH_2,
\] (43)
where $\partial \Omega$ is oriented counter-clockwise.

Proof. We recall that, if $\partial \Omega$ is parametrized by a (piecewise) $C^1$ curve $\gamma$, parametrized as $\gamma : [a, b] \to \mathbb{R}^2$ then
\[
\int_{\partial \Omega} f \, dx + g \, dy = \int_a^b f(\gamma(t)) \gamma_1'(t) + g(\gamma(t)) \gamma_2'(t) \, dt.
\]
Observe now that, defining the vector field $\mathbf{F} = (g, -f)$, the right hand side above can be written as
\[
\int_a^b \left( f(\gamma(t)) \frac{\gamma_1'(t)}{|\gamma'(t)|} + g(\gamma(t)) \frac{\gamma_2'(t)}{|\gamma'(t)|} \right) |\gamma'(t)| \, dt
\]
\[
= \int_a^b \left( g(\gamma(t)) \frac{\gamma_2'(t)}{|\gamma'(t)|} + ((-f(\gamma(t))) \frac{-\gamma_1'(t)}{|\gamma'(t)|} + \right) |\gamma'(t)| \, dt
\]
\[
= \int_{\partial \Omega} \langle \mathbf{F}, \nu \rangle \, dH_1,
\]
since, given the orientation of $\partial \Omega$, the outward normal $\nu$ equals $\left( \frac{\gamma_2'(t)}{|\gamma'(t)|}, -\frac{\gamma_1'(t)}{|\gamma'(t)|} \right)$. Thus, applying the Divergence Theorem to the term on the right hand side we see that
\[
\int_{\partial \Omega} \langle \mathbf{F}, \nu \rangle \, dH_1 = \int_{\Omega} \text{div} \mathbf{F} \, dH_2 = \int_{\Omega} \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \, dH_2;
\]
hence the conclusion. □
Definition 6.18. (†) The Laplace operator, or laplacian, in $\mathbb{R}^n$ is the second order partial differential operator

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$ 

Notice that, if $f$ is twice differentiable in $\mathbb{R}^n$ then $\Delta f = \text{div} \nabla f$.

Theorem 6.19. (†) (Green’s identities) Let $\Omega \subseteq \mathbb{R}^2$ be a $C^1$-regular domain, $A \supset \overline{\Omega}$ an open set, $\omega = f dx + g dy$ a 1-form in $\mathbb{R}^2$ with coefficients in $C^1(A)$. Then, denoting by $\frac{\partial u}{\partial \nu} := \nu \cdot \nabla u$ the normal derivative of $u$ on $\partial \Omega$,

\begin{align*}
\int_{\Omega} v\Delta u + \langle \nabla v, \nabla u \rangle \, dH_n &= \int_{\partial \Omega} v \frac{\partial u}{\partial \nu} \, dH_{n-1}; \\
\int_{\Omega} (u\Delta v - v\Delta u) \, dH_n &= \int_{\partial \Omega} \left( u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \, dH_{n-1}; \\
\int_{\Omega} (u\Delta u + |\nabla u|^2) \, dH_n &= \int_{\partial \Omega} u \frac{\partial u}{\partial \nu} \, dH_{n-1}.
\end{align*}

Proof. All identities follow easily from the divergence theorem. We observe, that

$$\text{div} \left( v \nabla u \right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( v \frac{\partial u}{\partial x_j} \right) = \langle \nabla v, \nabla u \rangle + v\Delta u,$$

and

$$\langle v \nabla u, \nu \rangle = v \frac{\partial u}{\partial \nu},$$

so that (44) follows at once.

Now (45) follows from (44) by taking a difference, and (45) follows from (44) by taking $v = u$. □
REFERENCES


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