NOTES FOR THE COURSE HARMONIC ANALYSIS

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A † indicates the parts that are not required for the exam.
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1. The Fourier transform

We begin by introducing some notation and basic definitions and recalling properties of a few function spaces.

We denote partial derivatives by $\partial_{x_j}$. We denote by $\mathbb{N}$ the set of non-negative integers. If $\alpha$ is a multi-index, i.e. $\alpha \in \mathbb{N}^n$, we write $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \sum_{j=1}^n \alpha_1 + \cdots + \alpha_n$, and $\alpha! = \prod_{j=1}^n \alpha_1 \cdots \alpha_n$.

Given a function $f$ on $\mathbb{R}^n$ we denote by $\tau_y$ the translation by $y$, that is,

$$\tau_y f(x) = f(x - y). \quad (1.1)$$

We recall that when $g$ is a continuous function, its support $\text{supp } g$ is the smallest closed set outside of which it vanishes, that is, it is the closure of $g^{-1}(\mathbb{C}\{0\})$. We denote by $C_c = C_c(\mathbb{R}^n)$ the space of continuous function with compact support and by $C_0 = C_0(\mathbb{R}^n)$ the continuous functions that vanish at infinity, that is, $\lim_{|x| \to \infty} g(x) = 0.1$

The following result is elementary and its proof is left as an exercise (see Exercise 2 in Appendix B).

**Lemma 1.1.** Endowed with the uniform norm, the space $C_0(\mathbb{R}^n)$ is a Banach space and $C_c(\mathbb{R}^n)$ is a dense subspace. Functions in $C_0$ are uniformly continuous.

1.1. **Convolutions.** If $f$ and $g$ are measurable functions on $\mathbb{R}^n$, their convolution $f \ast g$ is the function defined as

$$f \ast g(x) = \int f(x - y)g(y) \, dy$$

for all $x$ for which the integral exists.

The question of measurability of the convolution deserves a discussion. For $f$ Lebesgue measurable, the function $F(x,y) = f(x-y)$ turns out to be measurable on $\mathbb{R}^n \times \mathbb{R}^n$. One can see this fact by writing $F = f \circ g$ where $g(x,y) = x - y$ and show that $g^{-1}(E)$ of a Lebesgue measurable set is still a Lebesgue measurable set.\(^2\)

We now set some notation that will be use sistematically throughout these notes.

Given $f$ on $\mathbb{R}^n$ and $t > 0$ we define

$$f_t(x) = t^{-n}f(x/t) \quad \text{and} \quad f^t(x) = f(tx). \quad (1.2)$$

Notice that, if $f \in L^1$ then

$$\int f_t(x) \, dx = \int t^{-n}f(x/t) \, dx = \int f(y) \, dy$$

and that, if the integral is well defined (for instance if $f, g \in L^2$)

$$\int f_t(x)g(x) \, dx = \int f(x)g^t(x) \, dx,$$
as a simple change of variables shows.

The next result is elementary, however we need to define the support of a function \( f \) that is locally integrable, but defined only a.e. When \( f \) is only locally integrable (that is, \( f \in L^1(K) \) for every compact set \( K \)), we define

\[
\text{supp } f = \{ x \in \mathbb{R}^n : A \text{ open, bounded and } \int_A f = 0 \}.
\]

(1.3)

Thus, \( \text{supp } f \) is a closed set and outside of which \( f = 0 \) a.e. For consistency, we need to check that if \( f \) is also continuous, the above definition coincides with the one given for continuous functions. To do so, we only need to check that if \( E \) is a closed set outside of which \( f \) vanishes, then \( E \) contains \( \text{supp } f \) as defined (1.3). But this is clear passing to the complementary sets (and reversing the inclusion).

**Proposition 1.2.** Assume that the integrals below exist. Then

(i) \( f \ast g = g \ast f \);

(ii) \( f \ast (g \ast h) = (f \ast g) \ast h \);

(iii) \( \tau_y(f \ast g) = (\tau_y f) \ast g = f \ast (\tau_y g) \);

(iv) \( \text{supp}(f \ast g) \subseteq \text{supp} f + \text{supp} g = \{ x \in \mathbb{R}^n : x = y + z, y \in \text{supp } f, z \in \text{supp } g \} \).

From Fubini's theorem it is clear that the convolution of two \( L^1 \)-functions is still in \( L^1 \). But more can be said. Now we prove the following.

**Theorem 1.3.** The following properties hold.

(i) (Young’s inequality) If \( f \in L^1 \) and \( g \in L^p \), with \( 1 \leq p \leq \infty \) then \( f \ast g \) exists for almost every \( x \), \( f \ast g \in L^p \) and

\[
\| f \ast g \|_p \leq \| f \|_1 \| g \|_p.
\]

(ii) (Generalized Young’s inequality) Let \( 1 \leq p, q, r \leq \infty \) be such that \( 1/p + 1/q = 1 + 1/r \). Let \( f \in L^p, q \in L^q \), then \( f \ast g \in L^r \) and

\[
\| f \ast g \|_r \leq \| f \|_p \| g \|_q.
\]

(iii) If \( f \in L^1 \) and \( g \in C^k \) and \( \partial^\alpha_x g \in L^\infty \) for \( |\alpha| \leq k \), then

\[
\partial^\alpha_x (f \ast g) = f \ast \partial^\alpha_x g
\]

for \( |\alpha| \leq k \).

**Proof.** Notice that (i) follows from (ii) by taking \( p = 1 \) and \( q = r \) in (ii). Now we prove (ii).

Notice that \( 1/p' + 1/q' + 1/r = 1 \). Then, by the generalized Hölder’s inequality\(^3\)

\[
|f \ast g(x)| \leq \int |f(x-y)||g(y)| \, dy
\]

\[
= \int |f(x-y)|^{p/r} |f(x-y)|^{1-p/r} |g(y)|^{q/r} |g(y)|^{1-q/r} \, dy
\]

\[
\leq \left( \int |f(x-y)|^{p} |g(y)|^{q} \, dy \right)^{1/r} \left( \int |f(x-y)|^{q'} (1-p/r) \, dy \right)^{1/q'} \left( \int |g(y)|^{p' (1-q/r)} \, dy \right)^{1/p'}
\]

\[
= \left( \int |f(x-y)|^{p} |g(y)|^{q} \, dy \right)^{1/r} \| f \|^{1-p/r} \| g \|^{1-q/r}.
\]

\(^3\)See Exercise 4.
where we have used the fact that $1/q' = 1/p - 1/r$ so that $q'(1 - p/r) = p$ and analogously $1/p' = 1/q - 1/r$ and $p'(1 - q/r) = q$. Then,

$$
\int |f * g(x)|^r dx \leq \int \left( \int |f(x-y)|^p |g(y)|^q dy \right)^{r/p} \|g\|^r_{q}^{r-q} dx
$$

$$
= \|f\|^r_p \|g\|_q^{r-q} \int \left( |g(y)|^{r/q} \int |f(x-y)|^p dx \right) dy
$$

$$
= \|f\|^r_p \|g\|^r_q
$$

from which the conclusion follows.

For part (iii) see we use differentiation under the integral sign, see Exercise 5. Let $|\alpha| = 1$, that is, $\alpha = e_j$. Then, $\partial_x g$ is bounded and $|f(y)\partial_x g(x-y)| \leq C |f(y)|$, where $f \in L^1$, independently of $x$. Then we can pass the derivative under the integral sign

$$
\partial_x \left( \int f(y)g(x-y) \, dy \right) = \int f(y)\partial_x g(x-y) \, dy,
$$

and finally proceed by induction. □

**Theorem 1.4.** Let $\varphi \in L^1$ and let $\int \varphi(x) \, dx = a$. Then,

(i) If $f \in L^p$, $1 \leq p < \infty$, then $f * \varphi_t \to af$ in the $L^p$-norm, as $t \to 0$.

(ii) If $f$ is bounded and uniformly continuous, then $\varphi_t * f \to af$ uniformly as $t \to 0$.

**Proof.** We have

$$
f * \varphi_t(x) - af(x) = \int (f(x-y) - f(x))\varphi_t(y) \, dy
$$

$$
= \int (f(x-tz) - f(x))\varphi(z) \, dz
$$

$$
= \int ((\tau_{tz} f)(x) - f(x))\varphi(z) \, dz.
$$

Now we apply Minkowski’s integral inequality:\footnote{See Appendix, Theorem A.8.}

$$
\|f * \varphi_t - af\|_p = \left\| \int ((\tau_{tz} f) - f)(\cdot)\varphi(z) \, dz \right\|_p \leq \int \|((\tau_{tz} f) - f)\|_p |\varphi(z)| \, dz.
$$

But $\|((\tau_{tz} f) - f)\|_p \leq 2\|f\|_p$ and tends to 0 as $t \to 0^+$ for each $z$ (see Exercise I.6). Thus, assertion (i) follows from the dominated convergence theorem.

If $f$ is bounded and uniformly continuous, then it is still true that $\|((\tau_{tz} f) - f)\|_{\infty} \leq 2\|f\|_{\infty}$ and tends to 0 as $t \to 0^+$ for each $z$. Thus, assertion (ii) also follows from the dominated convergence theorem. □

**Definition 1.5.** We denote by $\mathcal{D}$ (or $C^\infty$) the space of $C^\infty$-functions with compact support, and more generally by $C^\infty_c(\Omega)$ the space of $C^\infty$-functions having compact support contained in $\Omega$.

The next result is elementary, however, we invite the readers to convince themselves of its validity.
Lemma 1.6. The space $\mathcal{D}$ is non-empty; for instance the function
\[ \varphi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases} \]
is an element of $\mathcal{D}$.

By translation and dilation, it is easy to produce an element $\psi$ of $\mathcal{D}$ with supp $\psi \subseteq B(x_0, \delta)$; it suffices to take $\varphi((x-x_0)/\delta)$.

Proposition 1.7. (C*-Urysohn’s Lemma) Let $K$ be a compact set in $\mathbb{R}^n$, $U$ an open set that contains $K$. Then there exists $f \in \mathcal{D}$ such that $f \geq 0$, $f = 1$ on $K$ and $f$ vanishes outside $U$.

Proof. Let $\delta = \text{dist}(K, \overline{U})$ and let $V = \{y : \text{dist}(y, K) < \delta/3\}$. Let $\psi \in \mathcal{D}$, $\int \psi = 1$, supp $\psi \subseteq \{x : |x| \leq \delta/3\}$. Notice that $\psi(x) = c\varphi(3x/\delta)$, with $\varphi$ as in the previous lemma, satisfies the requirements for a suitable constant $c$. Then we set $f = \chi_V * \psi$. We have that,
\[ \text{supp } f \subseteq V + \text{supp } \psi \subseteq \{\text{dist}(x, K) < 2\delta/3\} \subseteq U, \]
Moreover, $0 \leq f \leq 1$, and notice that for $x \in K$, $y \in V$, $|x - y| < \delta/3$ and
\[ f(x) = \int \chi_V(y)\psi(x-y)\,dy = \int_{V \cap \{y : |x-y| < \delta/3\}} \psi(x-y)\,dy = \int_{\{y : |x-y| < \delta/3\}} \psi(x-y)\,dy = 1. \]
This proves the lemma. \qed

Corollary 1.8. The space $\mathcal{D}$ is dense in $L^p$, $1 \leq p < \infty$.

Proof. We have shown that, if $f \in L^p$, $1 \leq p < \infty$, and $\varphi \in \mathcal{D}$, $\int \varphi = 1$, then $f * \varphi_t \to f$ in the $L^p$-norm. Let $f \in L^p$ and let $\varepsilon > 0$ be fixed. Let $K$ be a compact set chosen in such a way that $\|f\chi_K\|_p < \varepsilon/2$. Next, let $t_0$ be such that then $\|f\chi_K - (f\chi_K) * \varphi_t\|_p < \varepsilon/2$ for $0 < t < t_0$. Notice that $(f\chi_K) * \varphi_t \in \mathcal{D}$ for all $t > 0$. Then
\[ \|f - (f\chi_K) * \varphi_t\|_p < \|f - f\chi_K\|_p + \|f\chi_K - (f\chi_K) * \varphi_t\|_p < \varepsilon. \]
We conclude this section with a result (Prop. 1.10) that we will not use, but that is quite useful in a variety of problems.

Lemma 1.9. Let $E$ be a compact set in $\mathbb{R}^n$ and let $\{U_j\}$, $j = 1, \ldots, N$ be a cover of $E$. Then, there exist $\varphi_1, \ldots, \varphi_N \in \mathcal{D}$ such that supp $\varphi_j \subseteq U_j$ for all $j$, $\sum_j \varphi_j(x) = 1$ for all $x$, and $\sum_j \varphi_j(x) = 1$ for all $x \in E$.

Proof. We first observe that there exist compact sets $E_j$ such that $E_j \subseteq U_j$ and $E = \bigcup_{j=1}^N K_j$. Indeed, for each $x \in E$ there exists a ball $B(x,r_x)$ such that $B(x,r_x) \subseteq U_j$ for some $j = 1, \ldots, N$. Since $E$ is compact, we can select $B(x_1,r_{x_1}), \ldots, B(x_M,r_{x_M})$ so that $E \subseteq \bigcup_{k=1}^M B(x_k,r_{x_k})$. Now, we let
\[ E_j = \bigcup \{B(x_k,r_{x_k}) : B(x_k,r_{x_k}) \subseteq U_j \}. \]
Next, Prop. 1.7 guarantees the existence of $\psi_j \in \mathcal{D}$ with support in $U_j$ and equal to 1 on $E_j$. Then we define $\varphi_j = \psi_j/(\sum_j \psi_j)$.
It is easy to see that \( \varphi_j \in \mathcal{D} \), \( 0 \leq \varphi_j \leq 1 \) and that if \( x \in E \), then \( x \in E_j \) for at least one \( j \), so that \( \varphi_j(x) \neq 0 \) (in fact \( \varphi_j(x) = 1 \)) for at least one \( j \). Therefore, if \( x \in E \),

\[
\varphi(x) = \sum_{j=1}^{N} \varphi_j(x) = \sum_{j=1}^{N} \frac{\psi_j(x)}{\sum_{k=1}^{N} \psi_k(x)} = 1. \quad \square
\]

**Proposition 1.10.** (Partition of unity) Let \( \{U_k\} \) be a locally finite open cover\(^5\) of \( \mathbb{R}^n \). Then, there exists \( \{\varphi_k\} \subseteq \mathcal{D} \) such that

- \( \text{supp } \varphi_k \subseteq U_k \) for all \( k \);
- \( 0 \leq \varphi_k \leq 1 \) for all \( k \);
- \( \sum_k \varphi_k(x) = 1 \) for all \( x \in \mathbb{R}^n \).

**Proof.** Let \( Q \) be the closed unit cube \( Q = \{x : 0 \leq x_j \leq 1, \ j = 1, \ldots, n\} \) and \( B \) the ball centered at the center of \( Q \) and radius 1, so that \( Q \subseteq B \). For \( \kappa \in \mathbb{Z}^n \) we set \( Q_\kappa = Q + \kappa \) and \( V_\kappa = B + \kappa \). Then, \( \mathbb{R}^n = \bigcup_{\kappa \in \mathbb{Z}^n} Q_\kappa \), \( V_\kappa \) is open containing \( Q_\kappa \) and each \( x \in \mathbb{R}^n \) belongs to at most \( C_n \) of the \( V_\kappa \)'s, with \( C_n \) independent of \( \kappa \).

For each \( \kappa \in \mathbb{Z}^n \), we can find a finite sub-collection \( \{U^{(\kappa)}_j\}_{j=1}^{N_\kappa} \) of \( \{U_k\} \) that covers \( Q_\kappa \).

For each \( \kappa \in \mathbb{Z}^n \) fixed, we now apply Lemma 1.9 in the case \( E = Q_\kappa \) and finite open cover \( \{U^{(\kappa)}_j\}_{j=1}^{N_\kappa} \), \( j = 1, \ldots, N_\kappa \) and obtain \( \{\psi_{\kappa,j}\}, \ j = 1, \ldots, N_\kappa \), such that \( \psi_{\kappa,j} \in \mathcal{D} \), \( \text{supp } \psi_{\kappa,j} \subset U^{(\kappa)}_j \), and \( 0 \leq \psi_{\kappa,j} \leq 1 \).

Finally we set

\[
\varphi_{\kappa,j}(x) = \frac{\psi_{\kappa,j}(x)}{\sum_{\kappa \in \mathbb{Z}^n} \sum_{j=1}^{N_\kappa} \psi_{\kappa,j}(x)}. \]

It is easy now to convince oneself that the functions \( \varphi_{\kappa,j}, \ \kappa \in \mathbb{Z}^n, \ j = 1, \ldots, N_\kappa \) satisfy the conclusions in the statement. \( \square \)

**1.2. The Fourier transform.** We now come to the main object of our study. For \( f \in L^1(\mathbb{R}^n) \) we define the **Fourier transform** of \( f \) as

\[
\hat{f}(\xi) = \int f(x)e^{-2\pi ix\xi} \, dx.
\]

We will also denote it by \( \mathcal{F}f \), so that to emphasise the action of the Fourier transform as a mapping.

It is clear that \( \hat{f} \) is well defined for \( f \in L^1 \) and that \( \|\hat{f}\|_\infty \leq \|f\|_1 \). Moreover, \( \hat{f} \) is continuous, as an immediate application of the dominated convergence theorem. Thus,

\[
\mathcal{F} : L^1(\mathbb{R}^n) \to C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \tag{1.4} \]

is bounded.

We now see the first elementary properties of the Fourier transform.

**Proposition 1.11.** Let \( f, g \in L^1 \). Then, the following hold.

(i) For any \( y, \eta \in \mathbb{R}^n \), \( \mathcal{F} (\tau_y f)(\xi) = e^{-2\pi i y \cdot \xi} \hat{f}(\xi) \) and \( \tau_\eta \hat{f}(\xi) = \mathcal{F} (e^{2\pi i x \cdot \eta} f)(\xi) \).

(ii) \( \mathcal{F} (f * g) = \hat{f} \hat{g} \).

(iii) If \( x^\alpha f \in L^1 \) for \( |\alpha| \leq k \), then \( \hat{f} \in C^k \) and \( \partial_\xi^\alpha \hat{f}(\xi) = \mathcal{F} ((-2\pi i x)^\alpha f)(\xi) \).

---

\(^5\) A collection \( \{U_k\} \) of sets is called a **locally finite** cover of \( X \) if (i) \( \bigcup_k U_k \supseteq X \), and (ii) for each \( x \in X \) there exist only finitely many \( U_{k_1}, \ldots, U_{k_m} \) that contains \( x \).
(iv) Suppose \( f \in C^k \), \( \partial_x^\alpha f \in L^1 \) for \( |\alpha| \leq k \), then \( \mathcal{F}(\partial_x^\alpha f)(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi) \).

**Proof.** (i) is elementary:
\[
(\tau_y f)(\xi) = \int f(x-y)e^{-2\pi i x \xi} \, dx = \int f(x)e^{-2\pi i (x+y) \xi} \, dx = e^{-2\pi i y \xi} \hat{f}(\xi),
\]
and analogously for the other part of the statement.

(ii) is also elementary. By Fubini’s theorem,
\[
(f * g)(\xi) = \int \int f(x-y)g(y) \, dy \, dx = \int \int f(x-y)e^{-2\pi i (x-y) \xi} \, dx \, dy = \hat{f}(\xi) \hat{g}(\xi).
\]

(iii) We argue as in the proof of Thm. 1.2 (iii). We begin with the case \( |\alpha| = 1 \) and observe that
\[
|\partial_{\xi_j} \left( f(x)e^{-2\pi i x \xi} \right) | \leq C |x_j f(x)| \in L^1(\mathbb{R}^n).
\]
Thus, we can pass the differentiation under the integral sign and then proceed by induction.

In order to prove (iv), we first assume that \( f \) is such that \( \partial_x^\alpha f \in C_0 \) for \( |\alpha| \leq k \) (in particular if \( f \) also has compact support). Under this assumption, assume that \( |\alpha| = 1 \) first. Then, since \( \partial_{x_j} f \) vanishes at \( \infty \), by integration by parts we have
\[
\int \partial_{x_j} f(x)e^{-2\pi i x \xi} \, dx = -\int f(x)(-2\pi i \xi_j)e^{-2\pi i x \xi} \, dx = 2\pi i \xi_j \hat{f}(\xi).
\]
The case when \( |\alpha| > 1 \) follows by induction. The general case now follows by approximating \( f \in L^1 \) such that \( \partial_x^\alpha f \in L^1 \) when \( |\alpha| \leq k \), as follows. Let \( \eta \in C^\infty, 0 \leq \eta \leq 1, \eta = 1 \) when \( |x| \leq 1 \), and \( \eta = 0 \) when \( |x| \geq 2 \). Clearly \( \eta^\alpha f \in L^1 \) having compact support, so the first part of the argument applies. Then, it is easy to see that \( \partial_x^\alpha (\eta^\alpha f) \to \partial_x^\alpha f \) in \( L^1 \) as \( \varepsilon \to 0^+ \), for every \( |\alpha| \leq k \). Since \( \mathcal{F} : L^1 \to C \cap L^\infty \) is bounded (recall (1.4)) we have that
\[
(2\pi i \xi)^\alpha (\eta^\alpha f)(\xi) \to (2\pi i \xi)^\alpha \hat{f}(\xi),
\]
but also
\[
(2\pi i \xi)^\alpha (\eta^\alpha f)(\xi) = \mathcal{F} \left( \partial_x^\alpha (\eta^\alpha f) \right)(\xi) \to \mathcal{F}(\partial_x^\alpha f)(\xi)
\]
uniformly. The conclusion thus follows. \( \Box \)

**Corollary 1.12.** (Riemann–Lebesgue Lemma) The Fourier transform maps \( L^1 \) continuously into \( C_0(\mathbb{R}^n) \):
\[
\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n).
\]

\( ^6 \)See also Exercise 8.
Proof. We only need to show that for \( f \in L^1 \), \( \hat{f} \) vanishes at \( \infty \) (that is \( \lim_{|x| \to +\infty} \partial_x^n f(x) = 0 \)). For \( f \in L^1 \cap C_c^1 \) we have that

\[
|\xi| |\hat{f}(\xi)| \leq \sum_{j=1}^n \left| \xi_j \int f(x) e^{-2\pi i x \xi} \, dx \right|
\leq c_0,
\]

i.e. \( |\xi| \hat{f} \) is bounded, that is \( \hat{f} \in C_0 \). Finally, the result follows by the density of \( L^1 \cap C_c^1 \) in \( L^1 \).

For, given \( g \in L^1 \) and \( \varepsilon > 0 \), let \( f_n \in L^1 \cap C_c^1 \) such that \( \| \hat{f}_n - \hat{g} \|_\infty \leq \| f_n - g \|_1 \to 0 \). Thus, \( \hat{f}_n \in C_0 \) and converges uniformly to \( \hat{g} \). The conclusion now follows. □

We now compute a fundamental Fourier transform.

Lemma 1.13. Let \( a > 0 \) and \( f(x) = e^{-a\pi |x|^2} \). Then

\[
\hat{f}(\xi) = \frac{1}{a^{n/2}} e^{-\pi |\xi|^2 / a}.
\]

Proof. We begin with the case \( n = 1 \). By the previous proposition, we can differentiate under the integral sign and get

\[
(\hat{f})'(\xi) = (-2\pi ixe^{-a\pi x^2})(\hat{f})(\xi) = \left( \frac{i}{a} \right) \left( (e^{-a\pi x^2})' \right)(\xi)
= \left( \frac{i}{a} \right) (2\pi i\xi) \hat{f}(\xi)
= -\frac{2\pi}{a} \xi \hat{f}(\xi).
\]

Therefore, the function \( e^{\pi \xi^2 / a} \hat{f}(\xi) \) satisfies the differential equation

\[
\frac{d}{d\xi} e^{\pi \xi^2 / a} \hat{f}(\xi) = 0,
\]

i.e. it is a constant. In order to compute the constant, we select \( \xi = 0 \) and recall that \( \int e^{-t^2} \, dt = \sqrt{\pi} \), so that

\[
\hat{f}(0) = \int e^{-a\pi x^2} \, dx = \frac{1}{\sqrt{a}}.
\]

The \( n \)-dimensional case follows from an application of Fubini’s theorem:

\[
\hat{f}(\xi) = \prod_{j=1}^n e^{-2\pi i x_j \xi_j e^{-a\pi |x_j|^2}} \, dx_1 \cdots dx_n
= \prod_{j=1}^n e^{-\pi \xi_j^2 / a} \left( \frac{1}{\sqrt{a}} \right)
= \frac{1}{a^{n/2}} e^{-\pi |\xi|^2 / a}.
\]
Remark 1.14. When we consider the two types of dilations defined in 1.2 we easily see have that
\[ \hat{f}_t(t \xi) = (\hat{f})^t(\xi) \quad \text{and} \quad \hat{f}^t(\xi) = \tau(n)^{-1} \hat{f}(\xi/t) = (\hat{f})_t(\xi), \] 
(Equation 1.5).

We now prove

Theorem 1.15. (Inversion Theorem) Let \( f \in L^1 \) be such that \( \hat{f} \in L^1 \). Then
\[ f(x) = \int e^{2\pi i \xi x} \hat{f}(\xi) \, d\xi. \]
Proof. Let \( t > 0 \) and set
\[ \psi(\xi) = e^{2\pi i \xi - \pi t |\xi|^2}. \]
Then,
\[ (\mathcal{F}\psi)(y) = \tau_x(\mathcal{F}e^{-\pi t^2 |\xi|^2})(y) = t^{-n} \tau_x(e^{-\pi |y|^2/t^2}) = \varphi_t(x - y), \]
where \( \varphi(x) = e^{-\pi |x|^2} \).

Next notice that, if \( f, \varphi \in L^1 \), then
\[ \int f(x) \varphi(x) \, dx = \int \int f(x) \varphi(\xi) e^{-2\pi i \xi \cdot x} \, d\xi \, dx = \int \hat{f}(\xi) \varphi(\xi) \, d\xi. \] 
(Exercise).

Therefore,
\[ \int e^{2\pi i \xi - \pi t |\xi|^2} \hat{f}(\xi) \, d\xi = \int \psi(\xi) \hat{f}(\xi) \, d\xi = \int \hat{\varphi}(y) f(y) \, dy \]
\[ = \int \varphi_t(x - y) f(y) \, dy = f \ast \varphi_t(x). \]

Since \( \int \varphi(x) \, dx = \int e^{-\pi |x|^2} \, dx = 1 \), we have that \( f \ast \varphi_t \to f \) in the \( L^1 \) norm as \( t \to 0 \). On the left hand side of the equalities above we can use the dominated convergence theorem to show that \( \int e^{2\pi i \xi - \pi t |\xi|^2} \hat{f}(\xi) \, d\xi \to \int e^{2\pi i \xi} \hat{f}(\xi) \, d\xi \) as \( t \to 0 \). \( \square \)

As an immediate consequence of the Inversion Theorem we have the following.

Corollary 1.16. If \( f \in L^1 \) and \( \hat{f} = 0 \), then \( f = 0 \).

We now extend the definition of the Fourier transform to functions in \( L^2 \).

Theorem 1.17. (Plancherel Theorem) Let \( f \in L^1 \cap L^2 \). Then \( \hat{f} \in L^2 \) and \( \mathcal{F}_{L^1 \cap L^2} \) extends uniquely to a unitary isomorphism of \( L^2 \).

Proof. Consider the space \( \mathcal{H} = \{ f \in L^1 : \hat{f} \in L^1 \} \). We use the facts that \( f \in L^1 \) implies \( \hat{f} \in L^\infty \) and that \( L^1 \cap L^\infty \subset L^2 \) to see that \( \mathcal{S} \subseteq \mathcal{H} \subseteq L^2 \). Since \( \mathcal{S} \) is dense in \( L^2 \), also \( \mathcal{H} \) is. Notice that, for \( g \in \mathcal{H} \), by the inversion theorem, setting \( \hat{g} \)
\[ \hat{h}(\xi) = \int e^{-2\pi i \xi \cdot x} \hat{g}(x) \, dx = \int \int e^{2\pi i \xi \cdot x} \hat{g}(x) \, dx = \hat{g}(\xi). \]

Therefore,
\[ \int f(\hat{g}) \, dx = \int \hat{f}(\hat{h}) \, dx = \int \hat{f}(\hat{h}) \hat{d}\xi = \int \hat{f}(\hat{g}) \, d\xi. \]
Thus, \( F \) restricted to \( \mathcal{H} \) preserves the \( L^2 \)-scalar product, so that, by taking \( f = g \) we see that \( \|f\|_2 = \|\hat{f}\|_2 \). Since \( F(\mathcal{H}) = \mathcal{H} \) and \( \mathcal{H} \) is dense in \( L^2 \), \( F \) extends by continuity to a unique mapping that is still an isometry. Thus, its image is closed and contains the dense subspace \( \mathcal{H} \), that is, \( F \) is a surjective isomorphism of \( L^2 \).

Finally, one has to show that such an extension coincides with \( F \) on \( L^1 \cap L^2 \). Suppose \( f \in L^1 \cap L^2 \) and let \( \varphi(x) = e^{-\pi |x|^2} \). Consider \( f * \varphi_t \). Then \( f * \varphi_t \in L^1 \cap L^2 \), \( F(f * \varphi_t) = \hat{f}F(e^{-\pi t |\cdot|^2}) \in L^1 \) (since \( \hat{f} \) is bounded). Therefore, \( f * \varphi_t \in \mathcal{H} \) and \( f * \varphi_t \to f \) both in \( L^1 \) and \( L^2 \) norms, and \( F(f * \varphi_t) \to \hat{f} \) both uniformly and in \( L^2 \) norm. \( \square \)

1.3. The Schwartz space. We recall a few basic facts about seminormed linear spaces.\(^7\)

Let \( \mathcal{V} \) be a complex linear space. A function \( \varrho : \mathcal{V} \to [0, +\infty) \) is called a seminorm if

(a) \( \varrho(\lambda v) = |\lambda|v \) for all \( v \in \mathcal{V} \) and \( \lambda \in \mathbb{C} \);

(b) \( \varrho(v_1 + v_2) \leq \varrho(v_1) + \varrho(v_2) \) for all \( v_1, v_2 \in \mathcal{V} \).

Notice that if we also require that \( \varrho(v) = 0 \) implies \( v = 0 \), then \( \varrho \) would be a norm.

Let \( \mathcal{V} \) be a linear space on which there exists a family \( \{\varrho_\alpha\}_{\alpha \in A} \) of seminorms with the following property: For each pair of points \( v_1, v_2 \in \mathcal{V} \) there exists \( \varrho_\alpha \) such that \( \varrho_\alpha(v_1) \neq \varrho_\alpha(v_2) \). In this case we say that the family of seminorms separates the points in \( \mathcal{V} \).

A linear space \( \mathcal{V} \) on which there exists a family \( \{\varrho_\alpha\}_{\alpha \in A} \) of seminorms that separates the points is called a seminormed space.

On a seminormed space \( \mathcal{V} \) with seminorms \( \{\varrho_\alpha\}_{\alpha \in A} \) we define a topology by defining a system of open sets

\[ U_{x,\alpha,\varepsilon} = \{y \in \mathcal{V} : \varrho_\alpha(x - y) < \varepsilon\} . \]

Let \( \mathcal{V} \) be a (complex) seminormed space and suppose that admits a countable family \( \mathcal{P} \) of seminorms \( \{\varrho_k\} \). Let \( \tau_\mathcal{P} \) be the topology defined above, which also is the coarsest topology that make continuous the identity

\[ i : (\mathcal{V}, \tau_\mathcal{P}) \to (\mathcal{V}, \varrho_k) \]

for all \( k \in \mathbb{N} \).

We say that \( \mathcal{P} \) is separating if for each \( v \in \mathcal{V} \) there exists \( k \in \mathbb{N} \) such that \( \varrho_k(v) > 0 \). We will always assume that \( \mathcal{P} \) is separating.

**Proposition 1.18.** With the above notation, the topology \( \tau_\mathcal{P} \) enjoys the following properties.

(i) The space \((\mathcal{V}, \tau_\mathcal{P})\) is a locally convex Hausdorff space.

(ii) The finite intersections of the sets

\[ B_{k,n} = \{v : \varrho_k(v) < 1/n\} \]

form a fundamental system of neighborhoods of 0.

(iii) The topology \( \tau_\mathcal{P} \) is induced by the metric distance

\[ d_\mathcal{P}(v, w) = \sum_k \frac{1}{2^k} \frac{\varrho_k(v - w)}{1 + \varrho_k(v - w)} . \]

\(^7\)For proofs and more on this topic, see G. B. Folland, *Real Analysis, Modern Techniques and Their Applications*, 2 Ed..
(iv) A sequence \( \{v_n\} \) in \( V \) converges to \( v \) in the \( \tau_P \) topology if and only if it converges to \( v \) in all the seminorms \( \varrho_k \), that is if
\[
\lim_{n \to +\infty} \varrho_k(v - v_n) = 0
\]
for all \( k \in \mathbb{N} \).

(v) A linear functional \( T \) on \( V \), that is, a linear operator from \( V \) to \( \mathbb{C} \), is continuous in the \( \tau_P \) topology if and only if there exist seminorms \( \varrho_{k_1}, \ldots, \varrho_{k_N} \) and constant \( C > 0 \) such that
\[
|T(v)| \leq C \sum_{j=1}^{N} \varrho_{k_j}(v)
\]
for all \( v \in V \).

The distance \( d_P \) is **invariant**, that is
\[
d_P(v + z, w + z) = d_P(v, w)
\]
for all \( v, w, z \in V \).

We also recall that if \( T \) a linear operator between two linear normed spaces \( \mathcal{X}, \mathcal{Y} \), then \( T \) is continuous if and only if it is bounded, that is there exists a constant \( C > 0 \) such that
\[
\|Tx\|_Y \leq C\|x\|_X ,
\]
for all \( x \in \mathcal{X} \). The proof of this fact is simple. If \( T \) linear is bounded, then it is Lipschitz, with Lipschitz constant less or equal to \( C \):
\[
\|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y \leq C\|x_1 - x_2\|_X ,
\]
which of course implies continuity. Conversely, if it continuous, the inverse image of the ball of radius \( r > 0 \) is contained in the ball of radius \( R > 0 \), so that, for some \( R' < R \), and all \( x \) with \( \|x\|_X \leq R' \),
\[
\|Tx\|_Y \leq r .
\]
But then, for all \( x \in \mathcal{X} \),
\[
\|Tx\|_Y = \left\| \frac{\|x\|_X}{R'} T\left( \frac{R'x}{\|x\|_X} \right) \right\|_Y \leq \frac{\|x\|_X}{R'} r .
\]

As a consequence of the results discussed so far, we have the following proposition involving the continuity (or **boundedness**) of a linear operator acting between seminormed spaces.

**Proposition 1.19.** Let \( V, W \) be seminormed linear spaces admitting family of seminorms \( \{\varrho_\alpha\} \), \( \{\sigma_\beta\} \), resp. Let \( T : V \to W \) be a linear mapping. Then \( T \) is continuous if and only if, for each seminorm \( \sigma_\beta \) on \( W \) there exist seminorms \( \varrho_{\alpha_1}, \ldots, \varrho_{\alpha_N} \) and a constant \( C = C_{\beta, \alpha_1, \ldots, \alpha_N} > 0 \) such that for all \( v \in V \) we have
\[
\sigma_\beta(Tv) \leq C \sum_{j=1}^{N} \varrho_{\alpha_j}(v) .
\]
A space of function of fundamental importance is the Schwartz space \( S(\mathbb{R}^n) \) (often denoted as \( S \)). We denote by \( \varrho(\alpha,\beta) \) the seminorm
\[
\varrho(\alpha,\beta)(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|
\]
and define
\[
S = \{ f \in C^\infty(\mathbb{R}^n) : \varrho(\alpha,\beta)(f) < \infty, \text{ for all multi-indices } \alpha, \beta \}.
\]
Notice that for every positive integer \( N \) there exists a constant \( C = C_N > 0 \) such that for all \( x \in \mathbb{R}^n \) we have
\[
\frac{1}{C} \sum_{|\alpha| \leq N} |x^\alpha| \leq (1 + |x|)^N \leq C \sum_{|\alpha| \leq N} |x^\alpha|.
\]
(We leave the simple proof as an exercise.)

Observe that if \( \varphi \in S \), for every positive integer \( N \)
\[
(1 + |x|)^N |\varphi(x)| \leq C \sum_{|\alpha| \leq N} \varrho_{\alpha,0}(\varphi) \leq C_N,
\]
so that
\[
|\varphi(x)| \leq \frac{C_N}{(1 + |x|)^N}.
\]
Thus, Schwartz functions are in \( C_0 \), hence uniformly continuous. Moreover, they decay faster than the reciprocal of any polynomial. This fact easily implies that \( S \) is contained in \( L^p \), for all \( p, 1 \leq p \leq \infty \), and in particular that for \( 1 \leq p < \infty \),
\[
\|\varphi\|^p_p = \int (1 + |x|)^N |\varphi(x)|^p (1 + |x|)^{-Np} dx \leq C \sum_{|\alpha| \leq [np]+1} \varrho_{\alpha,0}(\varphi)(1 + |x|)^{-Np} dx,
\]
which is finite if \( N \) is chosen so that \( np - (n - 1) > 1 \).

It is clear that \( S \) contains \( D \), so it is not empty. However, such inclusion is proper since \( e^{-|x|^2} \), \( 1/\cosh x \) are examples of funtions in \( S \) that do not have compact support.

**Lemma 1.20.** Let \( p \) be a polinomial of degree \( d \) and \( \gamma \) be a multi-index. Then, the mapping
\[
T : S \ni \varphi \to \partial^\gamma_x (p\varphi) \in S
\]
is bounded (i.e. continuous).

**Proof.** For a given seminorm \( \varrho(\alpha,\beta) \) and any \( \varphi \in S \) we have
\[
\varrho(\alpha,\beta)(\partial^\gamma_x \varphi) = \varrho(\alpha,\beta + \gamma)(\varphi),
\]
so \( \varphi \to \partial^\gamma_x \varphi \) is a continuous mapping of \( S \) into itself.

On the other hand, if \( p(x) = \sum_{|\alpha'| \leq d} a_{\alpha'} x^{\alpha'} \), then
\[
\partial^\gamma_x (p\varphi) = \sum_{|\alpha'| \leq d, |\beta'| \leq |\beta|} b_{\alpha'} x^{\alpha'} \partial^\beta_x (\varphi)
\]
for some coefficients $b_{\alpha'}$, so that
\[
\varrho_{(\alpha,\beta)}(p\varphi) \leq C \sum_{|\alpha'| \leq \delta, |\beta'| \leq |\beta|} \varrho_{(\alpha+\alpha',\beta')} (p\varphi).
\]
Thus, also $\varphi \to p\varphi$ is a continuous mapping of $\mathcal{S}$ into itself, and the conclusion follows as composition of continuous functions. \hfill $\square$

**Proposition 1.21.** The space $\mathcal{S}$ is a complete metric space (i.e. a Fréchet space) in the topology defined by the seminorms $\varrho_{(\alpha,\beta)}$. Moreover, $C_0^\infty$ is dense in $\mathcal{S}$.

**Proof.** Using (iii) of the previous Prop. 1.18, we only need to prove that $\mathcal{S}$ is complete in the given topology.

Let $\{f_k\}$ be a Cauchy sequence in $\mathcal{S}$, then $\varrho_{(\alpha,\beta)}(f_j - f_k) \to 0$ as $j, k \to +\infty$ for all $(\alpha, \beta)$. In particular, $\{\partial^\beta f_k\}$ converges uniformly to a function $g_\beta$ for all $\beta$. Denote by $e_j$ the $j$-th element of the canonical basis in $\mathbb{R}^n$. Notice that
\[
g_0(x + te_j) - g_0(x) = \lim_{k \to +\infty} f_k(x + te_j) - f_k(x) = \lim_{k \to +\infty} \int_0^t \partial_{x_j} f_k(x + se_j) \, ds = \int_0^t g_{e_j}(x + se_j) \, ds.
\]
Therefore, $\partial_{x_j} g_0 = g_{e_j}$, and by induction on $|\beta|$ we obtain that $g_\beta = \partial^\beta g_0$. Finally,
\[
|x|^\alpha |\partial_x^\beta f_k(x) - \partial_x^\beta g_0(x)| = \lim_{j \to +\infty} |x|^\alpha |\partial_x^\beta f_k(x) - \partial_x^\beta f_j(x)|
\]
\[
\leq \lim_{j \to +\infty} \varrho_{(\alpha,\beta)}(f_k - f_j)
\]
\[
\leq \varepsilon,
\]
for $j, k \geq k_0$. Then $\varrho_{(\alpha,\beta)}(f_k - g_0) \to 0$ as $k \to +\infty$.

In order to show that $C_0^\infty$ is dense in $\mathcal{S}$, observe that, if $\eta \in C_c^\infty$ with $\eta(0) = 1$ and $\varphi \in \mathcal{S}$, then, for every $\varepsilon > 0$, $\eta^\varepsilon \varphi$ is in $C_c^\infty$, where $\eta^\varepsilon(x) = \eta(\varepsilon x)$. Now it is easy to check that $\eta^\varepsilon \varphi \to \varphi$ in the $\mathcal{S}$-topology, as $\varepsilon \to 0$. For, $\eta^\varepsilon \varphi - \varphi$ is identically 0 for $|\xi| \leq 1/\varepsilon$, so that
\[
\varrho_{\alpha,\beta}(\eta^\varepsilon \varphi - \varphi) = \sup_{|x| \geq 1/\varepsilon} |x^\alpha \partial_x^\beta (\eta^\varepsilon \varphi - \varphi)(x)|
\]
\[
\leq \sup_{|x| \geq 1/\varepsilon} \left( |x^\alpha \partial_x^\beta (\eta^\varepsilon \varphi)(x)| + |x^\alpha \partial_x^\beta \varphi(x)| \right)
\]
\[
\leq C_{\varepsilon} \sum_{|\gamma| \leq |\beta|} \sup_{|x| \geq 1/\varepsilon} |x^\alpha \partial_x^\gamma \varphi(x)|.
\]
Using the fact that $x^\alpha \partial_x^\gamma \varphi \in C_0$ for all $\alpha$ and $\gamma$, for a fixed $\varepsilon$, we can find $\varepsilon_0$ such that the right hand side above is less or equal to $\varepsilon$. The conclusion now follows. \hfill $\square$

A consequence of Prop. 1.11 and the Inversion Theorem is the following.

**Proposition 1.22.** The Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a continuous bijection, with continuous inverse.
We remark that, as a consequence of these facts, we obtain a simple proof that the convolution of two Schwartz functions is still a Schwartz function. For, \( \varphi \ast \psi = \mathcal{F}^{-1}(\hat{\varphi} \hat{\psi}) \), where \( \hat{\varphi} \hat{\psi} \in \mathcal{S} \).

**Proof.** In order to prove the continuity of \( \mathcal{F} \), and therefore the one of \( \mathcal{F}^{-1} \), we need to show that for every semi-norm \( \varrho_{\alpha,\beta} \), there exist multi-indices \( \alpha_1, \beta_1, \ldots, \alpha_N, \beta_N \) and \( C > 0 \) such that for every \( f \in \mathcal{S} \),

\[
\varrho_{(\alpha,\beta)}(\mathcal{F}(f)) \leq C \sum_{j=1}^{N} \varrho_{(\alpha_j,\beta_j)}(f).
\]

But,

\[
\xi^\alpha \partial^\beta_\xi (\hat{f}) = c_{\beta} \xi^\alpha \mathcal{F}(x^\beta f) = c_{\alpha,\beta} \mathcal{F}(\partial^\alpha_\xi (x^\beta f)).
\]

\[
\varrho_{(\alpha,\beta)}(\mathcal{F}(f)) = \sup_{\xi \in \mathbb{R}^n} |\xi^\alpha \partial^\beta_\xi (\hat{f})(\xi)| = c_{\alpha,\beta} \sup_{\xi \in \mathbb{R}^n} |\mathcal{F}(\partial^\alpha_\xi (x^\beta f))(\xi)| \leq C \|\partial^\alpha_\xi (x^\beta f)\|_1 \leq C \sum_{|\alpha'| \leq N} \varrho_{(\alpha',0)}(\partial^\alpha_\xi (x^\beta f)).
\]

It follows that \( \mathcal{F} \) is bounded from \( \mathcal{S} \) into itself. The Inversion theorem, Thm. 1.15, shows that \( \varphi \mapsto \hat{\varphi}(-\cdot) \) is its inverse. So, \( \mathcal{F} \) is one to one, onto and its inverse is also continuous. \( \square \)

We conclude this part by observing that, using the bounds in (1.7), if we set

\[
\tilde{\varrho}_{(N,\beta)}(f) = \sup_{x \in \mathbb{R}^n} (1 + |x|^N) |\partial^\beta_\xi f(x)|,
\]

for all non-negative integers \( N \) and multi-indices \( \beta \), we obtain a countable family of seminorms \( \{\tilde{\varrho}_{N,\beta}\} \) that defines a topology in \( \mathcal{S} \) equivalent to the given one.

1.4. **The space of tempered distributions.** We define the space \( \mathcal{S}' \) of tempered distributions as the dual space of \( \mathcal{S} \). We endow \( \mathcal{S}' \) with the weak-* topology, that is, the weakest topology that makes the elements of \( \mathcal{S} \) continuous on \( \mathcal{S}' \), that is, the functionals

\[
\mathcal{S}' \ni u \mapsto u(\varphi)
\]

for \( \varphi \in \mathcal{S} \) fixed. A ngbh basis for the weak-* topology is given by the sets

\[
\mathcal{I}_{\varphi_1,\ldots,\varphi_N,\varepsilon}(u) = \{ v \in \mathcal{S}' : |u(\varphi_j) - v(\varphi_j)| < \varepsilon, j = 1, \ldots, N \}, \quad (1.8)
\]

where \( \varphi_1, \ldots, \varphi_N \) are in \( \mathcal{S} \) and \( \varepsilon > 0 \).

There are many noticeable examples of tempered distributions.

**Examples 1.23.** (i) Functions in \( \mathcal{S} \), in any \( L^p \) class, \( 1 \leq p \leq \infty \) give rise to bounded linear functionals on \( \mathcal{S} \), by setting:

\[
L_f(\varphi) = \int f \varphi \, dx \quad \varphi \in \mathcal{S}, f \in L^p \text{ (or } \mathcal{S})
\]
For,

\[ |L_f(\varphi)| \leq \int |f\varphi| \, dx \leq \|f\|_{L^p} \|\varphi\|_{L^{p'}} \]

\[ = \|f\|_{L^p} \left( \int_{\mathbb{R}^n} (1 + |x|)^{-Np'} (1 + |x|)^{Np'} |\varphi(x)|^{p'} \, dx \right)^{1/p'} \]

\[ \leq C \|f\|_{L^p} \sup_x \{ (1 + |x|)^N |\varphi(x)| \} \]

\[ \leq C \sum_{|\alpha| \leq N} \varrho(\alpha, 0)(\varphi) , \]

dove \( N \) è scelto \( \geq n + 1 \).

(ii) A function \( f \) is called \emph{tempered} if there exists \( N > 0 \) such that \( (1 + |x|)^{-N} f \in L^1 \). Then, the formula

\[ L_f(\varphi) = \int \varphi f \, dx \]

defines a tempered distribution, since

\[ |\int \varphi f \, dx| \leq \sup_x \{ (1 + |x|)^N |\varphi(x)| \} \int (1 + |x|)^{-N} |f| \, dx \leq C \sum_{|\alpha| \leq N} \varrho(\alpha, 0)(\varphi) . \]

Analogously, a \emph{Borel measure} \( \mu \) is called \emph{tempered} if there exists \( N > 0 \) such that \( \int (1 + |x|)^{-N} d|\mu| < \infty \). Then, the pairing

\[ L_\mu(\varphi) = \int \varphi \, d|\mu| \]

defines a tempered distribution. Then, Dirac deltas are examples of tempered distributions.

Notice however, not every \( C^\infty \) function defines a tempered distribution; for instance one with exponential growth.

\textbf{Remark 1.24.} On \( S' \) we can introduce a few operations, besides the ones that define the vector space structure.

(I) Differentiation. For \( u \in S' \) and \( \alpha \) a multi-index we set

\[ \partial^\alpha u(\psi) = u((-1)^{|\alpha|} \partial^\alpha \psi) . \]

Notice that, if \( u \in S \subset S' \), then \( \partial^\alpha u \in S \) and therefore it defines again an element of \( S' \) by the integral pairing, and integrating by parts the boundary terms equal 0 (since both \( u \) and \( \psi \) are Schwartz functions)

\[ (\partial^\alpha u)(\psi) = \int (\partial^\alpha u)\psi \, dx = - \int (\partial^{\alpha-\epsilon_1} u)(\partial_x^{-\epsilon_1} \psi) \, dx \]

\[ = (-1)^{|\alpha|} \int u\partial^\alpha \psi \, dx . \]

Hence, the definition of derivative of a distribution coincides with the classical definition if the distribution admits classical derivatives.
(II) Multiplication by a smooth function of polynomial growth. Given \( f \in C^\infty \), we say that \( f \) is of moderate growth if for each multi-index \( \alpha \) there exist an integer \( N \) and a positive constant \( C \) such that
\[
|\partial_x^\alpha f(x)| \leq C(1 + |x|)^N.
\]
Clearly such a function defines an element of \( S' \), and if \( \varphi \in S \), then \( f\varphi \in S \). Moreover, if \( u \in S' \), we can define another element \( fu \) of \( S' \) by setting
\[
(fu)(\varphi) = u(f\varphi).
\]

(III) Fourier transform. Given \( u \in S' \) we define the tempered distribution \( \hat{u} \) by setting
\[
\hat{u}(\psi) = u(\hat{\psi}),
\]
for any \( \psi \in S \). Observe that, using identity (1.6), this definition extends the definition of the Fourier transform given on \( S \); that is, if \( u \) above is in fact an element of \( S \), then the equality above is justified by (1.6).

**Proposition 1.25.** The Fourier transform \( F \) is a surjective topological isomorphism of \( S' \) onto itself.

**Proof.** Clearly \( F \) maps \( S' \) into itself, and using the fact that \( F \) is a bijection on \( S \), it is easy to see that it is a bijection on \( S' \) too. For, if \( \hat{u}(\varphi) = 0 \) for all \( \varphi \), then \( u(\hat{\varphi}) = 0 \) for all \( \varphi \in S \), that is, \( u(\psi) = 0 \) for all \( \psi \in S \), i.e. \( u = 0 \). In order to see that \( F \) is onto, given \( u \in S' \), define \( v \in S' \) by setting \( v(\psi) = u(\mathcal{F}^{-1}\psi) \). Then, \( v \) is well defined and
\[
u(\varphi) = u(\mathcal{F}^{-1}\varphi) = v(\hat{\varphi}) = \hat{v}(\varphi),\]
for all \( \varphi \in S \). Then, \( u = Fv \), that is, \( F \) is onto. Notice also that we have shown that the inverse Fourier transform on tempered distributions is given by
\[
\mathcal{F}^{-1}u = u \circ \mathcal{F}^{-1}.
\]

The fact that \( F \) and \( F^{-1} \) are continuous on \( S' \) follows from the description of the weak-* topology: a nbhb basis of \( u \in S' \) is given by the sets
\[
\mathcal{I}_{\varphi_1,\ldots,\varphi_N;\varepsilon}(u) = \{ v \in S' : |u(\varphi_j) - v(\varphi_j)| < \varepsilon, \ j = 1,\ldots,N \},
\]
where \( \varphi_1,\ldots,\varphi_N \) are in \( S \) and \( \varepsilon > 0 \). Let \( v \in \mathcal{I}_{\varphi_1,\ldots,\varphi_N;\varepsilon}(u) \). Then,
\[
|\hat{u}(\varphi_j) - \hat{v}(\varphi_j)| = |u(\hat{\varphi}_j) - v(\hat{\varphi}_j)| < \varepsilon,
\]
if \( v \in \mathcal{I}_{\hat{\varphi}_1,\ldots,\hat{\varphi}_N;\varepsilon}(u) \), where \( \mathcal{I}_{\hat{\varphi}_1,\ldots,\hat{\varphi}_N;\varepsilon}(u) \) has the obvious meaning. Therefore,
\[
\mathcal{F}(\mathcal{I}_{\hat{\varphi}_1,\ldots,\hat{\varphi}_N;\varepsilon}(u)) \subset |\hat{u}(\varphi_j) - \hat{v}(\varphi_j)|
\]
and this shows that \( F : S' \to S' \) is continuous and we are done. \( \square \)

(IV) Convolution with a Schwartz function \( \varphi \). We first define the operator \( \tau_x \) on functions by setting \( \tau_x \varphi(x) = \varphi(-x) \). Then, for \( \varphi \in S \) and \( u \in S' \) we set\(^8\)
\[
(u \ast \varphi)(x) = u(\tau_x \varphi).
\]

**Proposition 1.26.** Let \( \varphi, \psi \in S \) and \( u \in S' \). Then

\(^8\)We adopt the convention that \( \tau_x \varphi = \varphi(\cdot - x) = \varphi(x - \cdot) \).
(i) \( u \ast \varphi \in C^\infty \) and for every multi-index \( \alpha \)
\[ \partial^\alpha (u \ast \varphi) = (\partial^\alpha u) \ast \varphi = u \ast (\partial^\alpha \varphi) ; \]

(ii) \( u \ast \varphi \) is of moderate growth, so that \( u \ast \varphi \in \mathcal{S}' \);

(iii) we have
\[ \int_{\mathbb{R}^n} u(\tau_x \tilde{\varphi}) \psi(x) \, dx = u \left( \int_{\mathbb{R}^n} (\tau_x \tilde{\varphi})(y) \psi(x) \, dx \right) ; \]

(iv) the action of \( u \ast \varphi \) on \( \mathcal{S} \) is given by the formula
\[ (u \ast \varphi)(\psi) = u(\tilde{\varphi} \ast \psi) ; \]

(v) \( F(u \ast \varphi) = \hat{u} \hat{\varphi} \) and \( F(u \varphi) = \hat{u} \ast \hat{\varphi} \);

(vi) \( (u \ast \varphi) \ast \psi = u \ast (\varphi \ast \psi) . \)

Notice that the pairing in (iv) extends the case when also \( u \in \mathcal{S} \) that we obtain by switching the integration order:
\[ (u \ast \varphi)(\psi) = \int \left( \int u(y) \varphi(x - y) \, dy \right) \psi(x) \, dx \]
\[ = \int u(y) \left( \int \varphi(x - y) \psi(x) \, dx \right) dy \]
\[ = \int u(y) (\tilde{\varphi} \ast \psi)(y) \, dy . \]

The same comments applies to the identity in (iv): if \( u \) is also in \( \mathcal{S} \), then the convolutions are given by absolutely convergent integrals and we have
\[ (u \ast \varphi) \ast \psi(x) = \int \left( \int u(z) \varphi(y - z) \, dz \right) \psi(x - y) \, dy = \int u(z) \left( \int \varphi(y - z) \psi(x - y) \, dy \right) dz \]
\[ = \int u(z) \left( \int \varphi(y') \psi(x - z - y') \, dy' \right) \, dz = \int u(z) (\varphi \ast \psi)(x - z) \, dz \]
\[ = u \ast (\varphi \ast \psi)(x) . \]

**Proof.** (i) We first claim that
\[ \varphi(t) := \frac{\varphi(\cdot + te_1) - \varphi}{t} = \frac{1}{t} (\tau_{-te_1} \varphi - \varphi) \rightarrow \partial_{x_1} \varphi \] (1.10)
as \( t \rightarrow 0 \) in \( \mathcal{S} \).

Assume the validity of the claim for the moment. Using the fact that the convolution commutes with translations, it follows that
\[ \frac{(u \ast \varphi)(x + te_1) - (u \ast \varphi)(x)}{t} = \frac{1}{t} \left[ u(\tau_{x+te_1} \varphi) - u(\tau_x \varphi) \right] \]
\[ = u \left( \frac{1}{t} [\tau_{te_1}(\tau_x \varphi) - \tau_x \varphi] \right) \]
\[ = u \left( - \partial_{y_1}(\tau_x \varphi) \right) \]
\[ = -u(\tau_x \partial_{y_1} \varphi) = u(\tau_x \partial_{y_1} \varphi)' \]
\[ = (u \ast \partial_{y_1} \varphi)(x) , \]
as \( t \rightarrow 0 \), by definition (1.9).
Iterating this argument we obtain that \( u \ast \varphi \) is \( C^\infty \) and

\[
\partial^\alpha (u \ast \varphi) = u \ast (\partial^\alpha \varphi),
\]

for all multi-indices \( \alpha \). Since

\[
u \ast (\partial^\alpha \varphi)(x) = u(\tau_x (\partial^\alpha \varphi)) = (-1)^{|\alpha|} u(\partial^\alpha (\tau_x \varphi)) = (\partial^\alpha u)(\tau_x \varphi) = (\partial^\alpha u) \ast \varphi(x),
\]

also the first equality in \((i)\) follows, modulo equality \((1.10)\).

In order to prove \((1.10)\) it suffices to prove that

\[
\mathcal{F}\left( \frac{1}{t} (\tau_{-te} \varphi - \varphi) - \partial_{x_1} \varphi \right) \to 0
\]

in \( S \), that is,

\[
\left( t^{-1} (e^{2\pi i \xi_1} - 1) - 2\pi i \xi_1 \right) \hat{\varphi} \to 0
\]

in \( S \), as \( t \to 0 \). It this is easy to check that the function \( m(\xi) = t^{-1} (e^{2\pi i \xi_1} - 1) - 2\pi i \xi_1 \) is \( C^\infty \), of moderate growth together with all its derivatives, uniformly so in \( t \), and they all tend to 0 uniformly as \( t \to 0^+ \); hence the conclusion. This proves \((i)\).

\((ii)\) In order to show that \( u \ast \varphi \) is of moderate growth we need to show that for each multi-index \( \alpha \) there exist an integer \( N \) and a positive constant \( C \) such that

\[|\partial^\alpha_x (u \ast \varphi)(x)| \leq C(1 + |x|)^N.\]

Since \( \partial^\alpha_x (u \ast \varphi) = u \ast (\partial^\alpha_x \varphi) \), and \( \partial^\alpha_x \varphi \) is again a Schwartz function, it suffices to prove the case \( \alpha = 0 \). Since \( u \in S' \), given \( \varphi \in S \) there exist semi-norms \( g_{(\alpha_1, \beta_1)}, \ldots, g_{(\alpha_N, \beta_N)} \) and \( C > 0 \) such that

\[|u(\tau_x \varphi)| \leq C \sum_{j=1}^N g_{(\alpha_j, \beta_j)}(\tau_x \varphi).\]

Now, for any semi-norm \( g_{(\alpha, \beta)} \),

\[g_{(\alpha, \beta)}(\tau_x \varphi) = \sup_{y \in \mathbb{R}^n} |y^\alpha \partial^\beta_y (\tau_x \varphi(y))| = \sup_{y \in \mathbb{R}^n} \left| y^\alpha \partial^\beta_y \varphi(x - y) \right| = \sup_{y \in \mathbb{R}^n} \left| (x + y)^\alpha \partial^\beta_y \varphi(y) \right|,
\]

which is clearly bounded by a polynomial in \( x \).

\((iii)\) We have the following

Claim. Let \( \eta \in C_0^\infty \). Then, the Riemann sums of the integral \( \int_{\mathbb{R}^n} (\tau_x \varphi)(y) \eta(x) \, dx \) converge in the topology of \( S \) to \( \varphi \ast \eta \).

For simplicity of notation assume \( n = 1 \), the general case is similar. If \( \eta \in C_0^\infty \), \( \text{supp} \ \eta \subseteq I \), denoting by \( \{x_0 < x_1 < \cdots < x_N \} \) a partition of the interval \( I \), \( \Delta_j = x_j - x_{j-1} \), \( \tilde{x}_j \) an arbitrary

\[\text{The reader should check that explicitly this polynomial is a constant times } (1 + |x|)^k \sum_{|\alpha'| \leq |\alpha|} g_{(\alpha', \beta)}(\varphi).\]
point in $I_j = (x_{j-1}, x_j)$, we have
\[ \int_I \varphi(x-y)\eta(x)\,dx = \lim_{N \to +\infty} \sum_{j=1}^N \Delta_j \varphi(\tilde{x}_j - y)\eta(\tilde{x}_j) \]
and we want to show that the sequence
\[ f_N(y) := \sum_{j=1}^N \Delta_j \varphi(\tilde{x}_j - y)\eta(\tilde{x}_j) \to \int_I \varphi(x-y)\eta(x)\,dx \] (1.11)
in $S$ as $N \to +\infty$. In fact,
\[ \int_I \varphi(x-y)\eta(x)\,dx - f_N(y) = \sum_{j=1}^N \int_{I_j} \left( \varphi(x-y)\eta(x) - \varphi(\tilde{x}_j - y)\eta(\tilde{x}_j) \right)\,dx \]
By the uniform continuity of the integrand, as a function of $x$, uniformly in $y$, given $\varepsilon > 0$ there exists $\delta > 0$ such that
\[ |\varphi(x-y)\eta(x) - \varphi(\tilde{x}_j - y)\eta(\tilde{x}_j)| < \varepsilon \]
when $\Delta_j < \delta$, for all $j = 1, \ldots, N$.
Therefore, choosing a partition of $I$ such that $\Delta_j < \delta$ for all $j$, that is $N \geq |I|/\delta$, we have
\[ |\int_I \varphi(x-y)\eta(x)\,dx - f_N(y)| \leq \varepsilon \sum_{j=1}^N \int_{I_j} = \varepsilon |I|. \]
This shows that
\[ \varrho_{(0,0)} \left( \int_I \varphi(x-\cdot)\eta(x)\,dx - f_N \right) \to 0 \]
as $N \to +\infty$. The cases of the seminorms $\varrho_{(\alpha,\beta)}$ for all other $\alpha$ and $\beta$ follow from this one.
In the case $|\beta| > 0$ we have
\[ \partial_\beta \left( \int_I \varphi(x-\cdot)\eta(x)\,dx - f_N \right) = (-1)^{|\beta|} \left( \int_I \partial_\beta \varphi(x-\cdot)\eta(x)\,dx - f_{\beta,N} \right), \]
where $f_{\beta,N}$ is defined as $f_N$ in (1.11) with $\varphi$ replaced by $\partial_\beta \varphi$. Hence,
\[ \varrho_{(0,\beta)} \left( \int_I \varphi(x-\cdot)\eta(x)\,dx - f_N \right) \leq C \varrho_{(0,0)} \left( \int_I \partial_\beta \varphi(x-\cdot)\eta(x)\,dx - f_{\beta,N} \right) \to 0 \]
as $N \to +\infty$.
In the case $|\alpha| > 0$,
\[ y^{\alpha} \left( \int_I \varphi(x-\cdot)\eta(x)\,dx - f_N \right) (y) \]
\[ = \int_I (y - x + x)^{\alpha} \varphi(x-y)\eta(x)\,dx - \sum_{j=1}^N \Delta_j (y - \tilde{x}_j + \tilde{x}_j)^{\alpha} \varphi(\tilde{x}_j - y)\eta(\tilde{x}_j) \]
\[ = \sum_{\alpha' + \alpha'' = \alpha} c_{\alpha',\alpha''} \left( \int_I (x-y)^{\alpha'} \varphi(x-y)\eta(x)\,dx - \sum_{j=1}^N \Delta_j (y - \tilde{x}_j)^{\alpha'} \varphi(\tilde{x}_j - y)\tilde{x}_j^{\alpha''} \eta(\tilde{x}_j) \right). \]
Therefore,

$$\varrho(\alpha, 0) \left( \int_I \varphi(x - \cdot) \eta(x) \, dx - f_N \right) \leq C \sum_{\alpha^\prime + \beta^\prime = \alpha} \varrho(0, 0) \left( \int_I \varphi_{\alpha^\prime}(x - \cdot) \eta_{\beta^\prime}(x) \, dx - f_{\alpha^\prime, \beta^\prime; N} \right) \to 0$$

as \( N \to +\infty \), where \( \varphi_{\alpha^\prime} = x^{\alpha^\prime} \varphi, \eta_{\beta^\prime} = x^{\beta^\prime} \eta \) and \( f_{\alpha^\prime, \beta^\prime; N} \) is defined as \( f_N \), with \( \varphi \) and \( \eta \) replaced by \( \varphi_{\alpha^\prime}, \eta_{\beta^\prime} \), resp.

The claim, at least in the case \( n = 1 \), now follows by combining the above cases. We leave the simple details to the reader.

Therefore, when \( \eta \in C_0^\infty(I) \),

$$u \left( \int_{\mathbb{R}} (\tau_x \varphi)(y) \eta(x) \, dx \right) = u \left( \int_I \varphi(x - y) \eta(x) \, dx \right)$$

$$= u \left( \lim_{N \to +\infty} \sum_{j=1}^N \Delta_j \varphi(\bar{x}_j - y) \eta(\bar{x}_j) \right)$$

$$= \lim_{N \to +\infty} \sum_{j=1}^N \Delta_j u \left( \varphi(\bar{x}_j - \cdot) \right) \eta(\bar{x}_j)$$

$$= \int_I u(\varphi(x - \cdot)) \eta(x) \, dx$$

$$= \int_{\mathbb{R}} u(\tau_x \varphi) \eta(x) \, dx.$$

Thus,

$$u \left( \int_{\mathbb{R}} (\tau_x \varphi)(\cdot) \eta(x) \, dx \right) = \int u(\tau_x \varphi) \eta(x) \, dx \quad (1.12)$$

when \( \eta \in C_0^\infty(I) \).

The case \( n > 1 \) can be proved with an analogous argument, which is just more involved notationally (because of the expression of the Riemann sums in several variables and the decomposition of the expressions of the form \((y - x + x)^\alpha\)). Again, we leave the details to the reader, and therefore we consider that (1.12) (has been proved and) it is valid in the case of \( \mathbb{R}^n \).

Next, for a generic \( \psi \in \mathcal{S} \), let \( \eta_k \in C_0^\infty \), \( \eta_k \to \psi \) in \( \mathcal{S} \), then, as \( k \to +\infty \),

- \( \int (\tau_x \varphi)(y) \eta_k(x) \, dx \to \int (\tau_x \varphi)(y) \psi(x) \, dx \) in \( \mathcal{S} \);
- \( \int u(\tau_x \varphi) \eta_k(x) \, dx \to \int u(\tau_x \varphi) \psi(x) \, dx \).

These conditions imply that

$$u \left( \int (\tau_x \varphi)(\cdot) \psi(x) \, dx \right) = \lim_{k \to +\infty} u \left( \int (\tau_x \varphi)(\cdot) \eta_k(x) \, dx \right)$$

$$= \lim_{k \to +\infty} \int u(\tau_x \varphi) \eta_k(x) \, dx$$

$$= \int u(\tau_x \varphi) \psi(x) \, dx.$$

This proves (iii).
In order to prove the identity (iv), notice that, by (iii)

\[(u * \varphi)(\psi) = \int_{\mathbb{R}^n} u(\tau_x \varphi) \psi(x) \, dx\]

\[= u\left(\int_{\mathbb{R}^n} (\tau_x \varphi)(y) \psi(x) \, dx\right)\]

\[= u\left(\int_{\mathbb{R}^n} \varphi(x - y) \psi(x) \, dx\right)\]

\[= u(\hat{\varphi} * \psi).

(v) For all \(\psi \in \mathcal{S}\), since \(u * \varphi \in \mathcal{S}'\), using (iv) we have

\[\mathcal{F}(u * \varphi)(\psi) = (u * \varphi)(\hat{\psi}) = u(\hat{\varphi} * \hat{\psi}) = u(\mathcal{F}(\hat{\varphi})\psi)\]

\[= \hat{u}(\hat{\varphi})\psi = (\hat{\varphi}^\ast \hat{u})(\psi) .

This shows that \(\mathcal{F}(u * \varphi) = \hat{u} \hat{\varphi}\). Next, we wish to show that \(\mathcal{F}(u \varphi) = \hat{u} \varphi\). For we have \(\psi \in \mathcal{S}\),

\[\mathcal{F}(u \varphi)(\psi) = (u \varphi)(\hat{\psi}) = u(\varphi \hat{\psi}) = u(\mathcal{F}(\varphi)\psi)\]

\[= \hat{u}(\varphi)\psi = (\varphi^\ast \hat{u})(\psi) = (\hat{u} \varphi)(\psi) .

This implies \(\mathcal{F}(u \varphi) = \hat{u} \varphi\), as we wished to show.

Finally (vi) follows at one from (v) by checking the equality of their Fourier transforms:

\[\mathcal{F}\left((u * \varphi) * \psi\right) = \mathcal{F}\left((u * \varphi)\right)(\psi) = (\hat{u} \varphi)(\hat{\psi}) = \mathcal{F}(u * (\varphi * \psi)) . \square\]

(V) We conclude this part with a simple observation. It follows from (i) and (iv) that if \(u \in \mathcal{S}'\)

\[(\partial^\alpha u)(\varphi) = (\partial^\alpha u)(\hat{\varphi}) = (-1)^{|\alpha|} u(\partial^\alpha \varphi) = u(\mathcal{F}(\partial(2\pi i \alpha) \varphi)) = (2\pi i \alpha)^\alpha \hat{u}(\varphi) .

A consequence of these facts is that \(\mathcal{S}\) is dense in \(\mathcal{S}'\).

**Proposition 1.27.** If \(\varphi \in \mathcal{S}\), \(\int \varphi = 1\) and \(u \in \mathcal{S}'\), then \(u * \varphi_t \to u\) in \(\mathcal{S}'\), as \(t \to 0\). Moreover, \(C_0^\infty\), hence \(\mathcal{S}\), is dense in \(\mathcal{S}'\) in its topology.

**Proof.** Given \(\psi \in \mathcal{S}\), we have

\[(u * \varphi_t)(\psi) = u(\varphi_t * \psi) \to u(\psi)\]

as \(t \to 0\), since \(\varphi_t * \psi \to \psi\) in \(\mathcal{S}\) and \(\int \varphi_t = 1\).

Next, arguing as in the proof of Prop. 1.21, let \(\eta \in C_0^\infty\), \(0 \leq \eta \leq 1\), \(\eta(x) = 1\) if \(|x| \leq 1\) and \(\eta(x) = 0\) if \(|x| \geq 2\). Since \(u * \varphi_t \in C^\infty\) (and it is of moderate growth), \(\eta''(u * \varphi_t) \in C_0^\infty\). For \(\psi \in \mathcal{S}\) we have,

\[(\eta''(u * \varphi_t))(\psi) = (u * \varphi_t)(\eta'' \psi) = (u * \varphi_t)(\psi) + (u * \varphi_t)(\eta'' \psi - \psi)\]

\[= u(\psi) + (u * \varphi_t - u)(\psi) + u(\eta'' \psi - \psi) + (u * \varphi_t - u)(\eta'' \psi - \psi) .

Given $\varepsilon > 0$ we can choose $t' > 0$ small enough so that $|u(\eta^t \psi - \psi)| < \varepsilon$, and then $t > 0$ small enough so that $|(u * \varphi_t - u)(\eta^t \psi - \psi)| < \varepsilon$ and also $|(u * \varphi_t - u)(\psi)| < \varepsilon$. Hence,

$$|\eta^t (u * \varphi_t - u)(\psi)| < 3\varepsilon.$$ 

The conclusion now follows. □

The next result, that we state without proof (for which we refer to [StWe, Thm. 3.16]) is an example of the significance of the space of tempered distributions.

**Theorem 1.28.** Let $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ be a bounded linear operator, $1 \leq p, q \leq \infty$, that commutes with translations (that is, $T(\tau_x f) = \tau_x (T f)$).

Then, there exists a tempered distribution $K$ such that, for $f \in \mathcal{S}$, $T f = f * K$. 

2. The Marcinkiewicz interpolation theorem and the Calderón–Zygmund decomposition

2.1. The Marcinkiewicz interpolation theorem. We recall that, given a measurable function \( f \) on \( \mathbb{R}^n \), its distribution function \( \alpha_f \) is the function, defined for \( \lambda \geq 0 \)

\[
\alpha_f(\lambda) = \left| \{ x \in \mathbb{R}^n : |f(x)| > \lambda \} \right|.
\]

Using Fubini’s theorem, it is easy to see that for \( p > 0 \)

\[
\|f\|_{L^p}^p = p \int_0^{+\infty} \lambda^{p-1} \alpha_f(\lambda) \, d\lambda.
\]

We also observed that

\[
\alpha_{f_1+f_2}(\lambda) \leq \alpha_{f_1}(\lambda/2) + \alpha_{f_2}(\lambda/2),
\]

since

\[
\{ x \in \mathbb{R} : |(f_1 + f_2)(x)| > \lambda \} \subseteq \{ x \in \mathbb{R} : |f_1(x)| > \lambda/2 \} \cup \{ x \in \mathbb{R} : |f_2(x)| > \lambda/2 \}.
\]

An operator \( T \) defined on measurable functions on \( \mathbb{R}^n \) is said to be sublinear if

- \( |T(f_0 + f_1)(x)| \leq |T(f_0)(x)| + |T(f_1)(x)| \);
- \( |T(\lambda f)(x)| = |\lambda||Tf(x)| \);

for all \( x \in \mathbb{R}^n, \lambda \in \mathbb{C} \). We remark that, if \( T \) is a linear operator, \(|T| \) is sublinear. Moreover, a typical example of an operator that is genuinely sublinear is the Hardy–Littlewood maximal function \( Mf(x) = \sup_{B(x,r),r>0} \frac{1}{|B|} \int_B |f(x)| \, dx \).

Definition 2.1. Given a sublinear operator \( T \) defined on measurable functions on \( \mathbb{R}^n \) is of weak-type \((p,q)\) if there exists a constant \( C > 0 \) such that

\[
\alpha_{Tf}(\lambda) = \left| \{ x \in \mathbb{R}^n : |Tf(x)| > \lambda \} \right| \leq \left( \frac{C}{\lambda} \|f\|_{L^p} \right)^q,
\]

while it is said to be strong-type \((p,q)\) if

\[
T : L^p \to L^q
\]

is bounded.

It is easy to see that if \( T \) is of strong-type \((p,q)\), then it is also of weak-type \((p,q)\). For,

\[
\alpha_{Tf}(\lambda) = \int_{\{ x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} \, dx \\
\leq \int_{\{ x \in \mathbb{R}^n : |Tf(x)| > \lambda \}} \left( \frac{|Tf(x)|}{\lambda} \right)^q \, dx \\
\leq \frac{1}{\lambda^q} \|Tf\|_{L^q}^q \\
\leq \left( \frac{C}{\lambda} \|f\|_{L^p} \right)^q.
\]
Theorem 2.2. (Marcinkiewicz interpolation theorem) Let $T$ be a sublinear operator defined on $L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$, $1 \leq p_j, q_j \leq +\infty$, $j = 0, 1$. Suppose that $T$ is of weak-type $(p_0, q_0)$ and of weak-type $(p_1, q_1)$. Then, $T$ is of strong-type $(p, q)$, where $p$ and $q$ are given by the relations
\[
\frac{1}{p} = \theta \frac{1}{p_0} + (1 - \theta) \frac{1}{p_1}, \quad \text{and} \quad \frac{1}{q} = \theta \frac{1}{q_0} + (1 - \theta) \frac{1}{q_1},
\]
and $0 < \theta < 1$.

Proof. We will prove the theorem only in the case $q_0 = p_0$ and $q_1 = p_1$. Then it suffices to assume that $p_0 < p < p_1$, without any reference to $\theta$. For the proof in the general case (that we need in Subsection 4.6) we refer the reader to [St1].

Let $f \in L^p$ and $\lambda > 0$ be given. For a constant $c > 0$ to be selected later, we decompose $f$ as $f = f_0 + f_1$, where
\[
f_0 = f \chi_{\{x: |f(x)| > c\lambda\}}, \quad \text{and} \quad f_1 = f \chi_{\{x: |f(x)| \leq c\lambda\}}.
\]
Notice that $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$, since $p_0 < p < p_1$. For,
\[
|f_0(x)|^{p_0} = |f(x)|^{p_0} - |f(x)|^{p_0} |f(x)|^p \chi_{\{x: |f(x)| > c\lambda\}}(x) \leq (c\lambda)^{p_0 - p} |f(x)|^p \chi_{\{x: |f(x)| > c\lambda\}}(x),
\]
so that $f_0 \in L^{p_0}$. Analogous reasoning shows that $f_1 \in L^{p_1}$:
\[
|f_1(x)|^{p_1} \leq (c\lambda)^{p_1 - p} |f(x)|^p \chi_{\{x: |f(x)| \leq c\lambda\}}(x).
\]

Then,
\[
|T(f_0 + f_1)(x)| \leq |T(f_0)(x)| + |T(f_1)(x)|
\]
and
\[
\alpha_T(f_0 + f_1)(\lambda) \leq \alpha_T f_0(\lambda/2) + \alpha_T f_1(\lambda/2).
\]

We distinguish two different cases. First assume that $p_1 < +\infty$. Then,
\[
\|Tf\|_{L^p}^p = p \int_0^{+\infty} \lambda^{p-1} \alpha_T f(\lambda) \, d\lambda
\]
\[
\leq p \int_0^{+\infty} \lambda^{p-1} \alpha_T f_0(\lambda/2) \, d\lambda + p \int_0^{+\infty} \lambda^{p-1} \alpha_T f_1(\lambda/2) \, d\lambda
\]
\[
\leq p \int_0^{+\infty} \lambda^{p-1} \left( \frac{2A_0}{\lambda} \|f_0\|_{L^{p_0}} \right)^{p_0} \, d\lambda + p \int_0^{+\infty} \lambda^{p-1} \left( \frac{2A_1}{\lambda} \|f_1\|_{L^{p_1}} \right)^{p_1} \, d\lambda
\]
\[
\leq p(2A_0)^{p_0} \int_0^{+\infty} \lambda^{p_0 - p_0 - 1} \int_{\{x: |f(x)| > c\lambda\}} |f(x)|^{p_0} \, dx \, d\lambda
\]
\[
+ p(2A_1)^{p_1} \int_0^{+\infty} \lambda^{p_1 - p_1 - 1} \int_{\{x: |f(x)| \leq c\lambda\}} |f(x)|^{p_1} \, dx \, d\lambda
\]
\[
\leq p(2A_0)^{p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0} \int_0^{+\infty} \lambda^{p_0 - p_0 - 1} \, d\lambda \, dx
\]
\[
+ p(2A_1)^{p_1} \int_{\mathbb{R}^n} |f(x)|^{p_1} \int_0^{+\infty} \lambda^{p_1 - p_1 - 1} \, d\lambda \, dx
\]
\[
= p \left( \frac{(2A_0)^{p_0}}{(p - p_0)(c^{p - p_0})} + \frac{(2A_1)^{p_1}}{(p_1 - p)(c^{p_1 - p_1})} \right) \|f\|_{L^p}^p.
\]
In the second case, that is, \( p_1 = +\infty \), then \( T : L^\infty \to L^\infty \) is bounded. We may choose \( c = 1/(2A_1) \), where \( A_1 \) is the \((L^\infty, L^\infty)\)-norm of \( T \), so that \( \alpha_{Tf_1}(\lambda/2) = 0 \).\(^{10}\) Then, we have that

\[
\|Tf\|_{L^p} \leq p \int_0^{+\infty} \lambda^{p-1} \left( \frac{2A_0}{\lambda} \|f_0\|_{L^{p_0}} \right)^{p_0} d\lambda
\]

\[
= p(2A_0)^{p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0} \int_0^{[f(x)]/c} \lambda^{p-p_0-1} d\lambda dx
\]

\[
= \frac{p}{p-p_0}(2A_0)^{p_0}(2A_1)^{p-p_1} \|f\|_{L^p}^p.
\]

This proves the theorem. \( \square \)

As an application, we show that the Hardy–Littlewood maximal operator \( M \) is bounded on \( L^p \), \( 1 < p \leq \infty \) and weak-type \((1, 1)\). We recall that, for \( f \in L^1_{\text{loc}} \),

\[ Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy. \]

We need a lemma. This type of result is called a covering lemma.

**Lemma 2.3.** Let \( \{B_1, B_2, \ldots, B_k\} \) be a finite collection of open balls in \( \mathbb{R}^n \). Then there exists a sub-collection \( \{B_{j_1}, \ldots, B_{j_\ell}\} \) of pairwise disjoint balls such that

\[
\sum_{i=1}^\ell |B_{j_i}| \geq 3^{-n} | \bigcup_{j=1}^k B_j |.
\]

**Proof.** We let \( B_{j_1} \) to be a ball of largest volume among the balls in the collection \( \{B_1, B_2, \ldots, B_k\} \). Next, we select \( B_{j_2} \) among the remaining balls that are disjoint from \( B_{j_1} \) to be of largest volume, and we proceed in the same fashion. Since we have a finite number of balls, this process will terminate after \( \ell \) steps.

Thus, we have constructed a sub-collection \( \{B_{j_1}, \ldots, B_{j_\ell}\} \) of pairwise disjoint balls. It \( B_m \) was not selected, that is, \( m \notin \{j_1, \ldots, j_\ell\} \), this means that \( B_m \) intersects one of the balls \( \{B_{j_1}, \ldots, B_{j_\ell}\} \), say \( B_{j_i} \), with \( |B_{j_i}| > |B_m| \). Then, \( 3B_{j_i} \supseteq B_m \), where \( 3B \) denotes the ball with the same center as \( B \), with radius 3 times as large. The conclusion now follows. \( \square \)

**Theorem 2.4.** The operator \( M \) is weak-type \((1, 1)\) and bounded on \( L^p \), \( 1 < p \leq \infty \).

**Proof.** The operator \( M \) is trivially bounded on \( L^\infty \). If we show that \( M \) is weak-type \((1, 1)\), the conclusion follows at once from Thm. 2.2.

Let \( \lambda > 0 \), then we wish to show that there exists \( C > 0 \) such that

\[
|\{x \in \mathbb{R}^n : |Mf(x)| > \lambda\}| \leq \frac{C}{\lambda} \|f\|_{L^1}.
\]

Let \( E \) be a compact subset of \( \Omega_\lambda := \{x \in \mathbb{R}^n : |Mf(x)| > \lambda\} \). For any \( x \in \Omega_\lambda \) there is an open ball \( B_x = B(x, r_x) \) such that \( x \in B_x \) and

\[
\frac{1}{|B_x|} \int_{B_x} |f(y)| dy > \lambda.
\]

\(^{10}\)For, \( |Tf_1(x)| \leq A_1 \|f_1\|_{L^\infty} \leq A_1 \lambda \leq \lambda/2 \), so that \( \{x : |Tf_1(x)| > \lambda/2\} = \emptyset \).
Since \( E \) is compact, finitely many such balls \( \{B_1, B_2, \ldots, B_k\} \) cover \( E \). Then, we can apply Lemma 2.3 to obtain a sub-collection \( \{B_{j_1}, \ldots, B_{j_\ell}\} \) of disjoint balls such that \(| \bigcup_{j=1}^\ell B_{j_i} | \leq 3^n \sum_{i=1}^\ell |B_{j_i}| \). Therefore, for every compact \( E \subseteq \Omega \lambda \) we have

\[
|E| \leq \left| \bigcup_{j=1}^\ell B_{j_i} \right| \leq 3^n \sum_{i=1}^\ell |B_{j_i}|
\leq 3^n \sum_{i=1}^\ell \frac{1}{\lambda} \int_{B_{j_i}} |f(y)| \, dy
\leq \frac{C}{\lambda} \|f\|_{L^1}.
\]

This proves the theorem. \( \square \)

### 2.2. The Calderón–Zygmund decomposition of an \( L^1 \)-function.

We now introduce a fundamental decomposition of the whole space \( \mathbb{R}^n \), the *dyadic decomposition*. We define the unit cube to be the set \([0, 1)^n\) and the set \( Q_0 \) to be the family of sets obtained by translating by \( me_j \), where \( m \in \mathbb{Z} \), and for \( j = 1, \ldots, n \), \( e_j \) is an element of the canonical basis, and all their possible combinations, that is,

\[
Q_0 = \{ Q = [0, 1)^n + \sum_{j=1}^n m_j e_j, m_j \in \mathbb{Z} \}.
\]

Analogously, for \( k \in \mathbb{Z} \) we define the family \( Q_k \) as

\[
Q_k = \{ Q = [0, 2^{-k})^n + \sum_{j=1}^n 2^{-k} m_j e_j, m_j \in \mathbb{Z} \}.
\]

The union of the families \( Q_k, k \in \mathbb{Z} \), is called the family of dyadic cubes in \( \mathbb{R}^n \).

It is immediate to see that these sets are cubes with sides parallel to the axes, that a cube in \( Q_k \) have vertices at adjacent points in the lattices \((2^{-k}\mathbb{Z})^n\). Moreover, the following properties are also easily checked.

1. For each \( k \) fixed, the cubes in \( Q_k \) are (mutually) disjoint and their union is all of \( \mathbb{R}^n \).
2. Given any two dyadic cubes, they are either disjoint, or one is contained in the other one.
3. Given \( j, k \in \mathbb{Z} \), with \( j < k \), then each cube in \( Q_k \) is contained in a unique cube in \( Q_j \) and contains \( 2^{k+1} \) cubes in \( Q_{k+1} \).

The proof of Thm. 3.6 uses an important and far-reaching result, called the *Calderón–Zygmund decomposition*, that here we state and prove in the setting of \( \mathbb{R}^n \).

**Theorem 2.5.** Let \( f \in L^1(\mathbb{R}^n) \) and non-negative. Given \( \lambda > 0 \), there exists a sequence \( \{Q_j\} \) of disjoint dyadic cubes such that

1. \( f(x) \leq \lambda \) for almost all \( x \not\in \bigcup_j Q_j \);
2. \( \left| \bigcup_j Q_j \right| \leq \frac{1}{\lambda} \|f\|_{L^1} \);
3. \( \lambda < \frac{1}{|Q_j|} \int_{Q_j} f(x) \, dx \leq 2^n \lambda \).
Proof. Let $\lambda > 0$ be fixed. Since $f \in L^1$, there exists $k_0 \in \mathbb{Z}$ so that

$$\frac{1}{|Q|} \int_Q f \, dx \leq \lambda$$

for all $Q \in \mathcal{Q}_{k_0}$.

Now we consider the cubes of the next generation, that are obtained from the cubes in $\mathcal{Q}_{k_0}$ by bisecting each cube of the collection $\mathcal{Q}_{k_0}$. We say that a cube $Q \in \mathcal{Q}_{k_0+1}$ belongs to the sequence we seek if

$$\frac{1}{|Q|} \int_Q f \, dx > \lambda$$

while (by construction),

$$\frac{1}{|Q'|} \int_{Q'} f \, dx \leq \lambda$$

for (the unique) $Q' \in \mathcal{Q}_{k_0}$ such that $Q' \supseteq Q_{k_0}$.

If a cube is not chosen, that is, $\frac{1}{|Q|} \int_Q f \, dx \leq \lambda$, we iterate this process and we bisect it by considering the dyadic cubes of the next generation that are contained in it. Then, we say that a cube $Q_j \in \mathcal{Q}_j$, $j > k_0$ is in the sequence if

$$\frac{1}{|Q_j|} \int_{Q_j} f \, dx > \lambda,$$

while,

$$\frac{1}{|Q'|} \int_{Q'} f \, dx \leq \lambda$$

for $Q' \in \mathcal{Q}_{j-1}$ such that $Q' \supseteq Q_j$.

We need to check that the sequence $\{Q_j\}$ so constructed satisfies the required conditions. By definition, if $Q_j \in \mathcal{Q}_j$ is a cube in the sequence and $Q \in \mathcal{Q}_{j-1}$ contains $Q_j$ then

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} f(x) \, dx \leq \frac{1}{2^{-n}|Q_j|} \int_Q f(x) \, dx \leq 2^n \lambda,$$

so that (iii) is satisfied.

Next,

$$|\bigcup_j Q_j| = \sum_j |Q_j| < \frac{1}{\lambda} \sum_j \int_{Q_j} f(x) \, dx$$

$$\leq \frac{1}{\lambda} \|f\|_{L^1},$$

so that (ii) also holds.

Finally, (i) follows from the Lebesgue differentiation theorem. If $x \in \mathcal{c}(\bigcup_j Q_j)$, then for all dyadic cubes $Q$ containing $x$ then

$$\frac{1}{|Q|} \int_Q f \, dx \leq \lambda.$$

Letting the side length of the cube containing $x$ tend to 0, by the Lebesgue differentiation theorem we obtain

$$f(x) = \lim_{\epsilon(Q) \to 0} \frac{1}{|Q|} \int_Q f \, dx \leq \lambda,$$

for a.a. $x \in \mathcal{c}(\bigcup_j Q_j)$. This proves the theorem. \qed
3. The Hilbert transform

Throughout this section we assume that \( n = 1 \) and consider the real line \( \mathbb{R} \). We are going to introduce and study perhaps the most important example of an integral operator that is \( L^p \)-bounded. By this we mean that there exists a (measurable) kernel \( K(x,y) \) defined a.e. on \( \mathbb{R} \times \mathbb{R} \) and such that for the integral operator

\[
Tf(x) = \int_{\mathbb{R}} f(y)K(x,y)\,dy
\]

defined on a suitable dense class of functions of \( L^p(\mathbb{R}) \) there exists a constant \( A = A_p \) depending on \( p \), but not on the given function \( f \), such that

\[
\|Tf\|_{L^p(\mathbb{R})} \leq A\|f\|_{L^p(\mathbb{R})}.
\]

Next we introduce a tempered distribution of fundamental importance.

**Definition 3.1.** We define the principal value of \( 1/x \) as the tempered distribution

\[
p.v.\frac{1}{x}(\varphi) = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} \, dx.
\]

**Definition 3.2.** We define the Poisson kernel \( P_y(x) \), defined for \( x \in \mathbb{R} \) and \( y > 0 \) as the kernel

\[
P_y(x) = \frac{1}{\pi} \frac{y}{y^2 + x^2}.
\]

We remark that the kernel \( P_y(x) \) can be viewed as the “dilated” of the \( L^1 \) function \( \frac{1}{\pi} \frac{1}{1+x^2} \), so that \( P_y(x) \) is a so-called summability kernel.

For a given function \( f \in L^p(\mathbb{R}) \), \( 1 \leq p < \infty \), we define the Poisson integral of \( f \) the function

\[
P_y * f(x) = \frac{1}{\pi} \int_{\mathbb{R}} f(x-t)P_y(t)\,dt = \frac{1}{\pi} \int_{\mathbb{R}} f(x-t)\frac{y}{y^2 + t^2} \, dt.
\]

**Definition 3.3.** We are also goint to consider the family of tempered distributions \( Q_y \), with \( y > 0 \), given by

\[
Q_y(x) = \frac{1}{\pi} \frac{x}{y^2 + x^2}.
\]

The kernel \( Q_y(x) \) is called the conjugate Poisson kernel.

For a function \( f \in L^p(\mathbb{R}) \), \( 1 \leq p < \infty \), we define the conjugate Poisson integral of \( f \) to be the function

\[
Q_y * f(x) = \frac{1}{\pi} \int_{\mathbb{R}} f(x-t)Q_y(t)\,dt = \frac{1}{\pi} \int_{\mathbb{R}} f(x-t)\frac{t}{y^2 + t^2} \, dt.
\]

We remark that \( Q_y \) is not an \( L^1 \) function, so that \( Q_y(t) \) is not a summability kernel (and also that here we are using an abuse of notation, since the subindex \( t \) does not represent the dilation defined in (1.2)). Nonetheless, the convolution \( Q_y * f \) is well defined and continuous (since instead \( Q_y \in L^q \), for \( 1 < q \leq \infty \)).

**Remark 3.4.** The three objects above arose naturally when trying to study the regularity of the mapping that associates to a given real harmonic function \( f \) on the upper half-plane (or on the unit disk) its harmonic conjugate.
To briefly illustrate this circle of ideas, suppose that \( f \) is a real function that is in \( L^p(T) \), where \( T \) is the unit circle, that is, the boundary of the unit disk in the complex plane (and \( T \) is identified with the one-dimensional torus, see [Ka]).

Consider its Poisson integral \( \mathcal{P}(f) \), a real harmonic function on the unit disk, that we still denote by \( f \). Then \( f \) admits a harmonic conjugate function \( \tilde{f} \). It can be proved that \( \tilde{f} \) defines a boundary function (still denoted by \( \tilde{f} \)).

It is important to notice that the Cauchy integral of \( f \)
\[
C(f)(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]
where the integral is the complex line integral over the boundary of the unit disk, produces a holomorphic function \( F \). Recall that the Poisson kernel (on the unit disk) satisfies the identity
\[
\mathcal{P} = C + \overline{C} - 1,
\]
that is, \( P = \text{Re} (2C - 1) \). Thus, the real part of \( 2C(F) - f(0) \) coincides with \( f \), and its imaginary part with \( \tilde{f} \).

Hence, the function \( \tilde{f} \) can be obtained as the integral of \( f \) against the imaginary part of the Cauchy kernel, kernel that is called the conjugate Poisson kernel \( Q \). Moreover, it can be proved that, as the variable \( z \) approaches the boundary, \( \tilde{f}(z) = Q(f)(z) \) tends to a function on the boundary, that we still denote by \( \tilde{f} \).

The correspondence of the boundary functions \( f \mapsto \tilde{f} \) is the “Hilbert transform”.

We collect in the next statement the definition and main properties of the Hilbert transform.

**Theorem 3.5.** The following properties hold:

(i) \[ |\text{p.v.} \frac{1}{x}(\varphi)| \leq C(\|\varphi'\|_\infty + \|x\varphi\|_\infty) = C(\varrho_{(0,1)}(\varphi') + \varrho_{(1,0)}(\varphi)); \]

(ii) in \( S' \), \[ \lim_{t \to 0} Q_t = \frac{1}{\pi} \text{p.v.} \frac{1}{x}; \]

(iii) \[ \mathcal{F}\left(\frac{1}{\pi} \text{p.v.} \frac{1}{x}\right)(\xi) = -i \text{sgn}(\xi). \]

The following expressions are equivalent and define the Hilbert transform \( Hf \) of a function \( f \in S' \):

(i') \[ Hf(x) = \left(\frac{1}{\pi} \text{p.v.} \frac{1}{x}\right) * f(x); \]

(ii') \[ Hf(x) = \lim_{t \to 0} Q_t * f(x); \]

(iii') \[ \mathcal{F}(Hf)(\xi) = -i \text{sgn}(\xi) \hat{f}(\xi). \]

**Proof.** Using the fact that \( \int_{\varepsilon \leq |x| \leq 1} 1/x \, dx = 0 \) we write

\[
\text{p.v.} \frac{1}{x} (\varphi) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |x| \leq 1} \frac{\varphi(x)}{x} \, dx + \int_{1 \leq |x|} \frac{\varphi(x)}{x} \, dx
\]
\[
= \lim_{\varepsilon \to 0} \int_{\varepsilon \leq |x| \leq 1} \frac{\varphi(x) - \varphi(0)}{x} \, dx + \int_{1 \leq |x|} \frac{\varphi(x)}{x} \, dx
\]
\[
= \int_{|x| \leq 1} \frac{\varphi(x) - \varphi(0)}{x} \, dx + \int_{1 \leq |x|} \frac{\varphi(x)}{x} \, dx.
\]
Now (i) follows using the mean value theorem.

In order to prove (ii) consider the functions \( u(\varepsilon)(x) = \frac{1}{\varepsilon} \chi_{\{|x| \geq \varepsilon\}} \). These are locally integrable and bounded, hence they define tempered distributions. By definition of the principal value distribution we have that

\[
\lim_{\varepsilon \to 0} u(\varepsilon) = \text{p.v.} \frac{1}{x}.
\]

Therefore, we wish to prove that

\[
\lim_{t \to 0} \left( Q_t - \frac{1}{\pi} u(t) \right) = 0
\]

in \( S' \).

Let \( \varphi \in S \) be fixed. We have

\[
(\pi Q_t - u(t))(\varphi) = \int_{\mathbb{R}} \frac{x \varphi(x)}{t^2 + x^2} \, dx - \int_{|x| \geq t} \frac{\varphi(x)}{x} \, dx
\]

\[
= \int_{|x| < t} \frac{x \varphi(x)}{t^2 + x^2} \, dx - \int_{|x| \geq t} \left( \frac{x}{t^2 + x^2} - \frac{1}{x} \right) \varphi(x) \, dx
\]

\[
= \int_{|x| < 1} \frac{x \varphi(tx)}{1 + x^2} \, dx - \int_{|x| \geq t} \frac{\varphi(tx)}{x(1 + x^2)} \, dx.
\]

Now we can pass to the limit as \( t \to 0 \), by applying the dominated convergence theorem, by majorizing the integrands using the fact that \( |\text{p.v.} 1_c(x)| \leq \|\varphi\|_{L^\infty} \). The limit equals \( \varphi(0) \) times

\[
\int_{|x| < 1} \frac{x}{1 + x^2} \, dx - \int_{|x| \geq 1} \frac{1}{x(1 + x^2)} \, dx = 0
\]

since the functions under the integral signs are odd and the region of integration symmetric with respect to the origin.

Next we compute the Fourier transform of \( \frac{1}{\pi} \text{p.v.} \frac{1}{x} \), and we do this using (ii) and the continuity of \( \mathcal{F} \) on \( S' \); that is, we use the equality

\[
\mathcal{F} \left( \frac{1}{\pi} \text{p.v.} \frac{1}{x} \right) = \lim_{t \to 0} \mathcal{F} Q_t.
\]

We observe that\(^{11}\)

\[
\mathcal{F}^{-1} \left( -i \text{sgn}(\xi) e^{-2\pi t|\xi|} \right)(x) = \int_{\mathbb{R}} -i \text{sgn}(\xi) e^{-2\pi t|\xi|} e^{2\pi i x \xi} \, d\xi
\]

\[
= i \int_{-\infty}^{0} e^{2\pi(t+ix)\xi} \, d\xi - i \int_{0}^{+\infty} e^{2\pi(-t+ix)\xi} \, d\xi
\]

\[
= \frac{i}{2\pi(t+ix)} + \frac{i}{2\pi(-t+ix)}
\]

\[
= \frac{1}{\pi} \frac{x}{t^2 + x^2} = Q_t(x).
\]

\(^{11}\)The Fourier transform of \( Q_t \) can be computed directly using the calculus of indefinite integrals in complex analysis that relies on the residue theorem. Here we use the inverse Fourier transform to avoid using complex analysis.
Therefore, we obtain that
\[
\mathcal{F}\left(\frac{1}{\pi} \text{p.v.} \frac{1}{x}\right) = \lim_{t \to 0} \mathcal{F}Q_t = \lim_{t \to 0} -i \text{sgn}(\xi) e^{-2\pi t|\xi|} = -i \text{sgn}(\xi),
\]
as we wish to show.\textsuperscript{12}

We define the \textit{Hilbert transform} \(Hf\) of a Schwartz function \(f\) as
\[
Hf(x) = \left(\frac{1}{\pi} \text{p.v.} \frac{1}{x}\right) * f(x),
\]
and observe that
\[
Hf(x) = \lim_{\varepsilon \to 0} u_{(\varepsilon)} * f(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\mathbb{R}} f(x - y) \frac{1}{y} \chi_{\{|y| \geq \varepsilon\}}(y) dy
= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x - y)}{y} dy. \tag{3.1}
\]

Finally, (i')-(iii') follow from the first part and Remark 1.24. \(\square\)

### 3.1. The \(L^p\)-boundedness of the Hilbert transform

The regularity result for the Hilbert transform is the following.

**Theorem 3.6.** The Hilbert transform \(H\), initially defined on Schwartz functions, can be extended to an operator, still denoted by \(H\), which is weak-type \((1,1)\) and strong-type \((p,p)\) when \(1 < p < \infty\). More precisely, there exists \(C > 0\) such that
\[
\left|\{x \in \mathbb{R} : |Hf(x)| > \lambda\}\right| \leq \frac{C}{\lambda} \|f\|_{L^1(\mathbb{R})},
\]
and
\[
\|Hf\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^p(\mathbb{R})}
\]
when \(1 < p < \infty\).

**Proof.** We first observe that \(H\) is bounded on \(L^2\), since by Plancherel’s theorem
\[
\|Hf\|_{L^2} = \|(Hf)^\vee\|_{L^2} = \|\text{sgn}(\xi) \hat{f}\|_{L^2} = \|f\|_{L^2}.
\]

\textsuperscript{12} Notice however that the limits above are in the \(\mathcal{S}'\)-sense, that is, for every \(\varphi \in \mathcal{S}\),
\[
\lim_{t \to 0} \left( -i \text{sgn}(\xi) e^{-2\pi t|\xi|}\right)(\varphi) = \lim_{t \to 0} \int_{\mathbb{R}} -i \text{sgn}(\xi) e^{-2\pi t|\xi|} \varphi(\xi) d\xi = \int_{\mathbb{R}} -i \text{sgn}(\xi) \varphi(\xi) d\xi = (-i \text{sgn}(\xi))(\varphi).
\]
Next, suppose that we have shown that $H$ is weak-type $(1, 1)$. Then, by Marcinkiewicz it follows that $H$ is strong-type $(p, p)$, for $1 < p \leq 2$. If $2 < p < \infty$ we use duality:

$$
\|Hf\|_{L^p} = \sup_{\|g\|_{L^{p'}} \leq 1} \left| \int_{\mathbb{R}} Hf(x)g(x) \, dx \right|
\leq \sup_{\|g\|_{L^{p'}} \leq 1} \left| \int_{\mathbb{R}} f(x)Hg(x) \, dx \right|
\leq \sup_{\|g\|_{L^{p'}} \leq 1} \|f(x)\|_{L^p} \|Hg\|_{L^{p'}}
\leq C\|f\|_{L^p},
$$

that is, $H$ is bounded on $L^p$ also when $2 < p < \infty$.

Hence, we have reduced ourselves to show the weak-type $(1, 1)$ inequality. Fix $\lambda > 0$ and we form the Calderón–Zygmund decomposition at height $\lambda$. We obtain a sequence of disjoint interval $\{I_j\}$ such that:

(i) $f(x) \leq \lambda$ for almost all $x \not\in \cup_j I_j$;

(ii) $|\cup_j I_j| \leq \frac{1}{\lambda} \|f\|_{L^1}$;

(iii) $\lambda < \frac{1}{|I_j|} \int_{I_j} f(x) \, dx \leq 2\lambda$.

Now we decompose $f$ as sum $f = g + b$, where

$$
g(x) = \begin{cases} 
    f(x) & \text{if } x \not\in \cup_j I_j \\
    \frac{1}{|I_j|} \int_{I_j} f(x) \, dx & \text{if } x \in I_j,
\end{cases}
$$

and

$$
b(x) = \sum_j b_j(x) = \sum_j \left( f(x) - \frac{1}{|I_j|} \int_{I_j} f \right) \chi_{I_j}(x).
$$

(The decomposition $f = g + b$ is what is called the Calderón–Zygmund decomposition of the non-negative $L^1$ function $f$.)

We observe that $0 \leq g(x) \leq \lambda$, if $x \not\in \cup_j I_j$, while $0 \leq g(x) = \frac{1}{|I_j|} \int_{I_j} f(x) \leq 2\lambda$ if $x \in I_j$. Hence, $g(x) \leq 2\lambda$, and also $g \in L^1$ since it is obviously integrable on $c(\cup_j I_j)$ and on each $I_j$ (that are disjoint) $\int_{I_j} g(x) \, dx = \int_{I_j} f(x) \, dx < \infty$. Notice in particular that $\int_{\mathbb{R}} g = \int_{\mathbb{R}} f$. On the other hand, $b$ has mean equal to 0, since each $b_j$ does.

We now proceed. Since $|Hf(x)| \leq |Hg(x)| + |Hb(x)|$, we have

$$
\|\chi \, \{x : |Hf(x)| > \lambda\}\| \leq \|\chi \, \{x : |Hg(x)| > \lambda/2\}\| + \|\chi \, \{x : |Hb(x)| > \lambda/2\}\|.
$$
Now, on the “good function” $g$ we use the $L^2$-boundedness of the Hilbert transform to obtain

$$\left| \{ x : |Hg(x)| > \lambda / 2 \} \right| = \frac{4}{\lambda^2} \int_{\mathbb{R}} |Hg(x)|^2 \, dx \leq \frac{4}{\lambda^2} \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} g(x)^2 \, dx \right] \, dx \leq \frac{8}{\lambda} \int_{\mathbb{R}} g(x) \, dx = \frac{8}{\lambda} \int_{\mathbb{R}} f(x) \, dx .$$

Next, let $I^*_j$ denote the interval with the same center $c_j$ as $I_j$ with twice the length. Then we estimate,

$$\left| \{ x : |Hb(x)| > \lambda / 2 \} \right| = \left| \{ x \in \bigcup_j I^*_j : |Hb(x)| > \lambda / 2 \} \right| + \left| \{ x \in \bigcup_j I^*_j : |Hb(x)| > \lambda / 2 \} \right| \leq \left| \bigcup_j I^*_j \right| + \left| \{ x \in \bigcup_j I^*_j : |Hb(x)| > \lambda / 2 \} \right| \leq \frac{2}{\lambda} \| f \|_{L^1} + \frac{2}{\lambda} \int_{\mathbb{R}\setminus \bigcup_j I^*_j} |Hb(x)| \, dx .$$

Since $|Hb(x)| \leq \sum_j |Hb_j(x)|$ a.e., it will suffice to prove that

$$\sum_j \int_{\mathbb{R}\setminus I^*_j} |Hb_j(x)| \, dx \leq C \| f \|_{L^1} . \tag{3.2}$$

Now, $b_j \notin \mathcal{S}$, nonetheless, for $x \notin I^*_j$ we have the formula

$$Hb_j(x) = \int_{I_j} \frac{b_j(y)}{x - y} \, dy .$$

Recall that $b_j$ has mean 0 and that we denote by $c_j$ the center of the interval $I_j$. Notice that $y \in I_j$ implies that $|y - c_j| \leq |I_j|/2$ and that $x \notin I^*_j$ and $y \in I_j$ imply that $|x - y| > |x - c_j|/2$. Then,

$$\int_{\mathbb{R}\setminus I^*_j} |Hb_j(x)| \, dx = \int_{\mathbb{R}\setminus I^*_j} \left| \int_{I_j} \frac{b_j(y)}{x - y} \, dy \right| \, dx \leq \int_{I_j} \left| b_j(y) \right| \int_{\mathbb{R}\setminus I^*_j} \frac{\left| y - c_j \right|}{|x - y| |x - c_j|} \, dx \, dy \leq \int_{I_j} \left| b_j(y) \right| \int_{\mathbb{R}\setminus I^*_j} \frac{|I_j|}{|x - c_j|^2} \, dx \, dy = 2 \int_{I_j} \left| b_j(y) \right| \, dy .$$

Therefore,

$$\sum_j \int_{\mathbb{R}\setminus I^*_j} |Hb_j(x)| \, dx \leq 2 \sum_j \int_{I_j} |b_j(y)| \, dy \leq 4 \| f \|_{L^1} ,$$

which gives (4.33). This concludes the proof. \qed
3.2. Further properties of the Hilbert transform. We now make a few observations.

(1) We have shown that, when \( f \in S \), then \( Hf \) satisfies a weak-type \((1,1)\) bound and a strong-type \((p,p)\) bound. We now extend the definition of \( Hf \) to all of \( L^p \) and all of \( L^1 \), with the same bounds.

If \( f \in L^p \), \( 1 < p < \infty \), then there exists a sequence \( \{f_n\} \) of Schwartz functions converging in \( L^p \) to \( f \):

\[
\lim_{n \to +\infty} \|f_n - f\|_{L^p} = 0
\]

Then, in order to define \( Hf \), notice that \( \|Hf_n - Hf_m\|_{L^p} \leq C\|f_n - f_m\|_{L^p} \), so that \( \{Hf_n\} \) is a Cauchy sequence in \( L^p \) and converges to \( g \in L^p \). We set then \( Hf = g \). It is immediate to see that this definition is well-given, in the sense that does not depend on the choice of the sequence \( \{f_n\} \) and that \( H \) then satisfies the bound \( \|Hf\|_{L^p} \leq C\|f\|_{L^p} \), with the same constant \( C \) as in Thm. 3.6.

(2) If \( f \in L^1 \), then there exists a sequence \( \{f_n\} \) of Schwartz functions converging in \( L^1 \) to \( f \). The weak \((1,1)\) inequality gives that, for all \( \varepsilon > 0 \) fixed,

\[
\lim_{n,m \to +\infty} \left| \left\{ x \in \mathbb{R} : \left| (Hf_n - Hf_m)(x) \right| > \varepsilon \right\} \right| \leq \frac{C}{\varepsilon} \|f_n - f_m\|_{L^1} = 0.
\]

Then, \( \{Hf_n\} \) is a Cauchy sequence in measure, so it converges to a measurable function \( g \) a.e.. We set, \( Hf = g \). It is easy to check that \( H \) satisfies the same weak-type \((1,1)\) bound on all of \( L^1 \).

(3) When \( p = 1 \), the strong-type inequality fails. For instance, if we take \( f = \chi_{[0,1]} \), then we have\(^{13}\)

\[
Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(y)}{x - y} \, dy = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{0}^{1} \frac{1}{x - y} \, dy
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{x-1}^{x-\varepsilon} \frac{1}{t} \, dt
\]

\[
= \frac{1}{\pi} \log \frac{|x|}{|x-1|}.
\]

It is clear that \( Hf \not\in L^1 \), while \( f \in L^1 \).

(4) Having defined the Hilbert transform of a function \( f \in L^p \), for \( 1 \leq p < \infty \) as limit in norm (when \( 1 < p < \infty \)) or in measure (when \( p = 1 \)), we now wish to defined it pointwise as well. In this part we will omit the proofs and we refer the reader to [Du]. Consider the truncated integrals

\[
H_\varepsilon f(x) = \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x-y)}{y} \, dy.
\]

**Proposition 3.7.** Let \( 1 < p < \infty \). Then, if \( f \in L^p \), \( Hf = \lim_{\varepsilon \to 0} H_\varepsilon f \) in the \( L^p\)-norm.

When \( p = 1 \) and \( f \in L^1 \), then \( Hf = \lim_{\varepsilon \to 0} H_\varepsilon f \) in measure.

**Proof.** Notice that the function \( \frac{1}{y} \chi_{\{|y| \geq \varepsilon\}} \) is in \( L^q \) for all \( 1 < q \leq \infty \). Then, the function

\[
\frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{1}{y} \chi_{\{|y| > \varepsilon\}}(y) f(x-y) \, dy = H_\varepsilon f(x),
\]

\(^{13}\)Here we use the equality at the end of the proof of Thm. 3.5.
is well defined for all $f \in L^p$, $1 \leq p < \infty$.

Now,

$$F \left( \frac{1}{y} \chi_{\{|y|>\epsilon\}} \right) (\xi) = \lim_{N \to +\infty} \int_{\epsilon < |y| < N} e^{-2\pi iy\xi} \frac{y}{y} dy$$

$$= \lim_{N \to +\infty} \int_{\epsilon < |y| < N} -i \sin(2\pi y\xi) \frac{y}{y} dy$$

$$= -2i \text{sgn}(\xi) \lim_{N \to +\infty} \int_{2\pi N |\xi|}^{2\pi N |\xi|} \frac{\sin x}{x} dx.$$ 

Therefore, $|F \left( \frac{1}{y} \chi_{\{|y|>\epsilon\}} \right) (\xi)| \leq 2\pi$, for all $\epsilon > 0$. This implies that $H_\epsilon$ is of strong-type $(2, 2)$, with constants uniform in $\epsilon$.

The weak-type $(1, 1)$ boundedness also follows with uniform constant, so the strong-type $(p, p)$ follows, with uniform constant, by interpolation and duality.

We consider now the case $1 < p < \infty$; the case $p = 1$ begin analogous. If $f \in L^p$, then let $\{f_n\}$ be in $S$ converging to $f$ in $L^p$. Then, using the uniform bound for the $L^p$-boundedness of $H_\epsilon$ (and the fact that $f_n \in S$), we have

$$Hf = \lim_{n \to +\infty} Hf_n = \lim_{n \to +\infty} \lim_{\epsilon \to 0} H_\epsilon f_n = \lim_{\epsilon \to 0} \lim_{n \to +\infty} H_\epsilon f_n = \lim_{\epsilon \to 0} H_\epsilon f.$$ 

This proves the lemma.

We conclude this section by stating the following result concerning the pointwise definition of the Hilbert transform.

**Theorem 3.8.** Let $1 \leq p < \infty$, and $f \in L^p$. Then

$$Hf(x) = \lim_{\epsilon \to 0} H_\epsilon f(x) \quad \text{a.e.} \ x \in \mathbb{R}.$$ 

We only mention that, similarly to the case of the Hardy–Littlewood maximal function and the Lebesgue differentiation theorem, the proof relies on the boundedness of a maximal function.

Set

$$H^* f(x) = \sup_{\epsilon > 0} |H_\epsilon f(x)|.$$ 

Then the following result holds true.

**Theorem 3.9.** The maximal Hilbert transform $H^*$ is weak-type $(1, 1)$ and strong-type $(p, p)$ for $1 < p < \infty$. 

4. Singular integrals

4.1. The Calderón–Zygmund theorem. Main result of this section is the following theorem. Given a function $K$ that is locally integrable in $\mathbb{R}^n \setminus \{0\}$, and it is also a tempered distribution, it makes sense to consider the convolution $K \ast f$ with a Schwartz function $f$. Therefore, we define the operator

$$Tf(x) = K \ast f(x),$$

initially defined on Schwartz functions.

The next result generalizes the theorem on the boundedness of the Hilbert transform.

Theorem 4.1. Let $K$ be a tempered distribution that coincides with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$. Suppose that

$$|\hat{K}(\xi)| \leq A,$$  \hspace{1cm} (4.1)

and furthermore

$$\int_{|x|>2|y|} |K(x-y) - K(x)| \, dx \leq B.$$  \hspace{1cm} (4.2)

Then, the operator $T$ is bounded from $L^p$ into itself, i.e., it is of strong-type $(p,p)$ for $1 < p < \infty$ and it is of weak-type $(1,1)$, that is,

$$\left| \left\{ x : |Tf(x)| > \lambda \right\} \right| \leq \frac{C}{\lambda} \|f\|_{L^1}.$$

Definition 4.2. A tempered distribution $K$ that satisfies the conditions in the theorem, that is, that coincides with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$ and satisfies (4.1) and (4.2) is called a Calderón–Zygmund convolution kernel.

Corollary 4.3. Let $K \in C^1(\mathbb{R}^n \setminus \{0\})$ be a tempered distribution that coincides with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$. Suppose that $K$ satisfies (4.1) and also

$$|\nabla K(x)| \leq \frac{B'}{|x|^{n+1}}.$$  \hspace{1cm} (4.3)

Then, for the operator $T$ the same conclusions hold as in the previous theorem.

Proof. In order to prove the corollary, assuming the validity of the theorem, it suffices to show that (4.3) implies (4.2). But this follows from the mean value theorem. Let $\sigma(x,y)$ denote the segment having endpoints in $x - y$ and $x$. Then, using the fact that $|x| > 2|y|$, it holds that $|x - y| \geq |x| - \frac{|x|}{2} = \frac{|x|}{2}$, so that, if $z \in \sigma(x,y)$ we have

$$|z| \geq \min\{|x - y|, |x|\} \geq \frac{|x|}{2}.$$

Therefore,

$$|K(x-y) - K(x)| \leq |y| \sup_{z \in \sigma(x,y)} |\nabla K(z)| \leq B' |y| \sup_{z \in \sigma(x,y)} \frac{1}{|z|^{n+1}} \leq \frac{2^{n+1}B'}{|x|^{n+1}} |y|.$$
Hence,\(^{14}\)
\[
\int_{|x|>2|y|} |K(x - y) - K(x)| \, dx \leq 2^{n+1} B'|y| \int_{|x|>2|y|} \frac{1}{|x|^{n+1}} \, dx
\]
\[
= 2^{n+1} B' \omega_n |y| \int_{2|y|}^{\infty} r^{-2} \, dr
\]
\[
= B.
\]
Thus, (4.3) implies (4.2) and the conclusion follows from Thm. 4.1. \(\square\)

Proof of Thm. 4.1. The proof goes along the same lines of the proof of Thm. 3.6.
By the assumption (4.1) and the Plancherel theorem it immediately follows that \(T\) is bounded on \(L^2\):
\[
\|Tf\|^2_{L^2} = \int_{\mathbb{R}^n} |\mathcal{F}(K * f)(\xi)|^2 \, d\xi = \int_{\mathbb{R}^n} |\hat{K}(\xi) \hat{f}(\xi)|^2 \, d\xi \leq A^2 \|f\|^2_{L^2}.
\]

We now observe that the adjoint operator \(T^*\) has kernel \(K^*(x) = K(-x)\), so that it also satisfies hypotheses (4.1) and (4.2). Thus, if we prove the weak-(1,1) inequality for \(T\), by interpolation it would follow that \(T\) is \(L^p\)-bounded for \(1 < p \leq 2\), and then, by duality is \(L^p\)-bounded also for \(2 \leq p < \infty\).\(^{15}\)

Hence, it suffices to prove the weak-(1,1) inequality. To do this, we may assume that \(f \geq 0\) and, let \(\lambda > 0\) be fixed. We decompose \(f\) according to Thm. 2.5, as in the proof of Thm. 3.6. We repeat it for simplicity: we a sequence of disjoint cubes \(\{Q_j\}\) such that:

(i) \(f(x) \leq \lambda\) for almost all \(x \notin \bigcup_j Q_j\);

(ii) \(|\bigcup_j Q_j| \leq \frac{1}{\lambda} \|f\|_{L^1}\);

(iii) \(\lambda < \frac{1}{|Q_j|} \int_{Q_j} f(x) \, dx \leq 2^n \lambda\).

Now we decompose \(f\) as sum \(f = g + b\), where
\[
g(x) = \begin{cases} 
    f(x) & \text{if } x \notin \bigcup_j Q_j \\
    \frac{1}{|Q_j|} \int_{Q_j} f(x) \, dx & \text{if } x \in Q_j,
\end{cases}
\]
and
\[
b(x) = \sum_j b_j(x) = \sum_j \left( f(x) - \frac{1}{|Q_j|} \int_{Q_j} f \right) \chi_{Q_j}(x).
\]

Arguing as in the proof of Thm. 3.6, we have
\[
|\{x : |Tf(x)| > \lambda\}| \leq |\{x : |Tg(x)| > \lambda/2\}| + |\{x : |Tb(x)| > \lambda/2\}|,
\]
and then
\[
|\{x : |Tg(x)| > \lambda/2\}| \leq C \int_{\mathbb{R}} f(x) \, dx.
\]

\(^{14}\)Here, and in the remaining of the notes, we denote by \(\omega_n\) the volume of the unit sphere \(S^{n-1}\) in \(\mathbb{R}^n\).

\(^{15}\)This boundedness is equivalent to the \(L^p\)-boundedness of \(T^*\) for \(1 < p \leq 2\).
Next,

\[ \left| \left\{ x : |Tb(x)| > \lambda/2 \right\} \right| \leq \frac{C}{\lambda} \|f\|_{L^1} + \frac{2}{\lambda} \int_{\mathbb{R} \setminus \bigcup_j Q_j^*} |Tb(x)| \, dx. \]

where \( Q_j^* \) denote the cube with the same center \( c_j \) as \( Q_j \) with side length \( 2\sqrt{n} \)-times longer. Then we reduce ourselves to show that

\[ \sum_j \int_{\mathbb{R} \setminus Q_j^*} |Tb_j(x)| \, dx \leq C \|f\|_{L^1}, \]

which, in turn, is implied by the inequality

\[ \int_{\mathbb{R} \setminus Q_j^*} |Tb_j(x)| \, dx \leq C \|b_j\|_{L^1}. \] (4.4)

We are going to use the Hörmander condition (4.2) and the fact that \( b_j \) has integral equal to 0. For \( x \not\in Q_j^* \),

\[ Tb_j(x) = \int_{Q_j} K(x - y)b_j(y) \, dy = \int_{Q_j} \left[ K(x - y) - K(x - c_j) \right] b_j(y) \, dy. \]

Therefore,

\[ \int_{\mathbb{R} \setminus Q_j^*} |Tb_j(x)| \, dx \leq \int_{Q_j} |b_j(y)| \int_{\mathbb{R} \setminus Q_j^*} |K(x - y) - K(x - c_j)| \, dx \, dy \]

\[ \leq B \int_{Q_j} |b_j(y)| \, dy, \]

since

\[ \mathbb{R} \setminus Q_j^* \subseteq \{ x \in \mathbb{R}^n : |x - c_j| > 2|y - c_j| \}, \]

so that

\[ \int_{\mathbb{R} \setminus Q_j^*} |K(x - y) - K(x - c_j)| \, dx \leq \int_{|x - c_j| > 2|y - c_j|} |K(x - y) - K(x - c_j)| \, dx \]

\[ = \int_{|x'| > 2|y - c_j|} \left| K(x' - (y - c_j)) - K(x') \right| \, dx' \]

\[ \leq B. \]

Thus, (4.4) follows and we are done. \( \square \)

4.2. Homogeneous distributions. In order to describe some classical and fundamental examples of singular integrals in \( \mathbb{R}^n \), examples that generalize the case of the Hilbert transform in the case of the real line, we need a preliminary discussion of the so-called \textit{homogenous distributions}.

We recall that a function \( f \) is said to be \textit{homogenous of degree} \( a \) if for all \( \lambda > 0 \) and all \( x \in \mathbb{R}^n \)

\[ f(\lambda x) = \lambda^a f(x). \]
Given another function \( \varphi \), we see that
\[
\int_{\mathbb{R}^n} f(x) \varphi(\lambda x) \, dx = \lambda^{-n} \int_{\mathbb{R}^n} f(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} f(\lambda x') \varphi(x') \, dx' = \lambda^a \int_{\mathbb{R}^n} f(x') \varphi(x') \, dx'.
\]

Therefore, we may extend the definition of homogeneity to tempered distributions as follows.

**Definition 4.4.** Given a tempered distribution \( u \), we say that \( u \) is homogenous of degree \( a > 0 \) if for all \( \varphi \in \mathcal{S} \) and \( \lambda > 0 \) we have
\[
u(\varphi \lambda) = \lambda^a u(\varphi).
\]

**Example 4.5.** It is immediate to see that the distribution \( \frac{1}{\pi} \text{p.v.} \frac{1}{x} \) on \( \mathcal{S}(\mathbb{R}) \) is homogenous of degree \(-1\), while \( m(\xi) = -i \text{sgn}(\xi) \) is a homogeneous (function) of degree 0.

We now wish to generalize the tempered distribution \( \frac{1}{\pi} \text{p.v.} \frac{1}{x} \) and the Hilbert transform to higher dimensions. A typical situation will be the following.

Let \( \Omega(x) \) be a function in \( \mathbb{R}^n \setminus \{0\} \) homogeneous of degree 0, that is, such that \( \Omega(\lambda x) = \Omega(x) \) for all \( \lambda > 0 \) and \( x \in \mathbb{R}^n \setminus \{0\} \). Notice that \( \Omega \), as all homogeneous functions, is uniquely determined by its values on the unit sphere \( S^{n-1} \). Suppose that \( \Omega \in L^1(S^{n-1}) \) and
\[
\int_{S^{n-1}} \Omega(x') \, d\sigma(x') = 0,
\]
where \( d\sigma \) denotes the Lebesgue surface measure on \( S^{n-1} \). We define the tempered distribution
\[
u(\varphi) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x)}{|x|^n} \phi(x) \, dx
\]
\[
= \text{p.v.} \int_0^{+\infty} \int_{S^{n-1}} \Omega(x') \phi(rx') \, d\sigma(x') r^{-1} \, dr
\]
\[
= \int_0^{+\infty} \int_{S^{n-1}} \Omega(x') (\phi(rx') - \phi(0)) \, d\sigma(x') r^{-1} \, dr,
\]
where we have used the fact that \( \Omega \) has zero integral on the unit sphere. Now, using the mean value theorem, as in the case of \( \frac{1}{\pi} \text{p.v.} \frac{1}{x} \), it is easy to see that the last integral above converges absolutely, for \( \varphi \in \mathcal{S}(\mathbb{R}^n) \).

**Lemma 4.6.** If \( u \in \mathcal{S}' \) is a tempered distribution homogeneous of degree \( a \), then its Fourier transform is a tempered distribution homogeneous of degree \(-n - a\).

**Proof.** Recall the identities (1.5). For \( \varphi \in \mathcal{S} \) and \( \lambda > 0 \) we have
\[
\hat{u}(\varphi \lambda) = u(\hat{\varphi} \lambda) = u((\hat{\varphi})^\lambda) = \lambda^{-n} u((\hat{\varphi})_{\lambda^{-1}})
\]
\[
= \lambda^{-n-a} u(\hat{\varphi}) = \lambda^{-n-a} \hat{u}(\varphi),
\]
as we wished to show. □

**Lemma 4.7.** Let \( 0 < a < n \). Then the function \(|x|^{-a}\) is locally integrable and defines a tempered distribution. The Fourier transform, as an element of \( \mathcal{S}' \) satisfies the equality
\[
(|x|^{-a})^{\wedge}(\xi) = c_{n,a} |\xi|^{a-n},
\]
where
\[
c_{n,a} = \frac{\Gamma(n-a)}{\Gamma(a-n)}.
\]
where
\[ c_{n,a} = \frac{\pi^{a-n/2} \Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}. \]  

**Proof.** We recall that the Fourier transform of a radial function is also radial.\(^{16}\) Suppose first that \(n/2 < a < n\), so that
\[ |x|^{-a} = |x|^{-a} \chi_{\{|x| \leq 1\}} + |x|^{-a} \chi_{\{|x| > 1\}} = f_1(x) + f_2(x), \]
where \(f_1 \in L^1\) and \(f_2 \in L^2\). Then we can compute the Fourier transform of \(|x|^{-a}\) as a function. Its Fourier transform is a radial function, and by the previous lemma, homogeneous of degree \(-n+a\). Hence, it is a constant multiple of \(|\xi|^{a-n}\), i.e.,
\[ \hat{(|x|^{-a})}(\xi) = c_{n,a} |\xi|^{a-n}. \]

We now compute the constant \(c_{n,a}\). Using the Parseval formula (1.6) and the identity \((e^{-\pi|x|^2})(\xi) = e^{-\pi|\xi|^2}\) of Lemma 1.13 we have
\[ \int_{\mathbb{R}^n} e^{-\pi|\xi|^2} |\xi|^{a-n} d\xi. \]  

Now, setting \(\pi r^2 = s\) we see that
\[ \int_0^{+\infty} e^{-\pi r^2} r^b dr = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-s\left(\frac{b}{2}\right)} \frac{1}{\sqrt{s}} ds \]
\[ = \frac{1}{2\pi^{(1+b)/2}} \int_0^{+\infty} e^{-s(s/b-1)/2} ds \]
\[ = \frac{1}{2\pi^{(1+b)/2}} \Gamma((1+b)/2). \]

From (4.6) it then follows
\[ c_{n,a} = \frac{1}{2\pi^{(n-a)/2}} \Gamma((n-a)/2) \left[ \frac{1}{2\pi^{a/2}} \Gamma(a/2) \right]^{-1} \]
\[ = \frac{\pi^{a-n/2} \Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}. \]

This proves the proposition in the case \(n/2 < a < n\).

In the case \(0 < a < n/2\) we use the first part, since in this case \(n/2 < n - a < n\), and the inversion formula, valid also in \(S'\). The case \(a = n/2\) follows by passing to the limit, and using the continuity of the Fourier transform in \(S'\). \(\square\)

---

\(^{16}\)This is immediate to check, using the fact that a function \(f\) is radial if and only if \(f(Ox) = f(x)\) for all \(x \in \mathbb{R}^n\) and all orthogonal transformations \(O\).
4.3. The Riesz transforms. We now define the main generalizations of the Hilbert transform, the Riesz transforms $R_j$, $j = 1, \ldots, n$.

Given $f \in S$, we set
\[
R_j f(x) = c_n \text{p.v.} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x - y) \, dy,
\]
where the constant $c_n$ is given by\(^{17}\)
\[
c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}.
\]

Notice that for $n = 1$ we have $c_1 = 1/\pi$ and we recover the definition of the Hilbert transform.

Lemma 4.8. For $f \in L^2(\mathbb{R}^n)$ and $j = 1, \ldots, n$ we have that
\[
(R_j f)(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi).
\]
Hence,
\[
\sum_{j=1}^{n} R_j^2 = -I.
\]

Proof. We wish to show that
\[
\mathcal{F}\left(c_n \text{p.v.} \frac{y_j}{|y|^{n+1}}\right)(\xi) = -i \frac{\xi_j}{|\xi|},
\]
as tempered distributions.

We claim that, again as tempered distributions,
\[
\partial_{x_j}|x|^{-n+1} = (1 - n) \text{p.v.} \frac{x_j}{|x|^{n+1}},
\]
assuming the claim we finish the proof. Using Lemma 4.7 and Remark 1.24 (V) we have
\[
\mathcal{F}\left(\text{p.v.} \frac{x_j}{|x|^{n+1}}\right)(\xi) = \frac{1}{1 - n} \mathcal{F}\left(\partial_{x_j}|x|^{-n+1}\right)(\xi)
\]
\[
= \frac{2\pi i \xi_j}{1 - n} \mathcal{F}\left(|x|^{-n+1}\right)(\xi)
\]
\[
= \frac{2\pi i \xi_j}{1 - n} \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} |\xi|^{-1}
\]
\[
= -i \frac{\pi^{(n+1)/2} \xi_j}{\Gamma((n+1)/2) |\xi|}
\]
\[
= -i \frac{\xi_j}{c_n |\xi|},
\]
as we wish to prove.

\(^{17}\)We recall that the volume of the unit ball $B_n$ and of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ are, resp., $|B_n| = \pi^{n/2}/\Gamma((n/2) + 1)$ and $\sigma(S^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$, resp.
It only remains to prove the claim. For \( \varphi \in \mathcal{S} \) we have
\[
(1 - n) \text{ p.v.} \frac{x_j}{|x|^{n+1}}(\varphi) = \lim_{\varepsilon \to 0^+} (1 - n) \int_{\mathbb{R}^n} \frac{x_j}{(\varepsilon^2 + |x|^2)^{(n+1)/2}} \varphi(x) \, dx
\]
\[
= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \frac{1}{(\varepsilon^2 + |x|^2)^{(n-1)/2}} \partial_{x_j} \varphi(x) \, dx
\]
\[
= - \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \frac{1}{|x|^{n-1}} \partial_{x_j} \varphi(x) \, dx
\]
\[
= (\partial_{x_j} |x|^{1-n})(\varphi).
\]
This proves the claims and therefore the lemma. \( \square \)

**Theorem 4.9.** The Riesz transforms \( R_j, j = 1, \ldots, n \), are of weak-type \((1, 1)\) and of strong-type \((p, p)\), for \( 1 < p < \infty \).

**Proof.** It suffices to show that the tempered distributions \( K_j(x) = c_n \text{ p.v.}(x_j/|x|^{n+1}) \) satisfy the hypotheses of Thm. 4.1, for \( j = 1, \ldots, n \), resp.

Clearly \( K_j \) coincides with a locally integrable function on \( \mathbb{R}^n \setminus \{0\} \). Moreover, by the lemma,
\[
\hat{K}_j(\xi) = -i \frac{\xi_j}{|\xi|},
\]
which is clearly bounded.

Finally, on \( \mathbb{R}^n \setminus \{0\} \)
\[
|\nabla K_j(x)| \leq c_n \frac{1}{|x|^{n+1}} + c_n \frac{n+1}{2} \frac{|x_j||x|}{|x|^{n+3}}
\]
\[
\leq C \frac{1}{|x|^{n+1}}.
\]
Thus, (4.2) is satisfied and we are done. \( \square \)

### 4.4. Solution of the Laplace equation.
In this section we consider the Laplace operator
\[
\Delta = \sum_{j=1}^n \partial^2_{x_j}, \quad (4.9)
\]
and prove a regularity result for the solution of the equation
\[
\Delta g = f.
\]

Given a partial differential operator \( P \) with constant coefficients, we say that a tempered distribution \( E \) is a fundamental solution for \( P \) if \( PE = \delta_0 \), the Dirac delta at the origin, that is, if
\[
P(f \ast E) = f \ast PE = f \ast \delta_0 = f,
\]
for all \( f \in \mathcal{S} \). Notice in particular that, for a given \( f \in \mathcal{S} \), the tempered distribution \( f \ast E \) solves the equation \( Pg = f \).

We now compute the fundamental solution for \( \Delta \).
Proposition 4.10. Let \( \omega_n = 2\pi^{n/2}/\Gamma(n/2) \) denote the volume of the unit sphere in \( \mathbb{R}^n \) and define
\[
E(x) = \frac{1}{(2 - n)\omega_n} \frac{1}{|x|^{n-2}} \quad \text{for } n > 2,
\]
and
\[
E(x) = \frac{1}{2\pi} \log |x| \quad \text{for } n = 2.
\]
Then \( E \) is a fundamental solution for \( \Delta \).

Proof. We consider a family of regularized tempered distributions, namely \( E^\varepsilon \), given by
\[
E^\varepsilon(x) = \frac{1}{(2 - n)\omega_n} \frac{1}{(\varepsilon^2 + |x|^2)^{(n-2)/2}} \quad \text{for } n > 2,
\]
and
\[
E^\varepsilon(x) = \frac{1}{4\pi} \log(\varepsilon^2 + |x|^2) \quad \text{for } n = 2.
\]
It is easy to see that \( E^\varepsilon \to E \) pointwise as \( \varepsilon \to 0 \), and that there exists \( g \in L^1_{\text{loc}} \) such that \( |E^\varepsilon| \leq g \) for all \( \varepsilon \leq 1 \), in both cases \( n = 2 \) and \( n > 2 \). (It suffices to take \( |E| \) when \( n > 2 \) and \( |\log |x|| + 1 \) when \( n = 2 \).) Hence, \( E^\varepsilon \to E \) in \( S' \) and therefore, also \( \Delta E^\varepsilon \to \Delta E \) in \( S' \).

Let \( \varphi \in S \), we wish to show that \( \Delta E^\varepsilon(\varphi) \to \varphi(0) \) as \( \varepsilon \to 0 \). We provide the details for \( n > 2 \). We have
\[
\partial_{x_j} E^\varepsilon(x) = \frac{1}{(2 - n)\omega_n} (2 - n)x_j \left(\frac{1}{(|x|^2 + \varepsilon^2)^{n/2}} - \frac{n x_j^2}{(|x|^2 + \varepsilon^2)^{(n+2)/2}}\right),
\]
so that
\[
\Delta E^\varepsilon(x) = \frac{1}{\omega_n} \sum_{j=1}^{n} \left(\frac{1}{(|x|^2 + \varepsilon^2)^{n/2}} - \frac{n x_j^2}{(|x|^2 + \varepsilon^2)^{(n+2)/2}}\right)
\]
\[
= \frac{n}{\omega_n} \frac{\varepsilon^2}{(|x|^2 + \varepsilon^2)^{(n+2)/2}}.
\]
Therefore,
\[
\Delta E^\varepsilon(x) = \psi_\varepsilon(x),
\]
where, as usual, \( \psi_\varepsilon(x) = e^{-n} \psi(x/\varepsilon) \) and
\[
\psi = \Delta E^1 = \frac{n}{\omega_n} \frac{1}{(|x|^2 + 1)^{(n+2)/2}}.
\]
Next we use the fact that \( \psi(x) = \psi(-x) \) to see that
\[
\Delta E^\varepsilon(\varphi) = \int \Delta E^\varepsilon(-x) \varphi(x) \, dx = \psi_\varepsilon * \varphi(0) \to (\int \psi) \varphi(0),
\]
by the properties of the summability kernels.

Thus, we only need to compute \( \int \psi \). But, by passing to polar coordinates and making the change of variables \( s = r^2/(1 + r^2) \), so that \( ds = 2(1 + r^2)^{-2}rdr \), and
\[
\frac{r^{n-1}}{(1 + r^2)^{(n+2)/2}} = \left(\frac{r^2}{1 + r^2}\right)^{(n-2)/2} \frac{r}{(1 + r^2)^2}.
\]
We then find
\[
\int \psi(x) \, dx = \frac{n}{\omega_n} \int \frac{1}{(|x|^2 + 1)^{(n+2)/2}} \, dx \\
= n \int_0^{+\infty} \frac{1}{(1+r^2)^{(n+2)/2}} r^{n-1} \, dr \\
= \frac{n}{2} \int_0^1 s^{(n-2)/2} \, ds = 1.
\]

This completes the proof (at least in the case \( n > 2 \), the case \( n = 2 \) being left to the reader). \( \square \)

We now have the regularity result for the solutions of the Laplace equation.

**Theorem 4.11.** Let \( f \in S \) be given. Define the operator
\[
Tf(x) = \partial^2_{x_j x_k} f \ast E = f \ast \partial^2_{x_j x_k} E,
\]
where \( j, k = 1, \ldots, n \). Then, \( T \) extends to an operator that is of weak-type \((1,1)\) and of strong-type \((p,p)\), for \( 1 < p < \infty \).

**Remark 4.12.** Notice that the result shows that, for a given \( f \in L^p \), it is possible to define the solution \( u = f \ast E \) of the equation \( \Delta u = f \) is such a way that the map \( T(f) = \partial^2_{x_j x_k}(f \ast E) \) is bounded on \( L^p \).

On the other hand, the map \( \mathcal{I} \) defined as \( f \mapsto u = f \ast E \) is bounded, when \( n \geq 3 \),
\[
\mathcal{I} : L^p \to L^q
\]
as a consequence of Thm. 4.19 of the next section, where
\[
\frac{1}{q} = \frac{1}{p} - \frac{2}{n}.
\]

**Proof of Thm. 4.11.** It suffices to show that the kernel \( K = \partial^2_{x_j x_k} E \) satisfies the hypotheses of Thm. 4.1. It is clear that, by Lemma 4.7,
\[
\mathcal{F}(\partial^2_{x_j x_k} E)(\xi) = -(2\pi)^2 \xi_j \xi_k \hat{E}(\xi) \\
= C_n \xi_j \xi_k |\xi|^{-2},
\]
so that (4.1) is satisfied.

On the other hand, also the condition (4.3) is also easily seen to be satisfied: if \( x \neq 0 \),
\[
|\nabla(\partial^2_{x_j x_k} E)(x)| \leq C \frac{1}{|x|^{n+1}},
\]
The result now follows from 4.1. \( \square \)

We now consider the **Dirichlet problem for the Laplacian on the half-space**
\[
\begin{cases}
\partial^2_t f(x,t) + \Delta u(x,t) = 0 & \text{for } (x,t) \in \mathbb{R}^{n+1}, \ t > 0 \\
u(x,0) = f(x) & \text{for } x \in \mathbb{R}^n.
\end{cases}
\] (4.10)

Here \( \Delta = \Delta_x = \sum_{j=1}^n \partial^2_{x_j} \), so that \( \partial^2_t + \Delta_x \) is the Laplacian in \( \mathbb{R}^{n+1} \). Moreover,
\[
\mathbb{R}^{n+1}_{+} := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}.
\]
We assume that the boundary data $g \in L^p(\mathbb{R}^n)$ and we look for a solution $f$ defined on $\mathbb{R}^{n+1}$ such that $f(\cdot,t) \in S'(\mathbb{R}^n)$ for all fixed $t > 0$. In this case, we can compute the Fourier transform w.r.t. the variable $x \in \mathbb{R}^n$. We have that

$$\mathcal{F}((\partial_t^2 + \Delta_x)f(\cdot,t))(\xi) = \partial_t^2 \hat{f}(\xi,t) + \mathcal{F}(\Delta_x)f(\cdot,t))(\xi) = \partial_t^2 \hat{f}(\xi,t) - |2\pi \xi|^2 \hat{f}(\xi,t).$$

For the time being we consider $\xi$ as a fixed parameter and write $\varphi = \hat{f}(t,\cdot)$. Then the Dirichlet problem (4.10) has been transformed into the Cauchy initial value problem

$$\begin{cases} 
\varphi''(t) - a^2 \varphi(t) = 0 \\
\varphi(0) = b,
\end{cases}$$

where $a = |2\pi \xi|$ and $b = \hat{g}(\xi)$. Since the characteristic equation has roots $\lambda_{\pm} = \pm |2\pi \xi|$, the solutions of the ODE are

$$\varphi(t) = c_+ e^{2\pi \xi|t|} + c_- e^{-|2\pi \xi|t}.$$ 

Then, the solution of (4.10) has to satisfy

$$\mathcal{F}^{-1}(c_+ e^{2\pi \xi|t|} + c_0 e^{-|2\pi \xi|t})(x)$$

and the only possibility for the (inverse) Fourier transform to make sense is that $c_+ e^{2\pi \xi|t|} + c_0 e^{-|2\pi \xi|t} \in S'(\mathbb{R}^n)$. This forces $c_+ = 0$ so that the solution of the Cauchy problem satisfies the initial condition

$$\varphi(0) = c_- = b = \hat{g}(\xi).$$

Therefore,

$$f(x,t) = \mathcal{F}^{-1}(\hat{g} e^{-|2\pi \cdot|t}) (x) = (g * \mathcal{F}^{-1}(e^{-|2\pi \cdot|t}) (x).$$

Thus, we need to compute $\mathcal{F}^{-1}(e^{-|2\pi \cdot|t}) (x)$. Observe that by Remark 1.14 we have that $\mathcal{F}^{-1}(e^{-|2\pi \cdot|t}) (x) = P_t(x)$, so it suffices to determine $\Psi$ such that $\hat{P}(x) = e^{-|2\pi \xi|}$.

We define the Poisson kernel for the half-space

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^\frac{n+1}{2}}$$

where $\omega_n = \frac{\Gamma(n+1)}{n+1}$. We have the following

**Proposition 4.13.** The function $P(x) = \omega_n \frac{1}{1 + (|x|^2)^\frac{n+1}{2}} \in L^1(\mathbb{R}^n)$ and

$$\hat{P}(\xi) = e^{-|2\pi \xi|}.$$ 

For the proof, see Exercise 9, Appendix B.\(^\text{18}\)

\(^\text{18}\)Check the constants $c_n$ and $\omega_n$. 
4.5. **The heat operator.** We now turn our attention to the heat operator in \( \mathbb{R}^{n+1} \) \( \partial_t - \Delta \). We begin by considering the Cauchy problem in the upper-half space \( \mathbb{R}^n \times (0, \infty) \), where we denote the variables \((x, t) \in \mathbb{R}^n \times \mathbb{R}\), called the Cauchy problem for the heat equation,

\[
\begin{aligned}
\partial_t u(x, t) - \Delta u(x, t) &= 0 & \text{for } (x, t) &\in \mathbb{R}^{n+1}, \ t > 0 \\
u(x, 0) &= f(x) & \text{for } x &\in \mathbb{R}^n.
\end{aligned}
\] (4.12)

We assume that \( f \in \mathcal{S}' \) and that the solution \( u \) is a tempered distribution in \( x \), for any \( t \) fixed. Then, by taking the Fourier transform in the \( x \)-variable, we obtain the ordinary differential equation Cauchy problem (for any \( \xi \in \mathbb{R}^n \) fixed)

\[
\begin{aligned}
\partial_t \hat{u}(\xi, t) + 4\pi^2 |\xi|^2 \hat{u}(\xi, t) &= 0 & \text{for } t > 0 \\
\hat{u}(\xi, 0) &= \hat{f}(\xi).
\end{aligned}
\]

We can easily solve this Cauchy problem and find that the solution is

\[\hat{u}(\xi, t) = \hat{f}(\xi)e^{-4\pi^2 |\xi|^2 t} \text{ for } t > 0.\]

Therefore, by undoing the Fourier transform we obtain

\[u(x, t) = f \ast K_{\sqrt{t}}(x),\]

where

\[K_{\sqrt{t}}(x) = K(x, t) = \mathcal{F}^{-1}(e^{-4\pi^2 |\xi|^2 t}) = \frac{1}{(4\pi t)^{n/2}}e^{-|x|^2/4t},\]

by Prop. 1.13. Notice that

\[K_{\sqrt{t}}(x) = t^{-n/2}K_1(x/t^{1/2}),\]

so that \( \int K_{\sqrt{t}} dx = \int K_1 dx = 1 \), for all \( t > 0.\)

We now compute a fundamental solution for the heat operator. We define \( K \) on all of \( \mathbb{R}^{n+1} \) by setting

\[K(x, t) = \begin{cases} (4\pi t)^{-n/2}e^{-|x|^2/4t} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0. \end{cases}\] (4.13)

Since \( \int_{\mathbb{R}^n} K(x, t) dx = 1 \) for \( t > 0 \), \( K(x, t) \) is integrable on every region of the form \( \mathbb{R}^n \times (-\infty, T] \).

**Proposition 4.14.** The function defined by (4.13) is a fundamental solution for the heat operator.

**Proof.** Let \( \varepsilon > 0 \) be given. Define \( K_\varepsilon(x, t) = K(x, t) \) for \( t > \varepsilon \) and \( K_\varepsilon(x, t) = 0 \) for \( t \leq \varepsilon \). Then \( K_\varepsilon \to K \) in the sense of distributions in \( \mathbb{R}^{n+1} \) as \( \varepsilon \to 0. \) Then, we wish to show that

\[\langle (\partial_t K_\varepsilon - \Delta_x K_\varepsilon), \varphi \rangle = \langle K_\varepsilon, (-\partial_t - \Delta_x)(\varphi) \rangle \to \varphi(0, 0)\]

as \( \varepsilon \to 0, \) for every \( \varphi \in \mathcal{D}(\mathbb{R}^{n+1}). \)

\[\text{The notation } K_{\sqrt{t}} \text{ is not standard, as this kernel is classically defined as } K_t. \text{ We have elected to use the former notation for consistency.}\]
Now, by integrating by parts we find that
\[
\langle K(\varepsilon), (-\partial_t - \Delta_x)(\varphi) \rangle = \int_\varepsilon^{+\infty} \int_{\mathbb{R}^n} K(x,t)(-\partial_t - \Delta_x)(\varphi)(x,t) \, dxdt
\]
\[
= \int_\varepsilon^{+\infty} \int_{\mathbb{R}^n} -\Delta_x K(x,t)\varphi(x,t) \, dxdt + \int_{\mathbb{R}^n} \int_\varepsilon^{+\infty} \partial_t K(x,t)\varphi(x,t) \, dx \, dt
\]
\[
+ \int_{\mathbb{R}^n} K(x,\varepsilon)\varphi(x,\varepsilon) \, dx
\]
\[
= \int_\varepsilon^{+\infty} \int_{\mathbb{R}^n} (\partial_t - \Delta_x)K(x,t)\varphi(x,t) \, dxdt + \int_{\mathbb{R}^n} K(x,\varepsilon)\varphi(x,\varepsilon) \, dx
\]
\[
= 0 + \int_{\mathbb{R}^n} K(-x,\varepsilon)\varphi(x,\varepsilon) \, dx
\]
since \( K \) is even in \( x \) and it is annihilated by the heat operator on \( \mathbb{R}^n \times (\varepsilon, +\infty) \). But, the right hand side above is (the convolution being in \( \mathbb{R}^n \))
\[
\int_{\mathbb{R}^n} K(-x,\varepsilon)\varphi(x,\varepsilon) \, dx = K(\varepsilon) * \varphi(\cdot, \varepsilon)(0)
\]
\[
= K(\varepsilon) * \varphi(0,0) + \left[ K(\varepsilon) * \varphi(\cdot, \varepsilon) - K(\varepsilon) * \varphi(\cdot, 0) \right](0).
\]
The former term tends to \( \varphi(0,0) \) as \( \varepsilon \to 0 \), while the latter one is bounded by
\[
\sup_{x \in \mathbb{R}^n} |\varphi(x,\varepsilon) - \varphi(x,0)|\|K(\varepsilon)\|_1 = \sup_{x \in \mathbb{R}^n} |\varphi(x,\varepsilon) - \varphi(x,0)|
\]
which tends to 0 as \( \varepsilon \to 0 \). \( \square \)

We conclude this part by solving the problem (4.12).

**Theorem 4.15.** Let \( 1 \leq p < +\infty \) and \( g \in L^p(\mathbb{R}^n) \) be given. Then the boundary value problem (4.12) admits a solution \( f(x,t) \) such that \( f(\cdot, t) \to g \) in \( L^p(\mathbb{R}^n) \) as \( t \to 0^+ \).

**Proof.** Setting \( f = g * K_{\sqrt{t}} \) we see that \( f \) solves the equation in \( \mathbb{R}^{n+1}_+ \) and \( f(\cdot, t) = g * K_{\sqrt{t}} \to g \) in \( L^p(\mathbb{R}^n) \) as \( t \to 0^+ \). \( \square \)

### 4.6. The Riesz potentials.

We now discuss some operators \( I_\alpha \), called the *Riesz potentials*, that arise in a natural way in connection with the Laplacian. The operators \( I_\alpha \) are integral operators whose kernel is homogenous of degree \( -(n - \alpha) \), then locally integrable when \( 0 < \alpha < n \), case to which we restrict ourselves.

**Definition 4.16.** Let \( 0 < \alpha < n \) and define the Riesz potential of order \( \alpha \) the integral operator
\[
I_\alpha(f)(x) = \frac{1}{(2\pi)^\alpha} \frac{1}{c_{n,n-\alpha}} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) \, dy,
\]
where \( c_{n,n-\alpha} \) is the constant defined in (4.5). (It will be sometimes convenient to set \( a = n - \alpha \). Notice that \( 0 < a < n \) exactly when \( 0 < \alpha < n \).)

We remark that the Laplacian satisfies the identity
\[
(-\Delta f)(\xi) = (2\pi|\xi|)^2 \hat{f}(\xi),
\]
for $f \in S$. By Lemma 4.7, it follows that
\[
(I_{\alpha}f)^{(\xi)} = \frac{1}{(2\pi)^\alpha} \frac{1}{c_{n,n-\alpha}} (|x|^{-(n-\alpha)} * f)^{(\xi)} = (2\pi|\xi|)^{-\alpha} \hat{f}(\xi).
\]
Hence, it is natural to define the fractional powers of $-\Delta$ by setting
\[
\left( (-\Delta)^{-\alpha/2} f \right)^{(\xi)} = (2\pi|\xi|)^{-\alpha} \hat{f}(\xi),
\]
that is, by setting
\[
(-\Delta)^{-\alpha/2} = I_{\alpha}.
\]
We now study the $(L^p, L^q)$-boundedness of the operators $I_{\alpha}$. We begin with a simple result that will give us a necessary condition on $p, q$ and $\alpha$. We set $D_{\lambda}f(x) = f(\lambda x) = f(\lambda x)$, for $\lambda > 0$.

**Lemma 4.17.** For $\lambda > 0$ we have the following identities:

(i) $D_{\lambda - 1}I_{\alpha}D_{\lambda} = \lambda^{-\alpha}I_{\alpha}$;

(ii) $\|D_{\lambda}f\|_{L^p} = \lambda^{-n/p}\|f\|_{L^p}$;

(iii) $\|D_{\lambda - 1}I_{\alpha}(f)\|_{L^q} = \lambda^{n/q}\|I_{\alpha}(f)\|_{L^q}$.

**Proof.** These are elementary. For a given function $f$, sufficiently regular, we have
\[
D_{\lambda - 1}I_{\alpha}D_{\lambda}(f)(x) = \frac{(2\pi)^\alpha}{c_{n,n-\alpha}} \int_{\mathbb{R}^n} \frac{1}{|\lambda^{-1}x - y|^{n-\alpha}} f(\lambda y) dy
\]
\[
= \frac{(2\pi)^\alpha}{c_{n,n-\alpha}} \lambda^n \int_{\mathbb{R}^n} \frac{1}{|\lambda^{-1}(x - z)|^{n-\alpha}} f(z) dz
\]
\[
= \lambda^{-\alpha}I_{\alpha}(f)(x),
\]
as we claimed.

(ii) and (iii) are simply a change of variables. \(\square\)

**Lemma 4.18.** Let $0 < \alpha < n$, $1 < p, q < \infty$ be given. Then, if $I_{\alpha} : L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ is bounded, then necessarily

\[
\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.
\]

**Proof.** If $I_{\alpha}$ is bounded, there exists a constant $C > 0$ such that
\[
\|I_{\alpha}(f)\|_{L^q} \leq C\|f\|_{L^p}
\]
for all $f \in L^p$. Then, for all $\lambda > 0$ it must hold that
\[
\|I_{\alpha}(D_{\lambda}f)\|_{L^q} \leq C\|D_{\lambda}f\|_{L^p}. \tag{4.15}
\]
Now, by Lemma 4.17,
\[
\|I_{\alpha}(D_{\lambda}f)\|_{L^q} = \|D_{\lambda}D_{\lambda - 1}I_{\alpha}(D_{\lambda}f)\|_{L^q} = \lambda^{-n/q}\lambda^{-\alpha}\|I_{\alpha}(f)\|_{L^q},
\]
Then, (4.15) is equivalent to
\[
\|I_{\alpha}(f)\|_{L^q} \leq C\lambda^{\frac{n}{q} + \alpha - \frac{n}{p}}\|f\|_{L^p}.
\]
Since this must hold for all \( \lambda > 0 \), if the exponent of \( \lambda \) in the inequality above is \( \neq 0 \), by letting \( \lambda \to 0 \), or \( \lambda \to +\infty \) we get a contradiction. The conclusion now follows. \( \square \)

We now prove the positive result, valid in the case \( 1 < p, q < \infty \).

**Theorem 4.19.** Let \( 0 < \alpha < n \), \( 1 \leq p, q < \infty \) satisfy
\[
\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},
\tag{4.16}
\]
and consider the fractional integral \( I_\alpha \). For \( f \in L^p \), the integral defining \( I_\alpha(f) \) converges absolutely. Moreover,

(i) if \( 1 < p, q < \infty \)
\[
I_\alpha : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)
\]
is strong-type \((p, q)\);

(ii) if \( p = 1 \), then \( I_\alpha \) is weak-type \((1, q)\).

**Proof.** Clearly, it suffices to consider the convolution operator \( T : f \mapsto K * f \), where \( K(x) = |x|^{-n+\alpha} \) and prove (i) and (ii) for this operator.

By the general form of the Marcinkiewicz interpolation theorem, if we show that \( T \) is weak-type \((p_1, q_1)\) and also weak-type \((1, q_0)\), both pairs satisfying the condition \((4.16)\), then it follows that it is strong-type \((p, q)\), where
\[
\frac{1}{p} = \frac{\theta}{p_1} + 1 - \theta, \quad \frac{1}{q} = \frac{\theta}{q_1} + 1 - \theta.
\]
Notice that
\[
\frac{1}{q} = \theta\left(\frac{1}{p_1} - \frac{\alpha}{n}\right) + (1 - \theta)\left(1 - \frac{\alpha}{n}\right)
\]
\[
= \frac{\theta}{p_1} + 1 - \theta - \frac{\alpha}{n} = \frac{1}{p} - \frac{\alpha}{n}.
\]

We write \( K = K_1 + K_\infty \), where
\[
K_1(x) = K(x)\chi_{\{|x| \leq R\}}, \quad \text{and} \quad K_\infty(x) = K(x)\chi_{\{|x| > R\}},
\]
and where \( R > 0 \) is a positive constant to be selected later.

We begin by observing that \( K_1 \in L^1(\mathbb{R}^n) \), since
\[
\int_{\mathbb{R}^n} |K_1(x)| \, dx = \int_{|x| \leq R} |x|^{-n+\alpha} \, dx = \omega_n \int_0^R r^{\alpha-1} \, dr
\]
\[
= \frac{\omega_n}{\alpha} R^\alpha.
\tag{4.17}
\]

On the other hand, \( K_\infty \in L^{p'} \), where \( p' \) is the exponent conjugate to \( p \). For,
\[
\int_{\mathbb{R}^n} |K_\infty(x)|^{p'} \, dx = \int_{|x| > R} |x|^{-(n+\alpha)p'} \, dx
\]
\[
= \omega_n \int_R^{+\infty} r^{n(1-p') + \alpha p' - 1} \, dr
\]
\[
= \frac{\omega_n}{n(p' - 1) - \alpha p'} R^{n(1-p') + \alpha p'}.
\tag{4.18}
and the integral is convergent since \( n(1 - p') + \alpha p' < 0 \), that is, \(-\frac{n}{p} + \alpha < 0\), which is equivalent to \( q < \infty\).

Now, let \( f \in L^p, 1 \leq p < \infty \). Then, the integral defining \( K \ast f \) converges absolutely since \( K_1 \in L^1 \) and \( K_\infty \in L^{p'} \). (Make sure you agree with this assertion.) This proves the first part of the statement.

In order to prove (i) and (ii), it suffices to show that, for \( 1 \leq p < q < \infty \) such that \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \), \( T \) is weak-type \((p,q)\), that is, 

\[
\left| \{ x : |K \ast f(x)| > \lambda \} \right| \leq \left( C \frac{\|f\|_{L^p}}{\lambda} \right)^q.
\]

Notice that we may assume that \( \|f\|_{L^p} = 1 \) (as it is easy to check, and we invite you to do so).

Then we have,

\[
\left| \{ x : |K \ast f(x)| > \lambda \} \right| \leq \left| \{ x : |K_1 \ast f(x)| > \lambda/2 \} \right| + \left| \{ x : |K_\infty \ast f(x)| > \lambda/2 \} \right|.
\]

Now, using (4.17) and the assumption \( \|f\|_{L^p} = 1 \), we have

\[
\left| \{ x : |K_1 \ast f(x)| > \lambda/2 \} \right| = \int \frac{|K_1 \ast f(x)|}{\lambda/2} \, dx \leq \int \left( \frac{|K_1 \ast f(x)|}{\lambda/2} \right)^p \, dx
= \frac{2^p \|K_1 \ast f\|_{L^p}^p}{\lambda^p} \leq \frac{2^p \|K_1\|_{L^1}^p \|f\|_{L^p}^p}{\lambda^p}
\leq C \left( \frac{R^\alpha}{\lambda} \right)^p.
\]

Next, using (4.18) we see that

\[
\|K_\infty \ast f\|_{L^\infty} \leq \|K_\infty\|_{L^{p'}} \|f\|_{L^p} = \|K_\infty\|_{L^{p'}} \leq CR^{\alpha-n/p}.
\]

We now choose \( R \) such that

\[
CR^{\alpha-n/p} = \frac{\lambda}{2}, \quad \text{that is,} \quad R = C' \lambda^{1/(\alpha-n/p)} = C' \lambda^{-q/n}.
\]

With choice of \( R \), by (4.21) it follows that

\[
\left| \{ x : |K_\infty \ast f(x)| > \lambda/2 \} \right| = 0.
\]

Therefore, by (4.21) (recalling that \( \|f\|_{L^p} = 1 \)),

\[
\left| \{ x : |K \ast f(x)| > \lambda \} \right| \leq C \left( \frac{R^\alpha}{\lambda} \right)^p \leq C \left( \lambda^{-q(\alpha/n + 1/q)} \right)^p
= C \lambda^{-q} = C \left( \frac{\|f\|_{L^p}}{\lambda} \right)^q.
\]

This concludes the proof of the theorem. \( \square \)
Construction of Calderón–Zygmund kernels. In order to prove the Mihlin-Hörmander multiplier theorem, Theorem 5.22 that follows, we describe a method, that is quite standard, to construct Calderón–Zygmund convolution kernels, as defined in Def. 4.2.

Definition 4.20. An $L^p$-function $f$ is said to satisfy the $L^p$-Lipschitz condition of order $\alpha$, $\alpha \in (0, 1)$, if there exists a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} |f(x-h) - f(x)|^p \, dx\right)^{1/p} \leq C|h|^\alpha$$

for every $h \in \mathbb{R}^n$. In this case we set

$$\|f\|_{\Lambda^p_\alpha} := \|f\|_{L^p(\mathbb{R}^n)} + \sup_{h \neq 0} |h|^{-\alpha} \left(\int_{\mathbb{R}^n} |f(x-h) - f(x)|^p \, dx\right)^{1/p}.$$

Remark 4.21. The following facts are easy to prove.\(^{20}\)

1. An $L^p$-function $f$ is in $\Lambda^p_\alpha$ if and only if for every $\lambda > 0$ we have

$$\sup_{0 < |h| < \lambda} |h|^{-\alpha} \left(\int_{\mathbb{R}^n} |f(x-h) - f(x)|^p \, dx\right)^{1/p} < \infty;$$

2. Let $\varphi \in C_0^\infty$ be identically 1 in a nghb of the origin and set $f = |x|^{n-n/p}\varphi$, then $f \in \Lambda^p_\alpha$.

Lemma 4.22. Let $f \in \Lambda^1_\alpha$. Then, there exists $C > 0$ such that

$$|\hat{f}(\xi)| \leq C(1 + |\xi|)^{-\alpha}\|f\|_{\Lambda^1_\alpha}.$$

Proof. Notice that

$$-\hat{f}(\xi) = e^{-\pi i} \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \xi} \, dx = \int_{\mathbb{R}^n} f(x) e^{-2\pi i (x + \xi / (2|\xi|^2)) \xi} \, dx$$

$$= \int_{\mathbb{R}^n} f(x - \xi / (2|\xi|^2)) e^{-2\pi i x \xi} \, dx.$$

Therefore,

$$|\hat{f}(\xi)| = \frac{1}{2} \left| \int_{\mathbb{R}^n} \left[ f(x) - f(x - \xi / (2|\xi|^2)) \right] e^{-2\pi i x \xi} \, dx \right|$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^n} \left| f(x) - f(x - \xi / (2|\xi|^2)) \right| \, dx$$

$$\leq C|\xi|^{-\alpha}\|f\|_{\Lambda^1_\alpha}.$$

On the other hand

$$|\hat{f}(\xi)| = \|f\|_{L^1} \leq \|f\|_{\Lambda^1_\alpha},$$

so that

$$(1 + |\xi|)^\alpha|\hat{f}(\xi)| \leq C(1 + |\xi|^\alpha)|\hat{f}(\xi)| \leq C\|f\|_{\Lambda^1_\alpha}.$$

This proves the lemma. \(\square\)

We will need a smooth decomposition of the function identically 1.

Lemma 4.23. There exists a function $\varphi \in C_0^\infty$ such that...

\(^{20}\)Exercise.
Moreover, there exists another function \( \varphi_0 \in C_0^\infty \) such that

(ii) \( \varphi_0(\xi) + \sum_{j=1}^{+\infty} \varphi(2^{-j}\xi) = 1 \quad \text{for all} \quad \xi \).

\begin{proof}
Let \( \varphi_0 \in C_0^\infty(\mathbb{R}^n) \) such that \( \varphi \geq 0 \), \( \varphi_0(\xi) = 1 \) if \( |\xi| \leq 1 \) and \( \varphi_0(\xi) = 0 \) if \( |\xi| \geq 2 \). (Such function exists by the \( C^\infty \)-Urysohn’s lemma, Lemma 1.7.) We set

\[ \varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi). \]

Notice that \( \varphi(\xi) = 0 \) if \( |\xi| \leq 1/2 \) or \( |\xi| \geq 2 \), that is,

\[ \text{supp} \varphi \subseteq \{ \xi : 1/2 \leq |\xi| \leq 2 \}. \]

By induction, it is elementary to prove that, for \( \ell \geq 1 \) and all \( \xi \),

\[ \varphi_0(\xi) + \sum_{j=1}^{\ell} \varphi(2^{-j}\xi) = \varphi_0(2^{-\ell}\xi). \quad (4.22) \]

Letting \( \ell \to +\infty \) we obtain (ii).

Next, again by induction on \( \ell' \), with \( \ell' \leq 0 \), it is easy to show that

\[ \sum_{j=\ell}^{\ell'} \varphi(2^{-j}\xi) = \varphi_0(2^{-\ell}\xi) - \varphi_0(2^{-\ell'+1}\xi). \quad (4.23) \]

Letting \( \ell \to +\infty \) and \( \ell' \to -\infty \) we obtain (i). (Notice that \( \varphi_0(2^{-\ell}\xi) - \varphi_0(2^{-\ell'+1}\xi) \to 1 \) only when \( \xi \neq 0 \).) \qed

\begin{corollary}
Let \( \varphi, \varphi_0 \) be as in Lemma 4.23, and set

\[ \psi(0) = \mathcal{F}\varphi_0, \quad \psi(j) = \mathcal{F}(\varphi(2^{-j} \cdot)) = (\mathcal{F}\varphi)_{2^{-j}}, \quad \text{for} \quad j = 1, 2, \ldots \quad (4.24) \]

where \( (\mathcal{F}\varphi)_{2^{-j}}(\xi) = 2^{jn}(\mathcal{F}\varphi)(2^j \xi) \) denotes the dilation defined in (1.2). Then the following properties hold:

(i) \( \sum_{j=0}^{N} \psi(j) \to \delta_0 \) in \( S' \) as \( N \to +\infty \);

(ii) \( \|\psi(j)\|_{L^1} \leq C, \|\psi(j)\|_{L^\infty} \leq C2^{nj}, \) for all \( j \);

(iii) for all \( j \), \( \int \psi(j) = 0 \).

\end{corollary}

\begin{proof}
It is immediate to see that the sum in Lemma 4.23 converges in \( S' \). Then

\[ \mathcal{F}\left( \varphi_0 + \sum_{j=1}^{N} \varphi(2^{-j} \cdot) \right) = \sum_{j=0}^{N} \psi(j) \to \delta_0 \]

as \( N \to +\infty \) in \( S' \).

Next,

\[ \|\psi(j)\|_{L^1} = \| (\mathcal{F}\varphi)_{2^{-j}} \|_{L^1} = \|\mathcal{F}\varphi\|_{L^1} \leq \|\mathcal{F}\varphi_0\|_{L^1} + \|\mathcal{F}(\varphi_0(2\cdot))\|_{L^1} \leq C. \]

Moreover,

\[ \|\psi(j)\|_{L^\infty} \leq \|\mathcal{F}\psi(j)\|_{L^1} = \|\varphi(2^{-j} \cdot)\|_{L^1} = 2^{jn}\|\varphi\|_{L^1} = C2^{jn}. \]

\end{proof}
Finally,
\[ \int \psi(j)(x) \, dx = F^{-1}(\psi(j))(0) = \varphi^{2^{-j}}(0) = \varphi(0) = 0. \]

This concludes the proof. \( \square \)

**Lemma 4.25.** Let \( \alpha \in (0, 1) \) and let \( p < (n + \alpha)/n \). Then, if \( f \in \Lambda^1_{\alpha} \) we have \( f \in L^p \) and \( \|f\|_{L^p} \leq C\|f\|_{\Lambda^1_{\alpha}} \).

**Proof.** Let \( f \in \Lambda^1_{\alpha} \), and let \( \psi(j) \) be as in the previous lemma.

Using (iii), observe that for \( j \geq 1 \),
\[ \|f \ast \psi(j)\|_{L^1} = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (f(x) - f(y)) \psi(j)(y) \, dy \right| \, dx \]
\[ \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x) - f(y)| \, dx \, |\psi(j)(y)| \, dy \]
\[ \leq \|f\|_{\Lambda^1_{\alpha}} \int_{\mathbb{R}^n} |y|^\alpha |\psi(j)(y)| \, dy \]
\[ = \|f\|_{\Lambda^1_{\alpha}} \int_{\mathbb{R}^n} |y|^\alpha |2^n \mathcal{F} \varphi(2^j y)| \, dy \]
\[ \leq 2^{-j\alpha} \|f\|_{\Lambda^1_{\alpha}} \int_{\mathbb{R}^n} |z|^\alpha |\mathcal{F} \varphi(z)| \, dz \]
\[ \leq C 2^{-j\alpha} \|f\|_{\Lambda^1_{\alpha}}. \]

Moreover, by (i)
\[ \|f \ast \psi(j)\|_{L^{\infty}} \leq \|f\|_{L^1} \|\psi(j)\|_{L^{\infty}} \leq 2^n \|f\|_{L^1} \leq C 2^n \|f\|_{\Lambda^1_{\alpha}}. \]

Hence,
\[ \|f \ast \psi(j)\|_{L^p} \leq \|f \ast \psi(j)\|_{L^{\infty}}^{p-1} \int_{\mathbb{R}^n} |f \ast \psi(j)(x)|^p \, dx \leq C 2^n (p-1) 2^{-j\alpha} \|f\|_{\Lambda^1_{\alpha}}^p. \]

Therefore,
\[ \sum_{j=0}^{+\infty} \|f \ast \psi(j)\|_{L^p} = \lim_{N \to +\infty} \sum_{j=0}^{N} \|f \ast \psi(j)\|_{L^p} \]
\[ \leq C \|f\|_{\Lambda^1_{\alpha}} \lim_{N \to +\infty} \sum_{j=0}^{N} 2^{j(n(p-1)-\alpha)/p} \]
\[ \leq C \|f\|_{\Lambda^1_{\alpha}}, \]

if \( n(p-1) - \alpha < 0 \), that is, \( p < (n + \alpha)/n \).

Consequently, \( \sum_{j=0}^{+\infty} f \ast \psi(j) \) converges in the \( L^p \)-norm, and since it converges in \( S' \) to \( f \) (at least when \( f \in S \)), we obtain that
\[ \|f\|_{L^p} \leq \sum_{j=0}^{+\infty} \|f \ast \psi(j)\|_{L^p} \]
\[ \leq C \|f\|_{\Lambda^1_{\alpha}}, \]
when \( p < (n + \alpha)/n \). □

We now obtain the reward for our work in this setting with the following result.

**Theorem 4.26.** Let \( k_{\langle j \rangle} \in L^1(\mathbb{R}^n) \), \( j \in \mathbb{Z} \), be functions such that there exist constants \( \varepsilon > 0 \), \( \alpha \in (0, 1) \) and \( C > 0 \) such that for all \( j \in \mathbb{Z} \),

1. \( \int_{\mathbb{R}^n} (1 + |x|^2) |k_{\langle j \rangle}(x)| \, dx \leq C; \)
2. \( \int_{\mathbb{R}^n} k_{\langle j \rangle}(x) \, dx = 0; \)
3. \( \|k_{\langle j \rangle}\|_{A^\varepsilon_1} \leq C. \)

Then the series \( \sum_{j \in \mathbb{Z}} (k_{\langle j \rangle})_{2^j} \) converges in \( S' \) to a Calderón–Zygmund kernel.

**Proof.** We wish to show that \( \sum_{j \in \mathbb{Z}} (k_{\langle j \rangle})_{2^j} \) converges in \( S' \) to a distribution \( K \) that coincides on \( \mathbb{R}^n \setminus \{0\} \) with a locally integrable function and that \( K \) satisfies the conditions (4.2) and (4.1).

**Step 1.** We begin by showing that the series of the Fourier transforms of the \( (k_{\langle j \rangle})_{2^j} \) converges absolutely, that is,

\[
\sum_{j \in \mathbb{Z}} |\mathcal{F}((k_{\langle j \rangle})_{2^j})(\xi)| \leq C. \tag{4.25}
\]

We split the sum in two, the first one being for \( j \) such that \( |2^j \xi| \geq 1 \). Using Lemma 4.22 and (3) we have\(^{21}\)

\[
\sum_{j: |2^j \xi| \geq 1} |\mathcal{F}((k_{\langle j \rangle})_{2^j})(\xi)| = \sum_{j: |2^j \xi| \geq 1} |\mathcal{F}(k_{\langle j \rangle})(2^j \xi)|
\leq C \sum_{j: |2^j \xi| \geq 1} (1 + |2^j \xi|)^{-\alpha} \|k_{\langle j \rangle}\|_{A^\varepsilon_1}
\leq C \sum_{j: |2^j \xi| \geq 1} (1 + |2^j \xi|)^{-\alpha}
\leq C \sum_{j: |2^j \xi| \geq 1} (2^j |\xi|)^{-\alpha}
= C |\xi|^{-\alpha} \sum_{j: |2^j \xi| \geq 1} 2^{-j\alpha}
\leq C. \tag{4.26}
\]

\(^{21}\) Here we also use the elementary estimates \( \sum_{j \geq j_0} q^j \simeq q^{j_0} \) if \( 0 < q < 1 \), and \( \sum_{j \leq j_0} q^j \simeq q^{j_0} \) if \( q > 1 \).
When $|2^j \xi| < 1$ we use (1), (2) and the estimate $|e^{-2\pi i t} - 1| \leq C|t|^{\varv}$ for $|t| < 1$ (and footnote 21 again). We have

$$
\sum_{j: |2^j \xi| < 1} |\mathcal{F}( (k(j))_{2j} )(\xi) | = \sum_{j: |2^j \xi| < 1} \left| \int_{\mathbb{R}^n} k(j)(x) (e^{-2\pi i 2^j \xi x} - 1) \, dx \right|
\leq C \sum_{j: |2^j \xi| < 1} \int_{\mathbb{R}^n} |k(j)(x)| |2^j \xi|^\varv \, dx
\leq C |\xi|^{\varv} \sum_{j: |2^j \xi| < 1} 2^{\varv j}
\leq C.
$$

Estimates (4.26) and (4.27) now give (4.25). We set

$$
u(\xi) = \sum_{j \in \mathbb{Z}} \mathcal{F}( (k(j))_{2j} )(\xi) \in L^\infty(\mathbb{R}^n).$$

Since, for each $f \in \mathcal{S}$, by the dominated convergence theorem,

$$
\int_{\mathbb{R}^n} \nu(\xi) f(\xi) \, d\xi = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \mathcal{F}( (k(j))_{2j} )(\xi) f(\xi) \, d\xi
$$

so that $\sum_{j \in \mathbb{Z}} \mathcal{F}( (k(j))_{2j} )$ converges in $\mathcal{S}'$. By continuity of the (inverse) Fourier transform in $\mathcal{S}'$,

$$
\sum_{j \in \mathbb{Z}} (k(j))_{2j} = K
$$

for some $K \in \mathcal{S}'$. This completes Step 1.

**Step 2.** We wish to prove that the distribution $K$ defined in (4.28) is a Calderón–Zygmund kernel.

We first show that $K$ coincides with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$, by showing that $\sum_{j \in \mathbb{Z}} (k(j))_{2j}$ converges in $L^1(E)$, for every compact $E \subseteq \mathbb{R}^n$.

It suffices to consider compact sets $E$ of the form $\{x : 2^m \leq |x| \leq 2^{m+1}\}$, for some $m \in \mathbb{Z}$. Notice that, with an obvious change of variables,

$$
\int_{\{x : 2^m \leq |x| \leq 2^{m+1}\}} |(k(j))_{2j}(x)| \, dx = \int_{\{x : 2^{m-j} \leq |x| \leq 2^{m-j+1}\}} |k(j)(x)| \, dx.
$$

Assume first that $j \leq m$. Then using (1) we have,

$$
\int_{\{x : 2^{m-j} \leq |x| \leq 2^{m-j+1}\}} |k(j)(x)| \, dx \leq 2^{(j-m)\varv} \int_{\{x : 2^{m-j} \leq |x| \leq 2^{m-j+1}\}} |x|^\varv |k(j)(x)| \, dx
\leq C 2^{(j-m)\varv}.
$$

Make sure you agree with this reduction.
On the other hand, if \( j > m \) we use (3) and Lemma 4.25 we have
\[
\int_{\{x: 2^{m-j} \leq |x| \leq 2^{m-j+1}\}} |k(j)(x)| \, dx \leq \left( \int_{\{|x| \leq 2^{m-j+1}\}} |k(j)(x)|^p \, dx \right)^{1/p} \left| \{ |x| \leq 2^{m-j+1} \} \right|^{1/p'} \\
\leq \|k(j)\|_{L^p} 2^{n(m-j+1)/p'} \\
\leq C 2^{n(m-j+1)/p'},
\]
for a suitable \( p \) (and \( p' \)).

Therefore,
\[
\sum_{j \in \mathbb{Z}} \|k(j)\|_{L^1(E)} \leq \sum_{j \leq m} C 2^{(j-m)\varepsilon} + \sum_{j > m} C 2^{n(m-j+1)/p'} \leq C,
\]
since both series converge.

We now show that \( K \) satisfies the two conditions (4.1) and (4.2). But, by construction and (4.25)
\[
|\hat{K}(\xi)| \leq \sum_{j \in \mathbb{Z}} |\mathcal{F}(\{(k(j)_{2j})(\xi)\})| \leq C,
\]
so that (4.1) holds.

Finally, to prove that \( K \) satisfies Hörmander’s condition, let \( h \in \mathbb{R}^n \setminus \{0\} \) and assume that \( 2^m < 2|h| \leq 2^{m+1} \). Then we have
\[
\int_{|x| > 2|h|} |K(x-h) - K(x)| \, dx \leq \sum_{j \in \mathbb{Z}} \int_{|x| > 2^m} \left| \{(k(j)_{2j})(x-h) - (k(j)_{2j})(x)\} \right| \, dx \\
= \sum_{j \in \mathbb{Z}} \int_{|x| > 2^m} \left| k(j)(x - 2^{-j}h) - k(j)(x) \right| \, dx \\
= \sum_{j < m} \int_{|x| > 2^{m-j}} \left| k(j)(x - 2^{-j}h) - k(j)(x) \right| \, dx \\
+ \sum_{j \geq m} \int_{|x| > 2^{m-j}} \left| k(j)(x - 2^{-j}h) - k(j)(x) \right| \, dx \\
\leq 2 \sum_{j < m} \int_{|x| > 2^{m-j-1}} \left| k(j)(x) \right| \, dx + \sum_{j \geq m} |2^{-j}h|^{\alpha} \|k(j)\|_{A^{\alpha}_h} \\
\leq 2 \sum_{j < m} 2^{(j-m)\varepsilon} \int_{|x| > 2^{m-j-1}} |x|^\varepsilon \left| k(j)(x) \right| \, dx + C \sum_{j \geq m} |2^{-j}h|^{\alpha} \\
\leq C \sum_{j < m} 2^{(j-m)\varepsilon} + C \sum_{j \geq m} |2^{-j}h|^{\alpha} \\
\leq C,
\]
where we have used conditions (1) and (3) in the hypotheses. This proves Step 2 and hence the theorem. \( \square \)
4.8. Vector-valued singular integrals. We conclude this part by extending Thm. 4.1 to the case of vector-valued functions. This extension will be used in Section 6, when developing the Littlewood--Paley theory.

Although this part of the theory works in the case of functions taking values in a separable Banach space, we restrict our attention to the case of functions with values in a separable Hilbert space $\mathcal{H}$.

We say that a function $f$ from $\mathbb{R}^n$ to $\mathcal{H}$ is measurable if for every $v \in \mathcal{H}$, the mapping
\[ \mathbb{R}^n \ni x \mapsto \langle f(x), v \rangle_{\mathcal{H}} \in \mathbb{C} \]
is measurable.

For $1 \leq p \leq \infty$ we define the space $L^p(\mathbb{R}^n, \mathcal{H})$ to be the space of measurable functions $f : \mathbb{R}^n \to \mathcal{H}$ such that
\[ \left( \int_{\mathbb{R}^n} \|f(x)\|^p_{\mathcal{H}} \, dx \right)^{1/p} < \infty. \]
Then, $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n, \mathbb{C})$, but we will still denote it by $L^p$ for simplicity.

A typical example of $L^p(\mathbb{R}^n, \mathcal{H})$ is a function of the form $f(x) = f_s(x)v$, where $f_s$ is a scalar-valued function in $L^p(\mathbb{R}^n)$ and $v$ is a fixed element of $\mathcal{H}$. The subset
\[ L^p \otimes \mathcal{H} = \{ f : f = \sum_{j=1}^N f_j v_j \text{ where } f_j \in L^p, \ v_j \in \mathcal{H} \}, \]
of finite linear combination of functions of the form above, is dense in $L^p(\mathbb{R}^n, \mathcal{H})$. (The proof of this fact is simple, for this and other facts about this topic, see [Ru].)

Given $f = \sum_{j=1}^N f_j v_j \in L^1 \otimes \mathcal{H}$, we define its integral to be the element of $\mathcal{H}$ given by
\[ \int_{\mathbb{R}^n} f(x) \, dx = \sum_{j=1}^N \left( \int_{\mathbb{R}^n} f_j(x) \, dx \right) v_j. \]
This map $f \mapsto \int f(x) \, dx$ extends to all of $L^1(\mathbb{R}^n, \mathcal{H})$ by density. Then, for $f \in L^1(\mathbb{R}^n, \mathcal{H})$, $\int f(x) \, dx$ is the unique element of $\mathcal{H}$ such that
\[ \left\langle \int f(x) \, dx, v \right\rangle_{\mathcal{H}} = \int_{\mathbb{R}^n} \langle f(x), v \rangle_{\mathcal{H}} \, dx \tag{4.29} \]
for all $v \in \mathcal{H}$.

Similarly reasoning works also in the case of $f \in L^p(\mathbb{R}^n, \mathcal{H})$ and $g \in L^{p'}(\mathbb{R}^n, \mathcal{H})$: notice that by applying the standard Hölder's inequality it follows that
\[ \int_{\mathbb{R}^n} |\langle f(x), g(x) \rangle_{\mathcal{H}}| \, dx \leq \left( \int_{\mathbb{R}^n} \|g(x)\|_{\mathcal{H}}^{p'} \, dx \right)^{1/p'} \left( \int_{\mathbb{R}^n} \|f(x)\|_{\mathcal{H}}^p \, dx \right)^{1/p} = \|g\|_{L^{p'}(\mathbb{R}^n, \mathcal{H})} \|f\|_{L^p(\mathbb{R}^n, \mathcal{H})}. \tag{4.30} \]
Therefore $\int_{\mathbb{R}^n} \langle f(x), g(x) \rangle_{\mathcal{H}} \, dx$ converges and it turns out that
\[ \|g\|_{L^{p'}(\mathbb{R}^n, \mathcal{H})} = \sup \left\{ \left| \int_{\mathbb{R}^n} \langle f(x), g(x) \rangle \, dx \right| : \|f\|_{L^p(\mathbb{R}^n, \mathcal{H})} = 1 \right\}. \]
This implies that $L^{p'}(\mathbb{R}^n, \mathcal{H}) \subseteq (L^p(\mathbb{R}^n, \mathcal{H}))^\ast$. In fact equality holds (in the Hilbert case), that is,
\[ (L^p(\mathbb{R}^n, \mathcal{H}))^\ast \equiv L^{p'}(\mathbb{R}^n, \mathcal{H}). \]
We conclude this preliminary discussion introducing the singular integral in the vector-value case.

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Suppose $f : \mathbb{R}^n \to \mathcal{H}_1$ and $\vec{K} : \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ are measurable functions. We may consider the integral operator

$$\vec{T}f(x) = \int_{\mathbb{R}^n \setminus \{0\}} K(y)f(x-y) \, dy \quad (4.31)$$

Here clearly the expression $\vec{K}(y)f(x-y)$ can only be interpreted as the element $\vec{K}(y)$ of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ acting on $f(x-y) \in \mathcal{H}_1$; hence, it $\vec{T}f(x)$ is well defined, it is an element of $\mathcal{H}_2$.

**Theorem 4.27.** Suppose $\vec{T}$ is a bounded linear operator from $L^r(\mathbb{R}^n, \mathcal{H}_1)$ to $L^r(\mathbb{R}^n, \mathcal{H}_2)$ for some $r, 1 < r < \infty$, defined by the integral operator with kernel $K$ as in (4.31). Further, assume that $K$ satisfies the (vector-valued) Hörmander condition

$$\int_{|x-y|>|2|y|} \|\vec{K}(x-y) - \vec{K}(x)\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \, dx \leq B. \quad (4.32)$$

Then $\vec{T}$ is bounded from $L^p(\mathbb{R}^n, \mathcal{H}_1)$ to $L^p(\mathbb{R}^n, \mathcal{H}_2)$ when $1 < p < \infty$ and it is weak-type $(1,1)$, that is,

$$\left| \left\{ x \in \mathbb{R}^n : \|\vec{T}f(x)\|_{\mathcal{H}_2} > \lambda \right\} \right| \leq \frac{C}{\lambda} \|f\|_{L^1(\mathbb{R}^n, \mathcal{H}_1)}. \quad (4.33)$$

**Proof.** For simplicity, we are going to drop the "-notation.

The proof does not follow from the scalar case, but, it follows from the same proof. We begin by observing that the adjoint of $T$ has kernel $K^*(-x)$, since

$$\langle Tf, g \rangle = \int_{\mathbb{R}^n} \left\langle \int_{\mathbb{R}^n} K(x-y)f(y) \, dy, g(x) \right\rangle_{\mathcal{H}_2} \, dx$$

$$= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \langle K(x-y)f(y), g(x) \rangle_{\mathcal{H}_2} \, dy \right) dx$$

$$= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \langle f(y), K^*(x-y)g(x) \rangle_{\mathcal{H}_2} \, dx \right) dy$$

$$= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} K^*(x-y)g(x) \, dx \right) \langle f(y), \rangle_{\mathcal{H}_1} dy .$$

If we show that $T$ is weak-type $(1,1)$, by Marcinkiewicz interpolation theorem it would follow that $T$ is of strong-type $(p,p)$ for $1 < p \leq r$. The operator $T^*$ satisfies the same assumption as $T$, with the assumption that $T^*$ is a bounded linear operator from $L^{r'}(\mathbb{R}^n, \mathcal{H}_2)$ to $L^{r'}(\mathbb{R}^n, \mathcal{H}_1)$, so that it is also of strong-type $(p,p)$ for $1 < p \leq r'$. Thus, by duality, $T$ is of strong-type $(p,p)$ for $r < p < \infty$, so it follows that $T$ is of strong-type $(p,p)$ for $1 < p < \infty$.

Thus, (as in the scalar case) it suffices to show that $T$ is weak-type $(1,1)$.

We now proceed as in the proof of Thm. 4.1. Hence, we only sketch the argument, indicating the main differences.

The main point is to generalize the Calderón–Zygmund decomposition of an $L^1$-function. In the proof of Thm. 3.6, as well as in Thm. 2.5, we assumed that $f$ is non-negative (since we wrote an arbitrary complex-valued function as $f = (\text{Re } f)_+ - (\text{Re } f)_+ + i[(\text{Im } f)_+ - (\text{Im } f)_-]$).

We provide the details for sake of completeness.
Given \( \lambda > 0 \), by applying the Calderón–Zygmund decomposition to the function \( \|f(x)\|_{\mathcal{H}_1} \), we find that there exists a sequence of disjoint cubes \( \{Q_j\} \) such that:

(i) \( 0 \leq \|f(x)\|_{\mathcal{H}_1} \leq \lambda \) for almost all \( x \not\in \bigcup_j Q_j \);

(ii) \( \left| \bigcup_j Q_j \right| \leq \frac{1}{\lambda} \|f\|_{L^1(\mathbb{R}^n, \mathcal{H}_1)} \);

(iii) \( \lambda < \frac{1}{|Q_j|} \int_{Q_j} \|f(x)\|_{\mathcal{H}_1} \, dx \leq 2^n \lambda \).

Now we decompose \( f \) as the sum \( f = g + b \), where

\[
g(x) = f(x)\chi_{\{c \cup \bigcup_j Q_j\}}(x) + \sum_j \left( \frac{1}{|Q_j|} \int_{Q_j} f(y) \, dy \right) \chi_{Q_j}(x),
\]

and

\[
b(x) = \sum_j b_j(x) = \sum_j \left( f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(y) \, dy \right) \chi_{Q_j}(x).
\]

Notice that, as \( f(x), g(x) \) and \( b(x) \) are vectors in \( \mathcal{H}_1 \), for a.a. \( x \in \mathbb{R}^n \), and that \( \frac{1}{|Q_j|} \int_{Q_j} f(y) \, dy \) is also a well-defined element of \( \mathcal{H}_1 \), since \( f \in L^1(\mathbb{R}^n, \mathcal{H}_1) \) and by the identity (4.29).

Notice that, similarly to the proof of Thm.'s 3.6 and 4.1, we have that

\[
\|g(x)\|_{\mathcal{H}_1} \leq \|f(x)\|_{\mathcal{H}_1} \chi_{\{c \cup \bigcup_j Q_j\}}(x) + \sum_j \left( \frac{1}{|Q_j|} \int_{Q_j} f(y) \, dy \right) \chi_{Q_j}(x)
\]

\[
\leq \lambda \chi_{\{c \cup \bigcup_j Q_j\}}(x) + \sum_j \left( \frac{1}{|Q_j|} \int_{Q_j} f(y) \, dy \right) \chi_{Q_j}(x)
\]

\[
\leq 2^n \lambda;
\]

and also that

\[
\int_{\mathbb{R}^n} \|g(x)\|_{\mathcal{H}_1} \, dx \leq \int_{\mathbb{R}^n} \|f(x)\|_{\mathcal{H}_1} \, dx.
\]

(This last inequality follows since \( g(x) = f(x) \) on \( c(\bigcup_j Q_j) \), while \( \int_{Q_j} g(x) \, dx = \int_{Q_j} f(x) \, dx \) for all \( Q_j \).)

On the other hand, \( \int_{\mathbb{R}^n} b(x) \, dx = 0_{\mathcal{H}_1} \), the zero element in \( \mathcal{H}_1 \).

We now proceed and in the proof of Thm. 4.1. Since

\[
\|Tf(x)\|_{\mathcal{H}_2} \leq \|Tg(x)\|_{\mathcal{H}_2} + \|Tb(x)\|_{\mathcal{H}_2},
\]

we have

\[
\left| \{x : \|Tf(x)\|_{\mathcal{H}_2} > \lambda \} \right| \leq \left| \{x : \|Tg(x)\|_{\mathcal{H}_2} > \lambda/2 \} \right| + \left| \{x : \|Tb(x)\|_{\mathcal{H}_2} > \lambda/2 \} \right|.
\]
Now, using the boundedness of $T : L^r(\mathbb{R}^n, \mathcal{H}_1) \rightarrow L^r(\mathbb{R}^n, \mathcal{H}_2)$,

$$\|x : \|Tg(x)\|_{\mathcal{H}_2} > \lambda/2\| = \int_{\{x: \|Tg(x)\|_{\mathcal{H}_2} > \lambda/2\}} dx \leq \int_{\{x: \|Tg(x)\|_{\mathcal{H}_2} > \lambda/2\}} \frac{\|Tg(x)\|_{\mathcal{H}_2}}{(\lambda/2)^r} dx$$

$$\leq \frac{C}{\lambda^r} \int_{\mathbb{R}^n} \|Tg(x)\|_{\mathcal{H}_2}^r dx$$

$$\leq \frac{C}{\lambda^r} \int_{\mathbb{R}^n} \|g(x)\|_{\mathcal{H}_1}^r dx \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} \|g(x)\|_{\mathcal{H}_1} dx$$

$$= \frac{C}{\lambda} \int_{\mathbb{R}^n} \|f(x)\|_{\mathcal{H}_1} dx.$$

Next, let $Q_j^*$ denote the cube with the same center $c_j$ as $Q_j$ with side length $\sqrt{2}n$ times longer. Then we estimate,

$$\|x : \|Tb(x)\|_{\mathcal{H}_2} > \lambda/2\| = \|x \in (\cup_j Q_j^*) : \|Tb(x)\|_{\mathcal{H}_2} > \lambda/2\|$$

$$+ \|x \in c(\cup_j Q_j^*) : \|Tb(x)\|_{\mathcal{H}_2} > \lambda/2\|$$

$$\leq \|\cup_j Q_j^*\| + \|x \in c(\cup_j Q_j^*) : \|Tb(x)\|_{\mathcal{H}_2} > \lambda/2\|$$

$$\leq \frac{2}{\lambda} \|f\|_{L^1(\mathbb{R}^n, \mathcal{H}_1)} + \frac{2}{\lambda} \int_{\mathbb{R}^n(\cup_j Q_j^*)} \|Tb(x)\|_{\mathcal{H}_2} dx.$$

Since $\|Tb(x)\|_{\mathcal{H}_2} \leq \sum_j \|Tb_j(x)\|_{\mathcal{H}_2}$ a.e., it will suffice to prove that

$$\sum_j \int_{\mathbb{R}^n Q_j^*} \|Tb_j(x)\|_{\mathcal{H}_2} dx \leq C \|f\|_{L^1(\mathbb{R}^n, \mathcal{H}_1)}.$$ (4.33)

Notice that, since $\int_{Q_j} b_j(y) dy = 0_{\mathcal{H}_1}$, also

$$\int_{Q_j} K(x - c_j)b_j(y) dy = K(x - c_j) \int_{Q_j} b_j(y) dy = 0_{\mathcal{H}_2},$$

so that

$$\int_{\mathbb{R}^n Q_j^*} \|Tb_j(x)\|_{\mathcal{H}_2} dx = \int_{\mathbb{R}^n Q_j^*} \| \int_{Q_j} K(x - y)b_j(y) dy \|_{\mathcal{H}_2} dx$$

$$= \int_{\mathbb{R}^n Q_j^*} \| \int_{Q_j} (K(x - y)b_j(y) - K(x - c_j)b_j(y)) dy \|_{\mathcal{H}_2} dx$$

$$\leq \int_{\mathbb{R}^n Q_j^*} \| \int_{Q_j} ((K(x - y) - K(x - c_j))b_j(y)) dy \|_{\mathcal{H}_2} dx$$

$$\leq \int_{\mathbb{R}^n Q_j^*} \| K(x - y) - K(x - c_j) \|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \|b_j(y)\|_{\mathcal{H}_1} dy dx$$

$$\leq \int_{Q_j} \|b_j(y)\|_{\mathcal{H}_1} \int_{\mathbb{R}^n Q_j^*} \| K(x - y) - K(x - c_j) \|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} dx dy$$

$$\leq B \int_{Q_j} \|b_j(y)\|_{\mathcal{H}_1} dy.$$
Therefore,

\[ \sum_j \int_{R \setminus Q_j^*} \|Tb_j(x)\|_{\mathcal{H}_2} \, dx \leq B \sum_j \int_{Q_j} \|b_j(y)\|_{\mathcal{H}_1} \, dy \leq 2B \|f\|_{L^1(R^n, \mathcal{H}_1)}, \]

estimates that completes the proof.  \( \square \)
5. FOURIER MULTIPLIERS

In this chapter we can to the analysis of the so-called Fourier multipliers, that is, of the operators, initially defined on Schwartz functions, of the form

$$T_m(f)(x) = \mathcal{F}^{-1}(m\hat{f})(x),$$

where $m \in S'(\mathbb{R}^n)$. We will often identify the operator $T_m$ with the distribution $m$ and call them indifferently a Fourier multiplier.

Observe that $m \in S'$ implies that $m = \hat{K}$ for some $K \in S'$, so that

$$T_m(f) = K * f.$$ Therefore, the singular integrals (of convolution type) that we have seen in the previous sections, are particular cases of Fourier multipliers.

Notice that $\hat{f} \in S$, so that $m\hat{f} \in S'$ and it makes sense to define its inverse Fourier transform.

We mention, without proving, the following result, due to Hörmander, that shows that every bounded linear operator between $L^p$-spaces that commutes with translations is given by the convolution with a tempered distribution $K$; hence it is a Fourier multiplier.

Recall that, if $a \in \mathbb{R}^n$ we define $[\tau_a f](x) = f(x + a)$. An operator $T$ is said to commute with translations if

$$\tau_a[T(f)](x) = T([\tau_a f])(x),$$

for all $a \in \mathbb{R}^n$, and a.a. $x$.

**Theorem 5.1.** Let $1 \leq p, q \leq \infty$. Suppose that $T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is a bounded operator that commutes with translations. Then there exists a unique $u \in S'$ such that

$$T(f) = u * f$$

for all $f \in S$.

For the proof we refer to [StWe], Thm. 3.16.

We will first show a few basic properties of these operators, and then describe some sufficient conditions on the function $m$ to ensure that $T_m$ is a bounded operator on $L^p$.

5.1. The space $\mathcal{M}_p$ of $L^p$-bounded Fourier multipliers. We define the space

$$\mathcal{M}_p = \{ m \in S' : T_m : L^p \to L^p \text{ is bounded} \},$$

and we set

$$\|m\|_{\mathcal{M}_p} = \|T_m\|_{(L^p,L^p)} = \sup_{\|f\|_{L^p}=1} \|T_m(f)\|_{L^p}.$$  

By Plancherel’s theorem we know that if $m$ is bounded then $T_m$ is bounded on $L^2$. The converse of this assertion is also true, as consequence of the following result.

**Lemma 5.2.** If $T_m$ is bounded on $L^p$ then $m$ is bounded. In particular, $T_m$ is bounded on $L^2$ is and only if $m \in L^\infty$ and, in this case, $\|T_m\|_{(L^2,L^2)} = \|m\|_{L^\infty}$. Finally, $\mathcal{M}_p = \mathcal{M}_{p'}$, if $p$ and $p'$ are conjugate exponents.
Proof. If \( m \) in not bounded, then it is easy to see that \( T_m \) cannot be bounded on \( L^2 \) (exercise). Now, it is easy to see that \( \| T_m \|_{(L^2,L^2)} = \| m \|_{L^\infty} \). For, by Plancherel theorem,
\[
\| T_m f \|_{L^2} = \| m \hat{f} \|_{L^2} \leq \| m \|_{L^\infty}\| \hat{f} \|_{L^2} = \| m \|_{L^\infty}\| f \|_{L^2},
\]
so that \( \| T_m \|_{(L^2,L^2)} \leq \| m \|_{L^\infty} \).

Next, if \( \varepsilon > 0 \) is fixed, let
\[
E_\varepsilon = \{ x : |x| \leq 1/\varepsilon \text{ and } |m(x)| > \| m \|_{L^\infty} - \varepsilon \}.
\]
Then \( E_\varepsilon \) has positive and finite measure, and let \( f \in L^2 \) be such that \( \hat{f} = \chi_{E_\varepsilon} \). Hence,
\[
\| T_m f \|_{L^2}^2 = \| m \hat{f} \|_{L^2}^2 = \int |m(\xi)\hat{f}(\xi)|^2 d\xi \\
> (\| m \|_{L^\infty} - \varepsilon)^2 \int |\hat{f}(\xi)|^2 d\xi \\
= (\| m \|_{L^\infty} - \varepsilon)^2 \| f \|_{L^2}^2.
\]
It follows that \( \| T_m \|_{(L^2,L^2)} \geq \| m \|_{L^\infty} \), and then equality holds.

Next we show that if \( T_m \) is bounded on \( L^p \), \( 1 \leq p < \infty \), then \( m \) is bounded. For, notice that the adjoint operator \( T_m^* \) is given by convolution with \( K \). Thus, \( T_m^* \) is also bounded on \( L^p \).

Since it is bounded on \( L^p \), by Marcinkiewicz interpolation theorem\(^{23}\) is also bounded on \( L^r \) for \( r \) between \( p \) and \( p' \); hence in particular on \( L^2 \). This implies that \( m \in L^\infty \) and we are done. \( \square \)

**Proposition 5.3.** For \( 1 < p < q < 2 \) we have the inclusion
\[
\mathcal{M}_1 \subset \mathcal{M}_p \subset \mathcal{M}_q \subset \mathcal{M}_2 \tag{5.1}
\]
and moreover,
\[
\| m \|_{L^\infty} \leq \| m \|_{\mathcal{M}_q} \leq \| m \|_{\mathcal{M}_p}.
\]

Proof. This follows easily from the previous lemma and the Marcinkiewicz theorem, if \( m \in \mathcal{M}_1 \), then \( m \) is also in \( \mathcal{M}_2 \) and therefore in \( \mathcal{M}_r \) for all \( 1 < r < 2 \). The same argument proves the other inclusions in (5.1).

Using again the Riesz–Thorin theorem, for \( 1 < p < 2 \) we have that
\[
\| T_m \|_{(L^2,L^2)} \leq \| T_m \|_{(L^p,L^p)}^{1/2}\| T_m \|_{(L^{p'},L^{p'})}^{1/2} = \| T_m \|_{(L^p,L^p)},
\]
since
\[
\frac{1}{2} = \frac{1/2}{p} + \frac{1/2}{p'}.
\]
A similar argument shows that, for \( 1 < p < q < 2 \), \( \| T_m \|_{(L^q,L^p)} \leq \| T_m \|_{(L^p,L^p)} \) and we are done. \( \square \)

**Proposition 5.4.** For \( 1 \leq p < \infty \) the space \( \mathcal{M}_p \) is a Banach space with respect to the norm \( \| \cdot \|_{\mathcal{M}_p} \). Moreover, \( \mathcal{M}_p \) is closed under pointwise multiplication and it is a Banach algebra.\(^{24}\)

\(^{23}\)Or, more simply by the Riesz-Thorin Theorem, [Ka] or [So].

\(^{24}\)An algebra \( \mathcal{A} \) which is a Banach space w.r.t. the norm \( \| \cdot \|_{\mathcal{A}} \) is called a Banach algebra if the product satisfies the inequality \( \| xy \|_{\mathcal{A}} \leq \| x \|_{\mathcal{A}}\| y \|_{\mathcal{A}} \) for all \( x, y \in \mathcal{A} \).
Proof. It suffices to consider the case $1 \leq p \leq 2$.

It is obvious that $\mathcal{M}_p$ is a linear space and that the equality $\|m\|_{\mathcal{M}_p} = \|T_m\|_{(L^p,L^p)}$ defines a norm on $\mathcal{M}_p$.

Next, if $m_1, m_2 \in \mathcal{M}_p$, then

$$(T_{m_1}m_2 f)(\xi) = m_1(\xi)m_2(\xi)\hat{f}(\xi) = (T_{m_1}(T_{m_2}f))(\xi),$$

so that $T_{m_1}m_2 = T_{m_1}T_{m_2}$. Moreover,

$$\|T_{m_1}m_2 f\|_{L^p} \leq \|T_{m_1}\|_{(L^p,L^p)}\|T_{m_2} f\|_{L^p} \leq \|T_{m_1}\|_{(L^p,L^p)}\|T_{m_2}\|_{(L^p,L^p)}\|f\|_{L^p},$$

that is,

$$\|m_1 m_2\|_{\mathcal{M}_p} = \|T_{m_1}m_2\|_{(L^p,L^p)} \leq \|T_{m_1}\|_{(L^p,L^p)}\|T_{m_2}\|_{(L^p,L^p)} = \|m_1\|_{\mathcal{M}_p}\|m_2\|_{\mathcal{M}_p}.$$ 

This proves that $\mathcal{M}_p$ is a Banach algebra w.r.t. the pointwise product.

We now show that $\mathcal{M}_p$ is complete. Let $\{m_j\}$ be a Cauchy sequence in $\mathcal{M}_p$. By Prop. 5.3 it follows that $\{m_j\}$ is a Cauchy sequence in $L^\infty$, so it converges, in the $L^\infty$-norm, hence a.e., to a function $m$. We wish to show that $m_j \to m$ in $\mathcal{M}_p$.

Let $f \in \mathcal{S}$ be fixed. Then, by Lebesgue’s dominated convergence theorem we have that

$$T_{m_j}(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)m_j(\xi)e^{2\pi i x \xi} \, d\xi \to \int_{\mathbb{R}^n} \hat{f}(\xi)m(\xi)e^{2\pi i x \xi} \, d\xi = T_m(f)(x)$$
a.e. By Fatou’s lemma it follows that

$$\int_{\mathbb{R}^n} |T_m(f)(x)|^p \, dx \leq \liminf_{j \to +\infty} \int_{\mathbb{R}^n} |T_{m_j}(f)(x)|^p \, dx \leq \liminf_{j \to +\infty} \|m_j\|_{\mathcal{M}_p}\|f\|_{L^p},$$

that is,

$$\|m\|_{\mathcal{M}_p} \leq \liminf_{j \to +\infty} \|m_j\|_{\mathcal{M}_p} \leq C^p$$

since $\{m_j\}$ is a Cauchy sequence, hence the sequence of the norms is bounded.

From the last inequality it also follows that

$$\|m - m_k\|_{\mathcal{M}_p} \leq \liminf_{j \to +\infty} \|m_j - m_k\|_{\mathcal{M}_p} < \varepsilon$$

if $k \geq k_0$, since $\{m_j\}$ is a Cauchy sequence. This proves the proposition. □

Examples 5.5.

1. The first example we consider is $m(\xi) = -i \text{sgn}(\xi)$. Then $T_m$ is the Hilbert transform, and we know that $T_m$ is bounded on $L^p$ for $1 < p < \infty$.

2. Consider the interval $(a, b)$, and assume for the moment that it is bounded. Notice that

$$\chi_{(a,b)}(\xi) = \frac{1}{2} \left[ \text{sgn}(\xi - a) - \text{sgn}(\xi - b) \right].$$

Let $a \in \mathbb{R}$ and observe that the operator (called the modulation operator)

$$M_a f(x) = e^{2\pi i ax} f(x)$$
is bounded on $L^p$ for all $p$, and also that $(M_af)'(\xi) = \hat{f}(\xi - a) = (\tau_a \hat{f})(\xi)$. Then notice that, for $f \in \mathcal{S}(\mathbb{R})$,

$$
\chi_{(a,b)}(\xi) \hat{f}(\xi) = \frac{1}{2} \left[ \text{sgn}(\xi - a) - \text{sgn}(\xi - b) \right] \hat{f}(\xi)
$$

$$
= \frac{1}{2} \left[ \tau_a (\text{sgn}(\xi) \tau_a(\hat{f})) - \tau_b (\text{sgn}(\xi) \tau_b(\hat{f})) \right](\xi)
$$

$$
= \frac{i}{2} \mathcal{F} \left( \left[ M_a \text{HM}_{-a} - M_b \text{HM}_{-b} \right] (f) \right)(\xi),
$$

(5.2)

where $H$ denotes the Hilbert transform. (This last equality can be easily be checked by computing the Fourier transform on the right hand side, recalling that $(Hf)'(\xi) = -i \text{sgn}(\xi) \hat{f}(\xi)$.)

Therefore,

$$
\mathcal{F}^{-1}(\chi_{(a,b)}) * f = \frac{i}{2} \left[ M_a \text{HM}_{-a} - M_b \text{HM}_{-b} \right] (f),
$$

so that

$$
T_{\chi_{(a,b)}}(f) = \frac{i}{2} \left[ M_a \text{HM}_{-a} - M_b \text{HM}_{-b} \right] (f).
$$

(5.3)

Next, suppose that the interval is unbounded. Let $(a, b) = (a, +\infty)$, the other case begin analogous. Then

$$
\chi_{(a, +\infty)}(\xi) = \frac{1}{2} \left[ 1 + \text{sgn}(\xi - a) \right],
$$

so that, arguing as in (5.2),

$$
\chi_{(a, +\infty)}(\xi) \hat{f}(\xi) = \frac{i}{2} \mathcal{F} \left( \left[ -iI + M_a \text{HM}_{-a} \right] (f) \right)(\xi),
$$

i.e.

$$
T_{\chi_{(a, +\infty)}}(f) = \frac{i}{2} \left[ M_a \text{HM}_{-a} - iI \right] (f).
$$

(5.4)

Therefore, it follows from (5.3) and (5.4) that, for all $a, b \in \mathbb{R} \cup \{\pm \infty\}$, $m = \chi_{(a,b)}$ is in $\mathcal{M}_p(\mathbb{R})$ for $1 < p < \infty$, with operator norms uniformly bounded in $a$ and $b$.

(3) Let $n > 1$. The Hilbert transform can be extended to define a family of $L^p$-bounded Fourier multipliers in $\mathbb{R}^n$. For $j = 1, \ldots, n$ define $m_j(\xi) = -i \text{sgn}(\xi_j)$ and hence $T_{m_j}(f) = \mathcal{F}^{-1}(m_j \hat{f})$. It is easy to see that

$$
T_{m_j}f(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{1}{t} f(x_1, \ldots, x_j - t, \ldots, x_n) \, dt.
$$

Now, taking $j = n$ for simplicity, using the boundedness of the Hilbert transform in the last variable we have

$$
\left\| T_{m_n}f \right\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \left| T_{m_n}f(x_1, \ldots, x_{n-1}, x_n) \right|^p \, dx_n \, dx_1 \cdots dx_{n-1}
$$

$$
\leq C \left( \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |f(x_1, \ldots, x_{n-1}, x_n)|^p \, dx_n \, dx_1 \cdots dx_{n-1} \right)^{\frac{1}{p}}
$$

$$
= C \left\| f \right\|_{L^p(\mathbb{R}^n)}^p.
$$

It also follows that, if $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ and

$$
E = \{ \xi \in \mathbb{R}^n : a \cdot \xi > b \},
$$

then $m = \chi_E \in \mathcal{M}_p$, $1 < p < \infty$. (Exercise.)
(4) Let \( m = \chi_{\{|\xi|<1\}} \) be the characteristic function of the unit ball in \( \mathbb{R}^n \). If \( n = 1 \) it follows from the previous discussion that \( m \in \mathcal{M}_p \) for all \( p, 1 < p < \infty \). On the other hand, the next result, the celebrated Fefferman’s theorem on the ball multiplier, Thm. 5.10 below, states that if \( n > 1 \) that \( \chi_{\{|\xi|<1\}} \in \mathcal{M}_p \) if and only if \( p = 2 \).

We conclude this section by stating the characterization of \( \mathcal{M}_1 \), a significant result, for whose proof we refer to [StWe] or [Gr]. We recall that a finite complex-valued Borel measure \( \mu \) defines a tempered distribution by setting \( \mu(f) = \int f(x) d\mu(x) \).

**Theorem 5.6.** A measurable function \( m \in \mathcal{M}_1 \) if and only if \( m \) is the Fourier transform of a finite complex-valued Borel measure \( \mu \).

### 5.2. Multiple Fourier series and convergence in the \( L^p \)-norm.

In this section we briefly illustrate an application of the theory of Fourier multipliers.

We define the \( n \)-dimensional torus to be the quotient group \( T^n = \mathbb{R}^n / \mathbb{Z}^n \). It is easy to convince oneself that \( T^n \) can be identified with the compact set \([0, 2\pi] \times \cdots \times [0, 2\pi] \equiv [0, 2\pi]^n\).

Given an integrable function \( f \) on \( T^n \) we define its (multiple) Fourier series to be the expression

\[
\sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i k \cdot t},
\]

where \( t \in T^n \) and

\[
\hat{f}(k) = \int_{T^n} f(t) e^{-2\pi i k \cdot t} \, dt.
\]

The main question is in what sense the expression in (5.5) represents \( f \), that is, if it converges and if it does converge, whether it converges to \( f \).

By elementary theory of Hilbert spaces applied to the case of \( H = L^2(T^n) \) it follows at once that, if \( f \in L^2(T^n) \) then the series in (5.5) converges (unconditionally) to \( f \) in the \( L^2(T^n) \)-norm.

If \( p \neq 2 \) and we are still interested in the norm convergence of the series in (5.5), when \( n > 1 \) we need to specify what we mean by the summation, since \( \mathbb{Z}^n \) lacks of a natural order. Therefore, we proceed as follows.

**Definition 5.7.** We call **convex body** an open convex set \( C \subset \mathbb{R}^n \) containing the origin. For \( \lambda > 0 \) we denote by \( \lambda C \) the diluted of \( C \) by \( \lambda \) and set

\[
S^\lambda_C(f) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i k \cdot t}.
\]

We say that the Fourier series converge in \( L^p(T^n) \), \( 1 \leq p < \infty \) in the summation method given by the convex body \( C \), or more briefly, in the \( C \)-sense, if

\[
\lim_{\lambda \to +\infty} \left\| S^\lambda_C(f) - f \right\|_{L^p(T^n)} = 0,
\]

for every \( f \in L^p(T^n) \cap C(T^n) \).

The following result is not difficult to prove, but for lack of adequate time we refer to [So], Ch. IV for its proof.
Theorem 5.8. The space $L^p(T^n)$, $1 \leq p < \infty$, admits norm convergence for the Fourier series in the summation method given by the convex body $C$, if and only if $\chi_C \in M_p$, that is, if and only if the characteristic function of $C$ is a bounded Fourier multiplier in $L^p(T^n)$.

A subset $S$ of $\mathbb{R}^n$ is called a half-space if there exist $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$S = \{x \in \mathbb{R}^n : a \cdot x > b\}.$$

A bounded subset $P \subset \mathbb{R}^n$ is called a convex polyhedron if it is convex, contains the origin, and it is the intersection of a finite number of half-spaces.

The most typical examples of convex bodies are the unit ball $B$, which has smooth boundary and nowhere vanishing curvature, and a convex polyhedron, that instead has only Lipschitz boundary a.e. flat.

Theorem 5.9. Let $P$ be a convex polyhedron. Then $\chi_P \in M_p$, $1 < p < \infty$. Hence, $L^p(T^n)$ admits norm convergence in the $P$-summation method.

Proof. If $n = 1$, then $P = (a, b)$ for some $a < 0 < b$ and the result follows from Example 5.5 (2).

When $n > 1$ we easily reduce ourselves to the boundedness of the Riesz transforms. For, if $P \cap \bigcap_{j=1}^N S_j$, where $S_j$ are half-spaces, then $\chi_P = \prod_{j=1}^N \chi_{S_j}$. Hence,

$$T_{\chi_P} = T_{\prod_{j=1}^N \chi_{S_j}} = T_{\chi_{S_1}} \cdots T_{\chi_{S_N}}.$$

Thus, it suffices to show that each $T_{\chi_{S_j}}$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Composing with a rotation and a translation, it suffices to show that $\chi_S \in M_p$, when

$$S = \{\xi \in \mathbb{R}^n : \xi_1 > 0\},$$

and this follows from Example 5.5 (3). $\Box$

5.3. The C. Fefferman multiplier theorem. The next result came as a big surprise. In the early ’70 C. Fefferman proved that the characteristic function of the unit ball in $\mathbb{R}^n$ with $n > 1$ is not a bounded Fourier multiplier in $L^p(\mathbb{R}^n)$, unless $p = 2$.

Theorem 5.10. Let $m = \chi_B$ be the characteristic function of the unit ball $B$ in $\mathbb{R}^n$, where $n > 1$. Then $m \in M_p$ if and only if $p = 2$.

The proof of this theorem is quite elaborated and for the time being we refer to [St2] Ch. X Section 2.5, or [So] Ch. 4, Section 3. (The latter proof is self-contained and it is essentially the original proof by C. Fefferman, while the latter one shows the deep connection of this problem with other important concepts and open problems in modern harmonic analysis.)

5.4. The Sobolev spaces $H^s$. We have remarked that the characteristic function of the unit ball in $\mathbb{R}^n$, with $n > 1$, defines a Fourier multiplier that is bounded on $L^p$ only for $p = 2$. It is clear that, besides the boundedness, the function $m$ needs to posses some regularity. In order to measure regularity we introduce a family of spaces, called the Sobolev spaces, whose elements have derivatives (in some weak sense) in $L^2$. 
We begin our brief introduction with the case of a non-negative integer $k$. We define $H^k = H^k(\mathbb{R}^n)$ to be the space of $L^2$ functions $f$ such that the distributional derivatives $\partial^\alpha f \in L^2$ for all $\alpha$ with $|\alpha| \leq k$. On $H^k$ we can define an inner product that makes $H^k$ into a Hilbert space:

$$
\langle f | g \rangle_k = \sum_{|\alpha| \leq k} (\partial^\alpha f)(\partial^\alpha g)^*. 
$$

We wish to extend this definition to non-integral values of $k$ and also to have a more “efficient” way to represent the inner product. We observe that, by Plancherel theorem, $f \in H^k$ if and only if $\xi^\alpha \hat{f} \in L^2$ for all $\xi$, $|\alpha| \leq k$. We claim that there exist positive constants $c_1, c_2$ such that, for all $\xi \in \mathbb{R}^n$,

$$
c_1(1 + |\xi|^2)^{k/2} \leq \sum_{|\alpha| \leq k} |\xi^\alpha| \leq c_2(1 + |\xi|^2)^{k/2}.
$$

For, if $\alpha$ is a multi-index with $|\alpha| \leq k$, if $|\xi| \geq 1$ then

$$
|\xi^\alpha| \leq |\xi|^k \leq (1 + |\xi|^2)^{k/2},
$$

while, if $|\xi| \leq 1$,

$$
|\xi^\alpha| \leq 1 \leq (1 + |\xi|^2)^{k/2}.
$$

On the other hand, since $|\xi|^k$ and $\sum_{j=1}^k |\xi_j|^k$ are both homogeneous of degree $k$ non-vanishing for $\xi \neq 0$, we have

$$
(1 + |\xi|^2)^{k/2} \leq c_0(1 + |\xi|^k) \leq c_0 \left(1 + c'_0 \sum_{j=1}^k |\xi_j|^k\right) \leq C \sum_{|\alpha| \leq k} |\xi^\alpha|.
$$

By the claim, it follows that $f \in H^k$ if and only if $(1 + |\xi|^2)^{k/2} \hat{f} \in L^2$. Moreover,

$$
c_1 \int (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi \leq \sum_{|\alpha| \leq k} \int |\xi^\alpha|^2 |\hat{f}(\xi)|^2 \, d\xi
$$

$$
\leq c \sum_{|\alpha| \leq k} \int |\partial^\alpha f(x)|^2 \, dx
$$

$$
\leq c_2 \int (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi,
$$

so that the two norms

$$
\left( \sum_{|\alpha| \leq k} \int |\partial^\alpha f(x)|^2 \, dx \right)^{1/2} \quad \text{and} \quad \left( \int (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}
$$

are equivalent, and the latter one is defined for all $k$ real, not just non-negative integer.

**Definition 5.11.** We define the Bessel potential of order $s \in \mathbb{R}$ of a distribution $f \in \mathcal{S}'$ as

$$
\Lambda_s f = \mathcal{F}^{-1} \left( (1 + |\cdot|^2)^{s/2} \hat{f} \right).
$$

---

25The fact that this inner product defines a Hilbert space, hence in particular complete, is left as an Exercise.
We endow $H$ with the inner product
\[
\langle f, g \rangle = \int (1 + |\xi|^2)^{s/2} \hat{f}(\xi) \hat{g}(\xi) d\xi
\]
and $(1 + |\xi|^2)^{s/2} \hat{\phi}(\xi)$ is again a Schwartz function.

**Definition 5.12.** Let $s \in \mathbb{R}$. We define the Sobolev space $H^s$ of order $s \in \mathbb{R}$ as the space of distributions $f \in S'$ such that $\Lambda_s f \in L^2$, that is
\[
H^s = \{ f \in S' : \int |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty \}.
\]
We endow $H^s$ with the inner product
\[
\langle f | g \rangle_s = \int (\Lambda_s f)(\Lambda_s g),
\]
which also gives the norm
\[
\|f\|_{H^s} = \left( \int |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2}.
\]

**Proposition 5.13.** The following properties are elementary.

(i) The Fourier transform $\mathcal{F}$ is a unitary isomorphism from $H^s$ to $L^2((1 + |\xi|^2)^s d\xi)$.

(ii) The Schwartz space $S$ is dense in $H^s$ for all $s \in \mathbb{R}$.

(iii) If $s > t$ then $\| \cdot \|_{H^t} \leq \| \cdot \|_{H^s}$ so that $H^s$ is a subspace of $H^t$ and it is also a dense subspace.

(iv) The Bessel potential $\Lambda_s$ is a unitary isomorphism from $H^t$ to $H^{t-s}$ for all $s, t \in \mathbb{R}$. Its inverse is $\Lambda_{-s}$.

(v) $H^0 = L^2$, so that $H^s \subset L^2$ for all $s > 0$. For $s < 0$ the elements of $H^s$ may not be functions.

(vi) The operators $\partial^\alpha$ are continuous from $H^s$ to $H^{s-|\alpha|}$, for all $s$ and $\alpha$.

**Proposition 5.14.** For $s \in \mathbb{R}$, the $L^2$ inner product $\langle \cdot | \cdot \rangle$ induces a unitary isomorphism between $H^{-s}$ and the dual space $(H^s)^*$ of $H^s$.

**Proof.** Let $f, \varphi \in S$. Then, by Plancherel's theorem,
\[
|\langle \varphi | f \rangle| = |\langle \hat{\varphi} | \hat{f} \rangle| \leq \int (1 + |\xi|^2)^{-s/2} |\hat{f}(\xi)| (1 + |\xi|^2)^{s/2} |\hat{\varphi}(\xi)| d\xi
\]
\[
\leq \left( \int (1 + |\xi|^2)^{-s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \left( \int (1 + |\xi|^2)^s |\hat{\varphi}(\xi)|^2 d\xi \right)^{1/2}
\]
\[
= \|f\|_{H^{-s}} \|\varphi\|_{H^s}.
\]
Therefore, the linear functional $\varphi \mapsto \langle \varphi | f \rangle$ extends to all of $H^s$ with norm $\leq \|f\|_{H^{-s}}$. But, we actually have equality of norms, since if $g = (\Lambda_{-s} f)$,
\[
\langle f | g \rangle = \int (1 + |\xi|^2)^{-s} |\hat{f}(\xi)|^2 d\xi = \|f\|^2_{H^{-s}}
\]
\[
= \|f\|_{H^{-s}} \|g\|_{H^s}.
\]
Notice that the pairing $\langle \hat{\varphi} | \hat{f} \rangle$ is well defined for all $f \in H^{-s}$ and $\varphi \in H^s$. 
Finally, let $L \in (H^s)^*$. Then $L \circ \mathcal{F}^{-1}$ is a bounded linear functional on $L^2((1 + |\xi|^2)^s d\xi)$, so that there exists $g \in L^2((1 + |\xi|^2)^s d\xi)$ such that

$$L(\varphi) = L \circ \mathcal{F}^{-1}(\hat{\varphi}) = \int \hat{\varphi}(\xi)g(\xi)(1 + |\xi|^2)^s d\xi = \int \hat{\varphi}(\xi)\mathcal{F}\Lambda_{2s}(\mathcal{F}^{-1}g)(\xi) d\xi = \langle \hat{\varphi} \mathcal{F}\Lambda_{2s}(\mathcal{F}^{-1}g) \rangle = \langle \hat{\varphi} f \rangle.$$

But, $f = \Lambda_{2s}(\mathcal{F}^{-1}g) \in H^{-s}$ (since $\mathcal{F}^{-1}g \in H^s$). This proves the proposition. □

We conclude this part with the Sobolev immersion theorem. We set

$$C^k_0 = \{ f \in C^k(\mathbb{R}^n) : \partial^\alpha f \in C_0, \text{ for } |\alpha| \leq k \}.$$

**Lemma 5.15.** Let $f$ be such that $\hat{f} \in H^s$, where $s > n/2$. Then $f \in L^1$ and for all $\varepsilon < s - n/2$ we have

$$\int_{\mathbb{R}^n} |f(x)|(1 + |x|)^\varepsilon dx \leq C_\varepsilon \|f\|_{H^s}.$$ 

**Proof.** Using the Cauchy-Schwarz inequality we have

$$\int |f(x)|(1 + |x|)^\varepsilon dx \leq 2^\varepsilon \int |f(x)|(1 + |x|^2)^{\varepsilon/2} dx \leq 2^\varepsilon \left(\int |f(x)|^2(1 + |x|^2)^{s} dx\right)^{1/2} \left(\int (1 + |x|^2)^{s-\varepsilon} dx\right)^{1/2}$$

$$\leq C_\varepsilon \|f\|_{H^s},$$

where $C_\varepsilon = 2^\varepsilon \left(\int (1 + |x|^2)^{s-\varepsilon} dx\right)^{1/2}$ is finite since $\varepsilon < s - n/2$. □

**Theorem 5.16.** (Sobolev Embedding Theorem) Let $t > k + (n/2)$. Then, $H^t$ embeds continuously in $C^k_0$.

**Proof.** Let $g \in H^t$ and $|\alpha| \leq k$. By the previous lemma, $\mathcal{F}^{-1}(\partial^\alpha g) \in L^1$, since $\partial^\alpha g \in H^s$, where $s = t - |\alpha| > n/2$, and

$$\int_{\mathbb{R}^n} |\mathcal{F}^{-1}(\partial^\alpha g)(x)|(1 + |x|)^\varepsilon dx \leq C_\varepsilon \|\partial^\alpha g\|_{H^s} \leq C_\varepsilon \|g\|_{H^t}.$$ 

By the Riemann-Lebesgue lemma, $\partial^\alpha g \in C_0$, for $|\alpha| \leq k$. □

For sake of clarity, we isolate the result that we need in the remaining of this part.

**Corollary 5.17.** Let $s > n/2$. Then, if $f \in H^s$ it follows that $f$ is continuous and

$$\|f\|_{L^\infty} \leq C\|f\|_{H^s}.$$ 

We conclude this part with a lemma that we are going to need later on.

**Lemma 5.18.** Let $f \in H^s$ and $\varphi \in S$. Then $f \varphi \in H^s$ and

$$\|f \varphi\|_{H^s} \leq C_{\varphi}\|f\|_{H^s}.$$ 

---

26We recall that we denote by $C_0$ the space of continuous functions that vanishes at infinity.
Proof. We begin with the preliminary observation that
\[ 1 + |\xi + \xi'|^2 \leq 1 + 2(|\xi|^2 + |\xi'|^2) \leq 2(1 + |\xi|^2)(1 + |\xi'|^2). \] (5.8)
Next, since \( \hat{f}\varphi = \hat{f} \ast \hat{\varphi} \), using the Cauchy-Schwarz inequality, we have
\[
\int_{\mathbb{R}^n} |(\hat{f}\varphi)(\xi)|^2 (1 + |\xi|^2)^s \, d\xi = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \hat{f}(\xi - \xi') \varphi(\xi') \, d\xi' \right|^2 (1 + |\xi|^2)^s \, d\xi
\leq \|\varphi\|_{L^1} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |\hat{f}(\xi - \xi')|^2 |\varphi(\xi')| \, d\xi' \right) (1 + |\xi|^2)^s \, d\xi
\leq \|\varphi\|_{L^1} \int_{\mathbb{R}^n} |\varphi(\xi')| \left( \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi + \xi'|^2)^s \, d\xi \right) \, d\xi'
\leq 2^s \|\varphi\|_{L^1} \int_{\mathbb{R}^n} |\varphi(\xi')|(1 + |\xi'|^2)^s \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi \, d\xi'
\leq C_\varphi \|f\|_{L^s}^2,
\]
where we have used the fact that \( \varphi \in S \). This proves the lemma. \( \square \)

5.5. The Mihlin–Hörmander multiplier theorem. In this section we prove the Mihlin–Hörmander multiplier theorem. The Mihlin–Hörmander condition that guarantees that a function gives rise to a Fourier multiplier that is bounded on \( L^p(\mathbb{R}^n) \) has the following features:

(i) it is invariant under the dilations \( m \mapsto m(r \cdot) \) for \( r > 0 \), in the sense that \( m \) satisfies this condition then also \( m(r \cdot) \) does;

(ii) the operator \( T_m : f \mapsto (\mathcal{F}^{-1}m) \ast f \) is an operator whose kernel \( K = (\mathcal{F}^{-1}m) \) is a Calderón–Zygmund convolution kernel, hence it is of weak-type \((1,1)\) and therefore bounded on \( L^p(\mathbb{R}^n), 1 < p < \infty; \) in order to show this we are going to appeal to Thm. 4.26.

We are now ready to define the space of Mihlin–Hörmander multipliers.

Definition 5.19. Let \( 0 < a_0 < a < b < b_0 \) and let \( \psi \in C_0^\infty \) be such that

(i) \( \text{supp} \psi \subseteq \{ \xi : a_0 \leq |\xi| \leq b_0 \} \);

(ii) \( \psi \geq 0 \) and \( \psi(\xi) = 1 \) for \( a \leq |\xi| \leq b \).

We call Mihlin-Hörmander multiplier a function \( m \) such that
\[
\|m\|_{MH^s} = \sup_{r > 0} \|m(r \cdot)\psi\|_{H^s}, \tag{5.9}
\]
for \( s > n/2 \). We denote the space of such multipliers by \( MH^s \).

Remark 5.20.

(1) Although we are not going to prove this assertion, and the one below, it is important to notice that the above definition is independent on the choice of \( \psi \) and that, a different \( \psi \) just gives rise to an equivalent norm.

(2) The norm \( \| \cdot \|_{MH^s} \) is invariant under dilation, in the sense that if \( m \in MH^s \) and \( m^r \) is given by \( m^r(\xi) = m(r\xi) \), where \( r > 0 \), then \( m^r \in MH^s \) and \( \|m^r\|_{MH^s} = \|m\|_{MH^s}. \)

Lemma 5.21. For \( s > n/2 \), if \( m \in MH^s \), then \( m \) is bounded and
\[
\|m\|_{L^\infty} \leq C\|m\|_{MH^s}. \tag{27}
\]

Exercise: Prove (1) and (2).
Proof. By assumption, the functions \( m(r \cdot \cdot) \psi \) are in \( H^s \) with \( s > n/2 \) and norms uniformly bounded. By Cor. 5.17, it follows that \( \| m(r \cdot \cdot) \psi \|_{L^\infty} = \| m(r^{-1} \cdot) \|_{L^\infty} \leq C\| m \|_{MH^s} \), with \( C \) independent of \( r > 0 \). Hence,

\[
\sup_{\xi} |m(\xi)|^2 = \sup_j \sup_{2^{-j} \leq |\xi| \leq 2^{j+1}} |m(\xi)|^2
\]

\[
= \sup_j \sup_{2^{-j} \leq |\xi| \leq 2^{j+1}} \sum_k |m(\xi)\psi(2^{-k}\xi)|^2
\]

\[
= \sup_j \sup_{2^{-j} \leq |\xi| \leq 2^{j+1}} \sum_{k \in \{j-1, ..., j+2\}} |m(\xi)\psi(2^{-k}\xi)|^2
\]

\[
\leq 4 \sup_k \sup_{\xi \in \mathbb{R}^n} |m(\xi)\psi(2^{-k}\xi)|^2
\]

\[
\leq C\| m \|_{MH^s}.
\]

This proves the lemma. \( \square \)

**Theorem 5.22. (Mihlin–Hörmander)** Let \( m \in MH^s \), with \( s > n/2 \). Then, the multiplier operator \( T_m \) is bounded on \( L^p \), \( 1 < p < \infty \) and it weak-type \((1, 1)\).

Before proving the theorem, we see a corollary.

**Corollary 5.23.** Let \( m \in C^k \setminus \{0\} \), with \( k > \lfloor n/2 \rfloor + 1 \), be such that

\[
\sup_{r > 0} r^{|\alpha|} \left( r^{-n} \int_{r/2 < |\xi| < 2r} |\partial^\alpha m(\xi)|^2 d\xi \right)^{1/2} < \infty, \tag{5.10}
\]

for all \( |\alpha| \leq k \). Then, \( m \in \mathcal{M}_p \), for \( 1 < p < \infty \).

In particular, \( m \in \mathcal{M}_p \), \( 1 < p < \infty \), if

\[
|\partial^\alpha m(\xi)| \leq C|\xi|^{-|\alpha|}, \tag{5.11}
\]

for all \( |\alpha| \leq k \).

**Proof.** We make the change of variables \( \zeta = \xi/r \) in (5.10) to obtain

\[
\sup_{r > 0} r^{|\alpha|} \left( r^{-n} \int_{r/2 < |\zeta| < 2r} |\partial^\alpha m(\zeta)|^2 d\zeta \right)^{1/2} = \sup_{r > 0} \left( \int_{1/2 < |\zeta| < 2} r^{2|\alpha|} |(\partial^\alpha m)(r\cdot)(\zeta)|^2 d\zeta \right)^{1/2}
\]

\[
= \sup_{r > 0} \left( \int_{1/2 < |\zeta| < 2} |(\partial^\alpha m(r \cdot))(\zeta)|^2 d\zeta \right)^{1/2}.
\]

This quantity is, by the assumption (5.10), finite. We prove that (5.9) is also satisfied. For, if \( |\alpha| \leq k \), by the Leibnitz rule, using the facts that \( \text{supp} \psi \subseteq \{1/2 \leq |\xi| \leq 2\} \) and that
\[ |\partial^\beta \psi(\xi)| \leq C, \text{ we see that} \]
\[ \|\partial^\alpha (m(r \cdot \psi))\|_{L^2} = \| \sum_{\beta \leq \alpha} c_{\alpha, \beta} \partial^\beta m(r \cdot) \partial^{\alpha-\beta} \psi\|_{L^2} \]
\[ \leq C \sum_{\beta \leq \alpha} \left( \int_{\mathbb{R}^n} |(\partial^\beta m(r \cdot))(\xi)\partial^{\alpha-\beta} \psi(\xi)|^2 \, d\xi \right)^{1/2} \]
\[ \leq C \sum_{|\beta| \leq k} \left( \int_{1/2 < |\xi| < 2} |(\partial^\beta m(r \cdot))(\xi)|^2 \, d\xi \right)^{1/2}, \]

where the constant \( C \) is independent of \( r > 0 \).

Therefore, \( m(r \cdot \psi) \in H^k \) with norm uniformly bounded in \( r > 0 \), that is, \( m \in MH^k \), with \( k > \lfloor n/2 \rfloor + 1 \). Applying Thm. 5.22 we obtain that \( m \in M_p \), for \( 1 < p < \infty \).

Finally, if \( m \) satisfies (5.11), then
\[ r^{\alpha} |(r^{-n} \int_{r/2 < |\xi| < 2r} |\partial^\alpha m(\xi)|^2 \, d\xi)^{1/2} \leq C r^{\alpha} \left( \int_{r/2 < |\xi| < 2r} |\xi|^{-2\alpha} \, d\xi \right)^{1/2} \leq C, \]

that is, (5.10) is also satisfied, and the corollary is proven. \( \square \)

5.6. †Proof of Thm. 5.22. Let \( m \in MH^s \) with \( s > n/2 \) and let \( \psi \) be as in Lemma 4.23 (i), that is, \( \psi \in C^\infty_0 \) and such that \( \sum_{j \in \mathbb{Z}} \psi(2^j \xi) = 1 \) for all \( \xi \neq 0 \).

We set \( m_j(\xi) = m(2^{-j} \xi)\psi(\xi) \), so that
\[ \sum_{j \in \mathbb{Z}} (m_j)^2(\xi) = \sum_{j \in \mathbb{Z}} m_j(2^j \xi) = m(\xi) \]
for all \( \xi \neq 0 \).

Moreover, we define
\[ K = \mathcal{F}^{-1} m, \quad \text{and} \quad k_{(j)} = \mathcal{F}^{-1} m_j. \]

Notice that then we have
\[ \mathcal{F} K = m = \sum_{j \in \mathbb{Z}} (m_j)^2 = \sum_{j \in \mathbb{Z}} (\mathcal{F} k_{(j)})^2 = \sum_{j \in \mathbb{Z}} \mathcal{F} \left( (k_{(j)})_{2^j} \right) = \mathcal{F} \left( \sum_{j \in \mathbb{Z}} (k_{(j)})_{2^j} \right), \]
that is,
\[ K = \sum_{j \in \mathbb{Z}} (k_{(j)})_{2^j}, \]
where the convergence is in \( S' \).

We wish to prove that the \( k_{(j)} \)'s satisfy the conditions (1)-(3) in Thm. 4.26.

We begin with (1). Using Lemma 5.15 we have
\[ \int (1 + |x|^\varepsilon |k_{(j)}(x)| \, dx = \int (1 + |x|^\varepsilon |\mathcal{F}^{-1} m_j(x)| \, dx \]
\[ \leq C_\varepsilon \| m_j \|_{H^s} = C_\varepsilon \| m(2^{-j} \cdot)\psi \|_{H^s} \]
\[ \leq C_\varepsilon \| m \|_{MH^s}. \]
Next,
\[ \int k_{(j)}(x) \, dx = \mathcal{F}k_{(j)}(0) = m_j(0) = 0, \]

since \( \psi(0) = 0 \); fact that proves (2).

Finally we prove (3). It suffices to show that \(^{28}\)
\[ \sup_{0 < |h| < 1} |h|^{-\alpha} \int |k_{(j)}(x - h) - k_{(j)}(x)| \, dx < \infty. \]

Since \( s > n/2 \), using the Cauchy-Schwarz inequality we have that for any \( \alpha \in (0, 1) \) and \( 0 < |h| < 1 \)
\[
|h|^{-\alpha} \int_{\mathbb{R}^n} |k_{(j)}(x - h) - k_{(j)}(x)| \, dx \\
\leq |h|^{-\alpha} \left( \int_{\mathbb{R}^n} |k_{(j)}(x - h) - k_{(j)}(x)|^2 (1 + |x|^2)^s \, dx \right)^{1/2} \\
= |h|^{-\alpha} \left( \int_{\mathbb{R}^n} \left| \int_0^1 h \cdot \nabla k_{(j)}(x - th) \, dt \right|^2 (1 + |x|^2)^s \, dx \right)^{1/2} \\
\leq |h|^{-\alpha} \left( \int_{\mathbb{R}^n} \left| \nabla k_{(j)}(x - th) \right|^2 dt (1 + |x|^2)^s \, dx \right)^{1/2} \\
\leq C \left( \int_{\mathbb{R}^n} \left| \nabla k_{(j)}(x) \right|^2 (1 + |x + th|^2)^s \, dx \right)^{1/2} \\
\leq C \left( \int_{\mathbb{R}^n} \left| \nabla k_{(j)}(x) \right|^2 (1 + |x|^2)^s \, dx \right)^{1/2} \\
\leq C \sum_{i=1}^n \| \mathcal{F}^{-1}(\partial_{\xi_i} k_{(j)}) \|_{H^s},
\]

where we have used an estimate as in (5.8).

Thus, we only need to prove that
\[ \sum_{i=1}^n \| \mathcal{F}^{-1}(\partial_{\xi_i} k_{(j)}) \|_{H^s} \leq C \| m \|_{MH^s}. \] (5.13)

Notice that
\[ \mathcal{F}^{-1}(\partial_{\xi_i} k_{(j)}) = 2\pi i \xi_i \mathcal{F}(k_{(j)})(\xi) = 2\pi i \xi_i m_j(\xi) = 2\pi i \xi_i m(2^j \xi) \psi(\xi) = (2\pi i \xi_i \eta(\xi)) m(2^j \xi) \psi(\xi), \]

where \( \eta \in C_0^\infty \) and it is identically 1 on the support of \( \psi \). Applying Lemma 5.18 we have then
\[ \sum_{i=1}^n \| \mathcal{F}^{-1}(\partial_{\xi_i} k_{(j)}) \|_{H^s} \leq C \sum_{i=1}^n \| (\xi_i \eta) m(2^j \cdot) \psi \|_{H^s} \]
\[ \leq C \| m(2^j \cdot) \psi \|_{H^s} \]
\[ \leq C \| m \|_{MH^s}, \]

thus proving (5.13) and therefore the theorem. □

\(^{28}\)See footnote 20.
6. Littlewood–Paley theory

In order to present the Littlewood–Paley theory we need the extension of the singular integrals to the case of vector-valued functions developed in Subsection 4.8.

6.1. An application. A simple application of the theory of vector-valued singular integrals is the following result. In this theorem the vector-valued kernel \( \vec{K} \) is a constant sequence. In this case it is easy to prove the boundedness of the corresponding vector-valued singular integral \( \vec{T} \).

**Theorem 6.1.** Let \( T \) be a convolution operator which is bounded \( L^2(\mathbb{R}^n) \) and whose integral kernel satisfy the Hörmander condition (4.2). Let \( 1 < p, r < \infty \). Then we have the strong \((p,p)\)-bound

\[
\left\| \left( \sum_j |Tf_j|^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left( \sum_j |f_j|^r \right)^{1/r} \right\|_p,
\]

and the weak \((1,1)\)-bound

\[
\left| \left\{ x \in \mathbb{R}^n : \left( \sum_j |Tf_j|^r \right)^{1/r} > \lambda \right\} \right| \leq \frac{C_r}{\lambda} \left\| \left( \sum_j |f_j|^r \right)^{1/r} \right\|_{L^1}.
\]

**Proof.** We have stated the theorem in the case of the \( \ell^r \)-norm, \( 1 < r < \infty \). Since we stated and proved Thm. 4.27 in the case of functions taking values in a Hilbert space (rather than in a separable, reflexive, Banach space), here we consider only the case \( r = 2 \).

We only have to check that the hypotheses of Thm. 4.27 are satisfied.

In the current situation we consider functions \( f : \mathbb{R}^n \to \ell^2 \), that is, \( f(x) = \{ f_j(x) \} \), where \( f_j \) are scalar valued functions. Then, we have that

\[
\vec{T}(f) = \vec{T}(\{f_j\}) = \{ K \ast f_j \}.
\]

Then \( \vec{T}(f) = \vec{K} \ast f \), where

\[
\vec{K}(x) : \ell^2 \to \ell^2
\]

\[
\{ a_j \} \mapsto \{ K(x) a_j \}.
\]

Hence, \( \vec{T} : L^2(\mathbb{R}^n, \ell^2) \to L^2(\mathbb{R}^n, \ell^2) \) is clearly bounded since, (as in the discussion prior to the theorem) if \( f \in L^2(\mathbb{R}^n, \ell^2) \),

\[
\left\| \vec{T}(f) \right\|_{L^2(\mathbb{R}^n, \ell^2)}^2 = \int_{\mathbb{R}^n} \sum_j |K \ast f_j(x)|^2 \, dx = \int_{\mathbb{R}^n} \sum_j |\hat{K}(\xi)|^2 |\hat{f}_j(\xi)|^2 \, d\xi
\]

\[
\leq A \| f \|_{L^2(\mathbb{R}^n, \ell^2)}^2.
\]

Next,

\[
\int_{|x| > 2|y|} |K(x - y) - K(y)| \, dx = \int_{|x| > 2|y|} |K(x) - K(y)| \, dx \leq B.
\]

The result now follows from Thm. 4.27.

In order to illustrate the Littlewood–Paley theorem we consider a collection \( \{ E_j \} \) of mutually disjoint measurable sets in \( \mathbb{R}^n \) and the operator \( S \) initially defined on Schwartz functions,

\[
Sf = F^{-1} \left( \sum_j (\chi_{E_j} \hat{f}) \right).
\]
Then, it is immediate to see that $S$ is bounded on $L^2$, since, by Plancherel’s theorem
\[
\|Sf\|_{L^2}^2 = \left\| \sum_j (\chi_{E_j} \hat{f}) \right\|_{L^2}^2 = \int_{\mathbb{R}^n} \left| \sum_j (\chi_{E_j} \hat{f})(\xi) \right|^2 d\xi = \sum_j \int_{E_j} |\hat{f}(\xi)|^2 d\xi = \|f\|_{L^2}^2. \tag{6.2}
\]

Starting from this simple observation we now state the following result. Notice that in this theorem, we restrict ourselves to the 1-dimensional case.

**Theorem 6.2.** Let $E_j = (-2^{j+1}, -2^j] \cup [2^j, 2^{j+1}) \subset \mathbb{R}$ and let $S_j$ be defined as

\[
(S_j f)^*(\xi) = \chi_{E_j}(\xi) \hat{f}(\xi).
\]

Then, for $1 < p < \infty$ there exists $C > 0$ such that, for all $f \in L^p(\mathbb{R})$

\[
\frac{1}{C} \|f\|_{L^p} \leq \left\| \left( \sum_j |S_j f|^2 \right)^{1/2} \right\|_{L^p} \leq C \|f\|_{L^p}.
\]

We will prove this theorem as a consequence of the Littlewood–Paley theorem, Thm. 6.5 in the next section, that holds true also in $\mathbb{R}^n$. In that theorem we will deal with an operator as the one in (6.1). Notice that

\[
\sum_j \chi_{E_j}(\xi) = \sum_j \chi_{E_0}(2^{-j}\xi) = 1
\]

for all $\xi \in \mathbb{R} \setminus \{0\}$. We will need a smooth decomposition of the function identically 1, and we then recall Lemma 4.23: There exists a function $\varphi \in C_0^\infty$ such that

(i) $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1$ for all $\xi \neq 0$.

Moreover, there exists another function $\varphi_0 \in C_0^\infty$ such that

(ii) $\varphi_0(\xi) + \sum_{j=1}^{+\infty} \varphi(2^{-j}\xi) = 1$ for all $\xi$.

A consequence of the lemma is the following.

**Corollary 6.3.** There exists $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp } \psi \subseteq \{ \xi : 1/2 \leq |\xi| \leq 2 \}$ and

\[
\sum_{j \in \mathbb{Z}} |\psi(2^{-j}\xi)|^2 = 1 \quad \text{for } \xi \neq 0. \tag{6.3}
\]

**Proof.** With $\varphi$ as in the Lemma, we define $\Phi$ by setting $(\hat{\Phi})^2 = \varphi$ so that

\[
(F(\Phi_{2^{-j}}))^2(\xi) = (\hat{\Phi}(2^{-j}\xi))^2 = \varphi(2^{-j}\xi),
\]

and therefore,

\[
\sum_{j \in \mathbb{Z}} |F(\Phi_{2^{-j}})(\xi)|^2 = \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1
\]

for $\xi \neq 0$.

Then, it suffices to set $\psi = \hat{\Phi}$. □

The next result is the first part of the Littlewood–Paley Thm. In a sense, it is the simplest, and least interesting, part. It shows that a certain decomposition on the Fourier transform side of a function gives rise to an operator that is bounded in $L^p$. It is more interesting to show that,
under an additional hypothesis, such an operator is also bounded from below– as we shall see in Thm. 6.5.

Theorem 6.4. (Littlewood–Paley Thm., part 1) Let \( \tilde{\Phi} \in \mathcal{S}(\mathbb{R}^n) \) be such that

\[
\sum_{j \in \mathbb{Z}} |\hat{\Phi}(2^{-j} \xi)|^2 \leq C
\]

for all \( \xi \neq 0 \), and define \( \tilde{\Delta}_j f = f \ast \tilde{\Phi}_{2^{-j}} \). Then, for \( 1 < p < \infty \), there exists \( C > 0 \) such that

\[
\left\| \left( \sum_{j \in \mathbb{Z}} |\tilde{\Delta}_j f|^2 \right)^{1/2} \right\|_{L^p} \leq C \| f \|_{L^p} .
\] (6.4)

Proof. If we set

\[
\tilde{T}(f) = \{ \tilde{\Delta}_j f \}_{j \in \mathbb{Z}}
\]

it suffices to show that \( \tilde{T} \) satisfies the hypotheses of Thm. 4.27, when

\[
\tilde{T} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n, \ell^2).
\]

As we have seen already, \( \tilde{T} \) is bounded when \( p = 2 \). For,

\[
\|\tilde{T} f\|_{L^2(\mathbb{R}^n, \ell^2)}^2 = \int_{\mathbb{R}^n} \|\tilde{T}(f)(x)\|_{\ell^2}^2 \, dx = \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |\tilde{\Delta}_j(f)(x)|^2 \, dx
\]

\[
= \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |\hat{\Phi}(2^{-j} \xi)|^2 |\hat{f}(\xi)|^2 \, d\xi
\]

\[
\leq C \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \, d\xi
\]

\[
= C \| f \|_{L^2}^2 .
\]

Thus, it will suffice to show that

\[
\|\nabla \tilde{K}(x)\|_{\ell(\mathbb{C}, \ell^2)} \leq \frac{C}{|x|^{n+1}} .
\]

Now,

\[
\nabla \tilde{K} = \{ \nabla \tilde{\Phi}_{2^{-j}}(x) \} = \{ \nabla (2^{nj} \tilde{\Phi}(2^j x)) \}
\]

\[
= \{ 2^{(n+1)j} (\nabla \tilde{\Phi})(2^j x) \} .
\]
Then, using the fact that $\tilde{\Phi}$ is a Schwartz function, for any integer $N > 0$ there exists a constant $C_N > 0$ such that

$$\|\nabla \tilde{K}(x)\|_{L^2} = \left( \sum_{j \in \mathbb{Z}} |2^{(n+1)j} (\nabla \tilde{\Phi})(2^j x)|^2 \right)^{1/2} \leq \sum_{j \in \mathbb{Z}} 2^{(n+1)j} |(\nabla \tilde{\Phi})(2^j x)| \leq C_N \sum_{j \in \mathbb{Z}} 2^{(n+1)j} \min(1, |2^j x|^{-N}) \leq C_N \left( \sum_{j \leq j_0} 2^{(n+1)j} + \sum_{j > j_0} 2^{(n+1)j}|2^j x|^{-N} \right) = C_N \left( \sum_{j \leq j_0} 2^{(n+1)j} + \frac{1}{|x|^{N}} \sum_{j > j_0} 2^{(n+1-N)j} \right),$$

with $j_0$ to be selected. Choosing $j_0$ so that $2^{j_0} \simeq |x|^{-1}$, and $N > n + 1$ we have\(^{29}\)

$$\|\nabla \tilde{K}(x)\|_{L^2} \leq C_N \left( 2^{(n+1)j_0} + \frac{2^{(n+1-N)j_0}}{|x|^{N}} \right) \leq C |x|^{-n+1},$$

as we wished to show. \(\square\)

### 6.2. The Littlewood–Paley theorem.

**Theorem 6.5.** (Littlewood–Paley theorem) Let $\psi \in S(\mathbb{R}^n)$ be as in (6.3). Let $\tilde{\Phi} = \mathcal{F}^{-1}\psi$ and set

$$\Delta_j f = f \ast \Phi_{2^{-j}}.$$

Then, for $1 < p < \infty$, there exists $C > 0$ such that

$$\frac{1}{C} \|f\|_{L^p} \leq \left\| \left( \sum_j |\Delta_j f|^2 \right)^{1/2} \right\|_{L^p} \leq C \|f\|_{L^p}.$$

**Proof.** The bound from above follows directly from Thm. 6.4, where we use only the estimate

$$\sum_{j \in \mathbb{Z}} |\psi(2^{-j}\xi)|^2 \leq C.$$

For the bound from below, we notice that

$$\|f\|_{L^2} = \left( \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{j \in \mathbb{Z}} |\psi(2^{-j}\xi)|^2 \, d\xi \right)^{1/2} = \left( \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |\hat{f}(\xi)\psi(2^{-j}\xi)|^2 \, d\xi \right)^{1/2} = \left\| \left( \sum_j |\Delta_j f|^2 \right)^{1/2} \right\|_{L^2}.$$

\(^{29}\)Recall footnote 21.
Having the identity \( \|f\|_{L^2} = \|\overline{T}(f)\|_{L^2(\mathbb{R}^n, \ell^2)} \), we can polarize it\(^3\), to obtain that for all \( f, g \in L^2(\mathbb{R}^n) \)
\[
\int f g = \int \sum_j \Delta_j f \overline{\Delta_j g}.
\]

Therefore,
\[
\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}} = 1} \left| \int f(x) \overline{g(x)} \, dx \right| = \sup_{\|g\|_{L^{p'}} = 1} \left| \int \sum_j \Delta_j f(x) \overline{\Delta_j g(x)} \, dx \right|
\leq \sup_{\|g\|_{L^{p'}} = 1} \left( \int \left( \sum_j |\Delta_j f(x)|^2 \right)^{p/2} \left( \sum_j |\Delta_j g(x)|^2 \right)^{p'/2} \, dx \right)^{1/p}
\leq \left( \int \left( \sum_j |\Delta_j f(x)|^2 \right)^{p/2} \, dx \right)^{1/p},
\]

where we have used the estimate from above. This proves the theorem. \( \square \)

We are now ready to prove Thm. 6.2. We recall that in this situation the space dimension is \( n = 1 \).

**Proof of Thm. 6.2.** Let \( \psi \in C_0^\infty(\mathbb{R}) \), supp \( \psi \subseteq \{ \xi : 1/2 \leq |\xi| \leq 4 \} \) and such that \( \psi(\xi) = 1 \) if \( 1 \leq |\xi| \leq 2 \), that is, if \( \xi \in E_0 \). Therefore,
\[
\psi(2^{-j} \xi) \chi_{E_j}(\xi) = \psi(2^{-j} \xi) \chi_{E_0}(2^{-j} \xi) = \chi_{E_0}(2^{-j} \xi) = \chi_{E_j}(\xi)
\]

for all \( \xi \). This implies that, if we define \( \hat{\Delta}_j f \) by setting
\[
(\hat{\Delta}_j f)(\xi) = \psi(2^{-j} \xi) \hat{f}(\xi),
\]

then
\[
\hat{\Delta}_j S_j f = S_j f.
\]

Of course we can write \( \hat{\Delta}_j f = \hat{\Phi}_{2^{-j}} * f \), where \( \hat{\Phi} = \psi \).

Notice that for any given \( \xi \), since supp \( \psi \subseteq \{ \xi : 1/4 \leq |\xi| \leq 4 \} \), there at most 3 indices \( j_1, j_2, j_3 \) such that \( \psi_j(\xi) \neq 0 \). Therefore,
\[
\sum_j |\psi(2^{-j} \xi)|^2 \leq C,
\]
for all \( \xi \neq 0 \).

We now make the following claim: For \( 1 < p < \infty \), there exists \( C > 0 \) such that for all \( g = \{ g_j \} \in L^p(\mathbb{R}^n, \ell^2) \)
\[
\left\| \left( \sum_j |S_j g_j|^2 \right)^{1/2} \right\|_{L^p} \leq C \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_{L^p} = C \|g\|_{L^p(\mathbb{R}^n, \ell^2)}.
\]

---

\(^3\)Recall the identity \( 4(f, g)_{\mathcal{H}} = \|f + g\|_{\mathcal{H}}^2 - \|f - g\|_{\mathcal{H}}^2 + i\|f + ig\|_{\mathcal{H}}^2 - i\|f - ig\|_{\mathcal{H}}^2 \) valid on any Hilbert space, and notice that in our case \( (f, g)_{\mathcal{H}} = \int \sum_j \Delta_j f \overline{\Delta_j g} \).
Assuming the claim we obtain the estimate from above, since
\[ \left\| \left( \sum_j |S_j f|^2 \right)^{1/2} \right\|_{L^p} = \left\| \left( \sum_j |S_j \tilde{\Delta} f|^2 \right)^{1/2} \right\|_{L^p} \]
\[ \leq C \left\| \left( \sum_j |\tilde{\Delta} f|^2 \right)^{1/2} \right\|_{L^p} \]
\[ \leq C \| f \|_{L^p} , \]
by Thm. 6.4.

The estimate from below follows by polarizing the identity (6.2) and using duality, as in the proof of Thm. 6.5, inequality (6.5), by replacing \( \Delta_j \) with \( S_j \).

Thus, we only need to prove the claim.

We now consider the vector-valued operator \( \vec{S} : L^p(\mathbb{R}^n, \ell^2) \to L^p(\mathbb{R}^n, \ell^2) \) defined as
\[ \vec{S}(g) = \{ S_j(g_j) \} \equiv \{ K_j \ast g_j \} \]
where \( K_j = F^{-1}(\chi_{E_j}) \). For this operator we wish to prove the bound (6.6).

Instead of proving that the operator \( \vec{S} \) and its kernel satisfy the hypotheses of Thm. 4.27, we proceed in a direct way.

Recall Example (5.5) (2). By (5.3) above we have that
\[ S_j(g_j)(x) = F^{-1}\left( \chi_{(-2^{-1},2^{-1})} + \chi_{[2^{-1},2]} \right) \ast g_j(x) \]
\[ = \frac{i}{2} \left[ M_{-2^{-1}} M_{2^{-1}} - M_{-2^{-1}} M_{2^{-1}} + M_{2^{-1}} H M_{-2^{-1}} - M_{2^{-1}} H M_{2^{-1}} \right] (g_j) , \]
so that
\[ |S_j(g_j)(x)| \leq |H(M_{2^{-1}} g_j)(x)| + |H(M_{2^{-1}} g_j)(x)| + |H(M_{-2^{-1}} g_j)(x)| + |H(M_{-2^{-1}} g_j)(x)| . \]
Therefore, in order to prove the claim it suffices to show that each of the four operators on the right hand side above satisfies (6.6), that is,
\[ \left\| \left( \sum_j |H(M_{a_j} g_j)|^2 \right)^{1/2} \right\|_{L^p} \leq C \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_{L^p} , \]
where \( \{ a_j \} \) is a sequence of real numbers.

Applying Thm. 6.1 we have that
\[ \left\| \left( \sum_j |H(M_{a_j} g_j)|^2 \right)^{1/2} \right\|_{L^p} \leq C \left\| \left( \sum_j |M_{a_j} g_j|^2 \right)^{1/2} \right\|_{L^p} \]
\[ = C \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_{L^p} . \]
This proves the claim and therefore the theorem. \( \square \)

We conclude this part with an extension of Thm. 6.2 to \( n \)-dimensions.
The dyadic cubes in $\mathbb{R}^n$ can be seen as the cartesian product of dyadic intervals of the same size; e.g. $\chi_{[0,1)^n}(x) = \chi_{[0,1)}(x_1) \cdots \chi_{[0,1)}(x_n)$. A dyadic rectangle $R$ in $\mathbb{R}^n$ is the cartesian product of dyadic intervals in $I_{j_1}, \ldots, I_{j_n}$ of possibly different scales:

$$\chi_R(x) = \chi_{I_{j_1}}(x_1) \cdots \chi_{I_{j_n}}(x_n).$$

We consider the operator

$$(S^1 \chi^{\ast}) = \chi_{I_{j_1}}(\xi_1) \hat{f}(\xi),$$

and more generally,

$$(S_j \chi^{\ast})(\xi) = (S^1 \cdots S^n_j \chi^{\ast})(\xi) = \chi_{I_{j_1}}(\xi_1) \cdots \chi_{I_{j_n}}(\xi_n) \hat{f}(\xi).$$

**Corollary 6.6.** Let $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then there exists a positive constant $C_p > 0$ such that

$$\frac{1}{C_p} \|f\|_{L^p} \leq \left( \sum_{j \in \mathbb{Z}^n} |S_j f|^2 \right)^{1/2} \|f\|_{L^p} \leq C_p \|f\|_{L^p},$$

where $\{I_{j_1}, \ldots, I_{j_n}\}$ are decompositions of the real line $\mathbb{R}$ into dyadic intervals of (possibly different) scales $2^{k_1}, \ldots, 2^{k_n}$, resp.

**Proof.** Notice that, if we set $R_j$, the collection $\{R_j\}_{j \in \mathbb{Z}^n}$ is a decomposition of $\mathbb{R}^n$ into mutually disjoint dyadic rectangles.

The estimate from above follows at once from Thm. 6.4. The estimate from below follows instead by applying Thm. 6.2 to the operators $S_j^k$, $k = 1, 2, \ldots, n$.

### 6.3. The Marcinkiewicz multiplier theorem

We conclude with another multiplier theorem, the *Marcinkiewicz multiplier theorem*, which is tightly connected with the theory of product spaces.

The simplest form of the Mihlin-Hörmander condition is given by the inequality (5.11) for $k \geq \lceil n/2 \rceil + 1$. We now consider the condition

$$|\partial^\alpha m(\xi)| \leq C|\xi_1|^{-\alpha_1} \cdots |\xi_n|^{-\alpha_n}. \quad (6.7)$$

Notice that when $n = 1$ it coincides with the Mihlin-Hörmander condition. But, when $n > 1$ the two conditions are different. In the theorem we give a more general condition for the $L^p$-boundedness of the multiplier operator. It is easy to see that the condition is satisfied if (6.7) holds, for $k \geq n$.

**Definition 6.7.** Let $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$. We define the product Sobolev space $H^s$ as the space of tempered distributions $f$ such that

$$\int_{\mathbb{R}^n} (1 + |\xi_1|^2)^{s_1} \cdots (1 + |\xi_n|^2)^{s_n} |\hat{f}(\xi)|^2 \, d\xi_1 \cdots d\xi_n < \infty.$$

Similarly to Lemma 5.15 one can easily prove the following.

**Lemma 6.8.** Let $g \in S'$ be such that $\hat{g} \in H^s$ with $s_i > 1/2$ for $i = 1, \ldots, n$. Then $g \in L^1(\mathbb{R}^n)$ and for $0 < \varepsilon < s_i - 1/2$ for all $i$

$$\int_{\mathbb{R}^n} (1 + |x_1|^2)^{\varepsilon} \cdots (1 + |x_n|^2)^{\varepsilon} |g(x)| \, dx \leq C_\varepsilon \|\hat{g}\|_{H^s}.$$
Definition 6.9. For $i = 1, \ldots, n$, let $\eta_i \in C^\infty(\mathbb{R})$ be a function as in the hypothesis of Lemma 4.23 and set $\eta(\xi) = \eta_1(\xi_1) \cdots \eta_n(\xi_n)$. For $\mathbf{r} = (r_1, \ldots, r_n) \in (\mathbb{R}^+)^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ we also set $\mathbf{r} \cdot \xi = (r_1 \xi_1, \ldots, r_n \xi_n)$.

A tempered distribution $m$ is said to be a Marcinkiewicz multiplier if

$$\sup_{\mathbf{r} \in (\mathbb{R}^+)^n} \|m(\mathbf{r} \cdot \eta)\|_{H^s} =: \|m\|_{M^s} < \infty.$$ 

Theorem 6.10. (Marcinkiewicz multiplier theorem) Let $m$ be a Marcinkiewicz multiplier in $H^s$, with $s_i > 1/2$ for $i = 1, \ldots, n$. Then $m$ is a bounded Fourier multiplier on $L^p$ for $1 < p < \infty$, that is, $m \in \mathcal{M}_p$ for $1 < p < \infty$.

The next result follows easily from the theorem.

Corollary 6.11. Let $m \in L^\infty(\mathbb{R}^n)$ be such that there exists a constant $B > 0$ such that for all $0 < k \leq n$

$$\sup_{I_{j_1} \times \cdots \times I_{j_k}} \int_{I_{j_1} \times \cdots \times I_{j_k}} |\partial_{\xi_{j_1}}^{k} \cdots \partial_{\xi_{j_k}}^{k} m(\xi)| \, d\xi_{j_1} \cdots d\xi_{j_k} \leq B,$$

where $I_{j_i}, i = 1, \ldots, k$, are dyadic intervals in $\mathbb{R}$.
Appendix A. Some results on $L^p$ spaces

We begin by recalling a few basic facts about the $L^p$ spaces in $\mathbb{R}^n$.

For proofs and further details of the facts and ideas presented here, we refer to [WZ], or also [Fo] and [Ru], for example.

We denote by $L^p = L^p(\mathbb{R}^n)$ the space of Lebesgue measurable functions, which are $p$-integrable functions, if $0 \leq p < \infty$, and for $p = \infty$ essentially bounded. These norms are

$$
\|f\|_p = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}
$$

$$
\|f\|_\infty = \inf_{\alpha \geq 0} \{ x \in \mathbb{R}^n : |f(x)| > \alpha \} = 0.
$$

It is clear that $L^p$ are vector spaces, since

$$
|f + g|^p \leq 2^p \left[ |f|^p + |g|^p \right].
$$

In order to prove that $\| \cdot \|_p$ is indeed a norm we need to prove the triangle inequality. We begin by showing that when $0 < p < 1$ instead the triangle inequality fails. For, if $A$ and $B$ are disjoint sets of positive, finite measure, then, since $0 < p < 1$,

$$
\|\chi_A + \chi_B\|_p = \left( |A| + |B| \right)^{1/p} > |A|^{1/p} + |B|^{1/p} = \|\chi_A\|_p + \|\chi_B\|_p.
$$

We now prove some basic inequalities concerning the $L^p$-norms. For $p \geq 1$ we denote by $p'$ its conjugate exponent (i.e. $1/p + 1/p' = 1$, which gives $p' = p/(p - 1)$ if $p > 1$, $p = 1$ and $p' = \infty$ being conjugate exponents).

**Theorem A.1.** (Hölder’s inequality) Let $1 < p < \infty$, and let $p'$ be its conjugate exponent. If $f$ and $g$ are measurable functions, we have

$$
\|fg\|_1 \leq \|f\|_p \|g\|_{p'}.
$$

Equality holds if and only if $\alpha |f|^p = \beta |g|^{p'}$ for some constants $\alpha$ and $\beta$.

**Proof.** The result is obvious if either $\|f\|_p = 0$ or $\|g\|_{p'} = 0$. In fact, in this case either $f = 0$ a.e. or $g = 0$ a.e., which gives $fg = 0$ a.e.

The result is also clear if either $\|f\|_p = \infty$ or $\|g\|_{p'} = \infty$.

Next we claim that for $a, b \geq 0$ and $0 < \lambda < 1$ then

$$
a^{1-\lambda}b^{\lambda} \leq \lambda a + (1 - \lambda)b. \tag{A.1}
$$

For, suppose $b = 0$, then the inequality holds. If $b \neq 0$, setting $t = a/b$, inequality (A.1) becomes

$$
t^{\lambda} \leq \lambda t + (1 - \lambda).
$$

Let $\varphi(t) = t^{\lambda} - \lambda t$, we obtain $\varphi'(t) = \lambda(t^{\lambda - 1} - 1)$, which is $> 0$ for $0 < t < 1$ and $< 0$ when $t > 1$. Thus, the maximum is attained when $t = 1$ with value $1 - \lambda$. This proves (A.1) and equality holds only if $t = 1$, i.e. $a = b$.

Now suppose that both $\|f\|_p$ and $\|g\|_{p'}$ are different from 0. We use the claim above with

$$
a = \left( |f(x)|/\|f\|_p \right)^p, \quad b = \left( |g(x)|/\|g\|_{p'} \right)^{p'} \quad \text{and} \quad \lambda = 1/p.
$$

We obtain

$$
\frac{|f(x)| |g(x)|}{\|f\|_p \|g\|_{p'}} \leq \frac{1}{p} \frac{|f(x)|}{\|f\|_p} + \frac{1}{p'} \frac{|g(x)|}{\|g\|_{p'}}.
$$
Integrating both sides of the inequality gives
\[ \frac{1}{\|f\|_p \|g\|_{p'}} \int |f(x)||g(x)| \, dx \leq \frac{1}{p} + \frac{1}{p'} = 1. \]
This proves the inequality. Finally equality holds if and only if \( a = b \), i.e.
\[ \left( \frac{|f(x)|}{\|f\|_p} \right)^p = \left( \frac{|g(x)|}{\|g\|_{p'}} \right)^{p'}, \]
that is \( \alpha |f|^p = \beta |g|^{p'} \). \( \square \)

**Theorem A.2.** (Minkowski’s inequality) Let \( 1 \leq p \leq \infty \), and let \( p' \) be its conjugate exponent. If \( f \) and \( g \) are measurable functions, we have
\[ \|f + g\|_p \leq \|f\|_p + \|g\|_{p'}. \]

**Proof.** The result is clear if \( p = 1 \) or \( p = +\infty \), and also if \( f + g = 0 \).
Suppose that \( 1 < p < \infty \), and \( f + g \neq 0 \) a.e. Then
\[ |f + g|^p \leq (|f| + |g|)^p |f + g|^{p-1} \]
and using Hölder’s inequality with conjugate exponents \( p \) and \( p' \) we obtain
\[ \int |f + g|^p \, dx \leq \|f\|_p \|f + g|^{p-1} \|g\|_{p'} + \|g\|_p \|f + g|^{p-1} \|f\|_{p'} \]
\[ = \left( \|f\|_p + \|g\|_p \right) \|f + g|^{p-1} \|f\|_{p'} \]
\[ = \left( \|f\|_p + \|g\|_p \right) \left( \int |f + g|^p \, dx \right)^{1/p'}. \]
Since \( 1/p' = 1 - 1/p \) it follows that
\[ \|f + g\|_p \leq \|f\|_p + \|g\|_{p'}. \] \( \square \)

**Theorem A.3.** For \( 1 \leq p \leq +\infty \), \( L^p \) is a Banach space.

**Proof.** We need to prove the completeness. We begin with the case \( p = +\infty \). Let \( \{f_k\} \) be a Cauchy sequence in \( L^\infty \). Then
\[ |f_k(x) - f_m(x)| \leq \|f_k - f_m\|_\infty \]
for all \( x \) outside a set \( E_{k,m} \) of measure 0. If \( E = \bigcup_{k,m=1}^\infty E_{k,m} \), then \( |E| = 0 \) and
\[ |f_k(x) - f_m(x)| \leq \|f_k - f_m\|_\infty \]
for all \( x \notin E \).
Thus, \( \{f_k\} \) converges uniformly to a bounded limit \( f \) outside a set of measure 0, i.e. \( \|f_k - f\|_\infty \to 0 \).

Now let \( 1 \leq p < \infty \). We use the following characterization of Banach spaces: A normed space \( X \) is complete if and only if every series converging absolutely it converges in norm (that is, if \( \sum_k \|x_k\|_X < \infty \) then there exists \( x \in X \) such that \( \lim_{N \to +\infty} \| \sum_{k=1}^N x_k - x \|_X = 0 \)).
Let \( \{f_k\} \) be a sequence of functions in \( L^p \) such that \( \sum_k \|f_k\|_{L^p} = C < \infty \). Let \( g_N = \sum_{k=1}^N |f_k| \) and \( g = \sum_{k=1}^\infty |f_k| \). By the monotone convergence theorem
\[ \|g\|^p_p = \lim_{N \to +\infty} \int |g_N(x)|^p \, dx \leq C^p. \]
Then \( g \in L^p \), and in particular \( g(x) < \infty \) a.e. This implies that the series \( \sum_k f_k \) converges a.e. to a function \( f \). For such a function, \( |f| \leq g \), so that \( f \in L^p \). Finally,

\[ |f - \sum_{k=1}^N f_k|^p \leq (g + g)^p = 2^p g^p, \]

so that by the dominated convergence theorem

\[ \lim_{N \to +\infty} \left\| f - \sum_{k=1}^N f_k \right\|_p = \lim_{N \to +\infty} \int |f(x) - \sum_{k=1}^N f_k(x)|^p \, dx = 0, \]

that is \( \sum_k f_k \) converges to \( f \) in the \( L^p \)-norm. \( \boxdot \)

**Proposition A.4.** For \( 1 \leq p < \infty \), the set of simple functions \( \{f : f = \sum_{k=1}^N \alpha_k \chi_{E_k}, \text{ with } |E_k| < \infty, \text{ for all } k\} \) is dense in \( L^p \).

**Proof.** We first show that if \( f \) is measurable, then there exists a sequence of simple functions \( s_m \) such that \( |s_1| \leq |s_2| \leq \cdots \leq |f|, s_m \to f \) pointwise, and \( s_m \to f \) uniformly on any set where \( f \) is bounded. By writing \( f = f_+ - f_- \) we may assume that \( f \geq 0 \).

Let \( m \) be a positive integer and let \( 0 \leq k \leq 2^{2m} - 1 \). It suffices to define

\[ E_k^m = f^{-1}\left((k2^{-m}, (k + 1)2^{-m})\right), \quad \text{and} \quad F_m = f^{-1}\left((2^m, +\infty]\right), \]

and set

\[ s_m = \sum_{k=0}^{2^{2m} - 1} k2^{-m} \chi_{E_k^m} + 2^m \chi_{F_m}. \]

It is not difficult to check that \( 0 \leq s_m \leq s_{m+1} \leq f \) for all \( m = 1, 2, \ldots \) and that \( 0 \leq f - s_m \leq 2^{-m} \) on the set where \( f \leq 2^{m} \). The claim now follows.

Now, let \( f \in L^p \), and let \( \{s_m\} \) be a sequence of simple measurable functions converging pointwise a.e. to \( f \), \( |s_m| \leq f \). Then, \( s_m \in L^p \) for all \( m \) and \( |f - s_m|^p \leq 2^p|f|^p \), which is a function in \( L^1 \). Then, by the dominated converge theorem, \( \|f - s_m\|_p \to 0 \) as \( m \to +\infty \). \( \boxdot \)

**Proposition A.5.** For \( 1 \leq p < \infty \) the set of continuous functions with compact support \( C_c \) is dense in \( L^p \).

**Proof.** Since the simple functions are dense in \( L^p \), it suffices to show that for any measurable set \( E \), with \( |E| < \infty \), we can approximate \( \chi_E \) in the \( L^p \)-norm with compactly supported, continuous functions. Given any measurable set \( E \), \( |E| < \infty \), we can find a (so-called Borel set) \( E' \), with \( E' \subseteq E \), \( |E \setminus E'| = 0 \), such that for any \( \varepsilon > 0 \) there exist an open set \( U \) and a compact set \( K \) such that

\[ K \subseteq E' \subseteq U, \quad |U \setminus K| < \varepsilon. \]

Now, by Urysohn’s lemma, we can find \( f \in C_c \) such that \( \chi_K \leq f \leq \chi_U \). Therefore,

\[ \|\chi_E - f\|_p = \|\chi_{E'} - f\|_p \leq \|U \setminus K\|^{1/p} \leq \varepsilon^{1/p}. \]

Next we wish to describe the dual space of \( L^p \), when \( 1 \leq p < \infty \).

**Theorem A.6.** (Reverse Hölder’s inequality) Let \( g \) be a measurable function, \( 1 \leq p < \infty \). Then

\[ \|g\|_{L^q} = \sup \left\{ \int f g \, dx : \|f\|_p = 1 \right\}, \]
Proof. Set $M(g) = \sup \{ |\int fg\,dx| : \|f\|_p = 1 \}$. From Hölder’s inequality we know that, if $g \in L^{p'}$,

$$|\int fg\,dx| \leq \|f\|_p \|g\|_{p'}$$

that is $M(g) \leq \|g\|_{p'}$. The statement holds true if $g = 0$ a.e. If $g \neq 0$ and $p' < \infty$, select

$$f = \text{sgn} \frac{|g|}{\|g\|_{p'}}^{p'-1}$$

where $(\text{sgn} g)(x) = g(x)/|g(x)|$ if $g(x) \neq 0$ and equals 0 if $g(x) = 0$. Then,

$$\|f\|_p^p = \frac{1}{\|g\|_{p'}^{p(p'-1)}} \int |g(x)|^{p(p'-1)}\,dx$$

$$= 1$$

while

$$M(g) \geq |\int fg\,dx| = \frac{1}{\|g\|_{p'}^{p(p'-1)}} \int |g(x)|^{p'}\,dx$$

$$\|g\|_{p'}.$$  

Finally, if $p' = \infty$, given any $\varepsilon > 0$, let $E$ be a set such that

$$E \subseteq \{x : |g(x)| > \|g\|_\infty - \varepsilon\}, \quad 0 < |E| < \infty.$$  

Select

$$f = \frac{1}{|E|} \text{sgn} g \chi_E.$$  

Then

$$M(g) \geq |\int fg\,dx| = \frac{1}{|E|} \int_E |g(x)|\,dx \geq \|g\|_\infty - \varepsilon.$$  

This proves the theorem. \qed

Theorem A.7. (Duality of the $L^p$ spaces) For $1 \leq p < \infty$ the dual of $L^p$ is $L^{p'}$, with equatity of norms.

Proof. By Hölder’s inequality is clear that any $f \in L^{p'}$ gives rise to a bounded linear functional $L_f$ on $L^p$, $1 \leq p < \infty$. Conversely, let $L$ be a bounded linear functional on $L^p$, $1 \leq p < \infty$. We wish to find an $f \in L^{p'}$ such that

$$L(g) = \int g \,f\,dx,$$

that is, $L = L_f$. The construction of such an $f$ goes a little beyond the scope of these notes and lectures. Therefore, we refer to [Fo], or [WZ] for a proof. \qed

We conclude this section with a few further inequalities.

Theorem A.8. (Minkowski’s integral inequality) Let $1 \leq p \leq \infty$, $f$ a measurable function defined on the product space $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Suppose that $f(\cdot, y) \in L^p$ for a.e. $y$ and that the function $y \mapsto \|f(\cdot, y)\|_p \in L^1$. Then the function $f(x, \cdot) \in L^1$ for a.e. $x$, the function $x \mapsto \int f(x, y)\,dy \in L^p$, and

$$\| \int f(\cdot, y)\,dy \|_p \leq \int \|f(\cdot, y)\|_p\,dy.$$
Proof. When \( p = 1 \) this is just Fubini’s theorem on iterated integrals. Let \( 1 < p < \infty \), then we use the duality between \( L^p \) spaces. Let \( p' \) be the conjugate exponent of \( p \), and \( g \in L^{p'} \). We have then

\[
| \left( \int f(x, y) \, dy \right) g(x) \, dx | \leq \int \left( \int | f(x, y) |^p \, dx \right)^{1/p} \| g \|_{p'} \, dy
\]

\[
= \| g \|_{p'} \int \| f(\cdot, y) \|_p \, dy.
\]

By taking the supremum over \( g \in L^{p'} \), \( \| g \|_{p'} = 1 \), the statement follows. \( \square \)

Theorem A.9. (Chebyshev’s inequality) Let \( 1 \leq p < \infty \). Then, for any \( \lambda > 0 \)

\[
| \{ x : | f(x) | > \lambda \} | \leq \left( \frac{\| f \|_p}{\lambda} \right)^p.
\]

Proof. Define \( E_\lambda = \{ x : | f(x) | > \lambda \} \). We have

\[
\| f \|_p^p = \int | f |^p \, dx \geq \int_{E_\lambda} | f |^p \, dx \geq \lambda^p \int_{E_\lambda} \, dx = \lambda^p | E_\lambda |.
\]

This gives the desired inequality. \( \square \)

Appendix B. Some exercises

1. Evaluate the following limits, justifying your answers:

\[
\lim_{k \to +\infty} \int_0^k \left( 1 - \frac{x}{k} \right)^k e^{x/2} \, dx \quad \lim_{k \to +\infty} \int_0^k \left( 1 + \frac{x}{k} \right)^k e^{-2x} \, dx.
\]

2. Prove Lemma 1.1.

3. (Measure of the ball and sphere in \( \mathbb{R}^n \).) Recall that for \( a, b > 0 \),

\[
\int_{\mathbb{R}^n} e^{-a|x|^2} \, dx = (\pi/a)^{n/2}, \quad \text{and} \quad \int_0^1 (1 - s)^{a-1} s^{b-1} \, ds = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)},
\]

(where the Euler’s Gamma function is defined as \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \)).

Use these facts and integration in polar coordinates to compute the surface measure of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \) and the volume of the ball of radius \( r \), \( B_r \):

\[
\sigma(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \quad |B_r| = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}.
\]

4. (Generalized Hölder’s inequality.) Show that if \( f_j \in L^{p_j}, j = 1, \ldots, m \) and \( \sum_{j=1}^m 1/p_j = 1 \) then \( \prod_{j=1}^m f_j \in L^1 \) and

\[
\int | \prod_{j=1}^m f_j(x) | \, dx \leq \prod_{j=1}^m \| f_j \|_{p_j}.
\]
5. (Differentiation under the integral sign.) Let \( f : \mathbb{R}^n \times [a, b] \) be measurable and denote by \((x, y)\) the variables in \( \mathbb{R}^n \times \mathbb{R} \). Suppose \( \partial_y f \) exists and suppose that there exists \( g \in L^1(\mathbb{R}^n) \) such that \(|\partial_y f(x, y)| \leq |g(x)|\) for all \((x, y) \in \mathbb{R}^n \times [a, b]\). Show that

\[
F(y) = \int_{\mathbb{R}^n} f(x, y) \, dx
\]

is differentiable and that \(F'(y) = \int_{\mathbb{R}^n} \partial_y f(x, y) \, dx\). Use this result to prove Theorem 1.3 (iii).

6. Show that the translation is a continuous operation in \( L^p \) when \( 1 \leq p < \infty \), but it is not continuous for \( p = \infty \).

7. Let \( f, g \in L^2 \). Show that both sides of the equality \( \mathcal{F}(f * g) = \hat{f} \hat{g} \) are well defined, and prove such an equality.

8. Let \( 1 \leq p < \infty \), \( f \in L^p \) such that \( \partial_\alpha x f \in L^p \) for \(|\alpha| \leq N\). Show that there exists a sequence of functions \( \{\varphi_k\} \) in \( C_0^\infty \) such that \( \partial_\alpha x \varphi_k \to \partial_\alpha x f \) in \( L^p \), for all \(|\alpha| \leq N\).

9. Prove that \( \mathcal{F}(\varphi)(x) = e^{-2\pi|\xi|} \), where \( \varphi(x) = \omega_n (1+|x|^2)^{-(n+1)/2} \) and \( \omega_n = \Gamma((n+1)/2)\pi^{-(n+1)/2} \), by proving: (i) for \( \beta \geq 0 \),

\[
e^{-\beta} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\beta t}}{1+t^2} \, dt;
\]

(ii) for \( \beta \geq 0 \),

\[
e^{-\beta} = \int_0^{+\infty} \frac{e^{-\beta s} e^{-s^2/4s}}{\sqrt{\pi s}} \, ds;
\]

(iii) set \( \beta = 2\pi|\xi| \) in (ii) and use Lemma 1.13.

10. Let \( G \) be an invertible linear transformation of \( \mathbb{R}^n \) and let \( g = f \circ G \), for \( f \in L^1 \). Express \( \hat{g} \) in terms of \( \hat{f} \) (that is, prove that

\[
\hat{g}(\xi) = \frac{1}{|\det G|} \hat{f} \left(G^{-1}(\xi) \right).
\]

11. Let \( \delta \) denote the Dirac delta at the origin, that is the distribution \( \langle \delta, \varphi \rangle = \varphi(0) \). Prove that \( \partial_\alpha \delta(\xi) = (2\pi i)^{\alpha} \) and that \( \langle x^\alpha \rangle = (-2\pi i)^{-\alpha} \partial_\alpha \delta \).

12. On \( \mathbb{R}^1 \), let \( \Phi \) be the distribution defined as

\[
\langle \Phi, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{1}{\pi x} \varphi(x) \, dx.
\]

Show that the equality above defines indeed a distribution and show that the limit equals

\[
\int_{|x| \leq 1} \frac{\varphi(x) - \varphi(0)}{\pi x} \, dx + \int_{|x| > 1} \frac{\varphi(x)}{\pi x} \, dx.
\]

Moreover, prove that \( \hat{\Phi} = -i\text{sgn} \xi \). Call \( g = \hat{\Phi} \). Prove that \( f \mapsto f * g \) extends to a unitary operator on \( L^2 \) (called the Hilbert transform).
REFERENCES


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