A FEW CHAPTERS FOR THE COURSE *HARMONIC ANALYSIS*

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Rispetto alla versione ha.pdf solo la dimostrazione del Teorema 3.2 ha subito una modifica sostanziale.

1. The Hilbert transform

1.1. Preliminary facts and notation. We briefly recall some preliminary facts that can be found, together with their proofs in Ch. 2 of [Soa].

For \( f \in L^1(\mathbb{R}^n) \) we defined the Fourier transform of \( f \) as

\[
\hat{f}(\xi) = \int f(x)e^{-2\pi i x \cdot \xi} \, dx.
\]

If \( f \) and \( \hat{f} \) are in \( L^1 \), then we have the inversion formula,

\[
f(x) = \int e^{2\pi i x \cdot \xi} \hat{f}(\xi) \, d\xi.
\]

Then, if \( f \in L^1 \) and \( \hat{f} = 0 \), then \( f = 0 \).

Given a function \( f \) on \( \mathbb{R}^n \) and \( t > 0 \) we define

\[
f_t(x) = t^{-n}f(x/t) \quad \text{and} \quad f^t(x) = f(tx).
\]

(1.1)

Notice that, if \( f \in L^1 \) then \( \int f_t(x) \, dx = \int t^{-n}f(x/t) \, dx = \int f(y) \, dy \). Moreover, we have that

\[
\hat{f}_t(\xi) = \hat{f}(t\xi) = \hat{f}^t(\xi) \quad \text{and} \quad \hat{f}^t(\xi) = t^{-n}\hat{f}(\xi/t) = (\hat{f})_t(\xi),
\]

(1.2)

see Thm.’s 2.11 and 2.13 in [So].

A space of functions of fundamental importance is the Schwartz space \( S(\mathbb{R}^n) \) (often denoted as \( S \)). We denote by \( \rho(\alpha,\beta) \) the seminorm

\[
\rho(\alpha,\beta)(f) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|
\]

and define

\[
S = \{ f \in C^\infty(\mathbb{R}^n) : \rho(\alpha,\beta)(f) < \infty, \ \text{for all multi-indices} \ \alpha,\beta \}.
\]

From the inversion formula it also follows that the Fourier transform \( \mathcal{F} : S \to S \) is a continuous bijection (see the next subsection for the question of the topology on \( S \)).

We remark that we have the Parseval formula: if \( f, g \in L^1 \),

\[
\int f(x)\hat{g}(x) \, dx = \int \int f(x)\hat{g}(\xi)e^{-2\pi i x \cdot \xi} \, dx d\xi = \int \hat{f}(\xi)\hat{g}(\xi) \, d\xi.
\]

(1.3)

Finally, we can extend the Fourier transform to a unitary operator on \( L^2 \), and obtain the Plancherel theorem.

**Theorem 1.1.** (Plancherel Theorem) Let \( f \in L^1 \cap L^2 \). Then \( \hat{f} \in L^2 \) and \( \mathcal{F}_{|L^1 \cap L^2} \) extends uniquely to a unitary isomorphism of \( L^2 \).

1.2. The Schwartz space and the space of tempered distributions. We now recall a few basic facts about seminormed linear spaces.\(^1\)

Let \( \mathcal{V} \) be a complex linear space. A function \( g : \mathcal{V} \to [0, +\infty) \) is called a seminorm if

(a) \( g(\lambda v) = |\lambda| v \) for all \( v \in \mathcal{V} \) and \( \lambda \in \mathbb{C} \);

(b) \( g(v_1 + v_2) \leq g(v_1) + g(v_2) \) for all \( v_1, v_2 \in \mathcal{V} \).

\(^1\)For proofs and more on this topic, see G. B. Folland, *Real Analysis, Modern Techniques and Their Applications, 2 Ed.*
Notice that if we also require that $\varrho(v) = 0$ implies $v = 0$, then $\varrho$ would be a norm.

Let $\mathcal{V}$ be a linear space on which there exists a family $\{\varrho_\alpha\}_{\alpha \in A}$ of seminorms with the following property: For each pair of points $v_1, v_2$ in $\mathcal{V}$ there exists $\varrho_\alpha$ such that $\varrho_\alpha(v_1) \neq \varrho_\alpha(v_2)$. In this case we say that the family of seminorms separates the points in $\mathcal{V}$.

A linear space $\mathcal{V}$ on which there exists a family $\{\varrho_\alpha\}_{\alpha \in A}$ of seminorms that separates the points is called a seminormed space.

On a seminormed space $\mathcal{V}$ with seminorms $\{\varrho_\alpha\}_{\alpha \in A}$ we define a topology by defining a system of open sets

$$U_{x,\alpha,\varepsilon} = \{y \in \mathcal{V} : \varrho_\alpha(x - y) < \varepsilon\}.$$

Let $\mathcal{V}$ be a (complex) seminormed space and suppose that admits a countable family $\mathcal{P}$ of seminorms $\{\varrho_k\}_k$. Let $\tau_\mathcal{P}$ be the topology defined above, which also is the coarsest topology that make continuous the identity

$$i : (\mathcal{V}, \tau_\mathcal{P}) \to (\mathcal{V}, \varrho_k)$$

for all $k \in \mathbb{N}$.

We say that $\mathcal{P}$ is separating if for each $v \in \mathcal{V}$ there exists $k \in \mathbb{N}$ such that $\varrho_k(v) > 0$. We will always assume that $\mathcal{P}$ is separating.

**Proposition 1.2.** With the above notation, the topology $\tau_\mathcal{P}$ enjoys the following properties.

(i) The space $(\mathcal{V}, \tau_\mathcal{P})$ is a locally convex Hausdorff space.

(ii) The finite intersections of the sets

$$B_{k,n} = \{v : \varrho_k(v) < 1/n\}$$

form a fundamental system of neighborhoods of 0.

(iii) The topology $\tau_\mathcal{P}$ is induced by the metric distance

$$d_\mathcal{P}(v, w) = \sum_k \frac{\varrho_k(v - w)}{2^k 1 + \varrho_k(v - w)}.$$

(iv) A sequence $\{v_n\}$ in $\mathcal{V}$ converges to $v$ in the $\tau_\mathcal{P}$ topology if and only if it converges to $v$ in all the seminorms $\varrho_k$, that is if

$$\lim_{n \to +\infty} \varrho_k(v - v_n) = 0$$

for all $k \in \mathbb{N}$.

(v) A linear functional $T$ on $\mathcal{V}$, that is, a linear operator from $\mathcal{V}$ to $\mathbb{C}$, is continuous in the $\tau_\mathcal{P}$ topology if and only if there exist seminorms $\varrho_{k_1}, \ldots, \varrho_{k_N}$ and constant $C > 0$ such that

$$|T(v)| \leq C \sum_{j=1}^N \varrho_{k_j}(v)$$

for all $v \in \mathcal{V}$.

The distance $d_\mathcal{P}$ is invariant, that is

$$d_\mathcal{P}(v + z, w + z) = d_\mathcal{P}(v, w)$$

for all $v, w, z \in \mathcal{V}$.

We also recall that if $T$ a linear operator between two linear normed spaces $\mathcal{X}, \mathcal{Y}$, then $T$ is continuous if and only if it is bounded, that is there exists a constant $C > 0$ such that

$$\|Tx\|_\mathcal{Y} \leq C\|x\|_\mathcal{X}.$$
for all $x \in \mathcal{X}$. The proof of this fact is simple. If $T$ linear is bounded, then it is Lipschitz, with Lipschitz constant less or equal to $C$:

$$\|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y \leq C\|x_1 - x_2\|_X,$$

which of course implies continuity. Conversely, if it continuous, the inverse image of the ball of radius $r > 0$ is contained in the ball of radius $R > 0$, so that, for some $R' < R$, and all $x$ with $\|x\|_X \leq R'$,

$$\|Tx\|_Y \leq r.$$

But then, for all $x \in \mathcal{X}$,

$$\|Tx\|_Y = \left\| \frac{\|x\|_X}{R'} T \left( \frac{R'x}{\|x\|_X} \right) \right\|_Y \leq \frac{\|x\|_X}{R'} r.$$

As a consequence of the results discussed so far, we have the following proposition involving the continuity (or boundedness) of a linear operator acting between seminormed spaces.

**Proposition 1.3.** Let $\mathcal{V}, \mathcal{W}$ be seminormed linear spaces admitting family of seminorms $\{\varrho_\alpha\}$, $\{\sigma_\beta\}$, resp. Let $T : \mathcal{V} \to \mathcal{W}$ be a linear mapping. Then $T$ is continuous if and only if, for each seminorm $\sigma_\beta$ on $\mathcal{W}$ there exist seminorms $\varrho_{\alpha_1}, \ldots, \varrho_{\alpha_N}$ and a constant $C = C_{\beta, \alpha_1, \ldots, \alpha_N} > 0$ such that for all $v \in \mathcal{V}$ we have

$$\sigma_\beta(Tv) \leq C \sum_{j=1}^N \varrho_{\alpha_j}(v).$$

**Proposition 1.4.** The space $\mathcal{S}$ is a complete metric space (i.e. a Fréchet space) in the topology defined by the seminorms $\rho(\alpha, \beta)$.

**Proof.** Using (iii) of the previous Prop. 1.2, we only need to prove that $\mathcal{S}$ is complete in the given topology.

Let $\{f_k\}$ be a Cauchy sequence in $\mathcal{S}$, then $\rho(\alpha, \beta)(f_j - f_k) \to 0$ as $j, k \to +\infty$ for all $(\alpha, \beta)$. In particular, $\{\partial^\beta f_k\}$ converges uniformly to a function $g_\beta$ for all $\beta$. Denote by $e_j$ the $j$-th element of the canonical basis in $\mathbb{R}^n$. Notice that

$$g_0(x + te_j) - g_0(x) = \lim_{k \to +\infty} f_k(x + te_j) - f_k(x)$$

$$= \lim_{k \to +\infty} \int_0^t \partial_{x_j} f_k(x + se_j) \, ds$$

$$= \int_0^t g_{e_j}(x + se_j) \, ds.$$ 

Therefore, $\partial_{x_j}g_0 = g_{e_j}$, and by induction on $|\beta|$ we obtain that $g_\beta = \partial^\beta g_0$. Finally,

$$|x^n| |\partial^\beta f_k(x) - \partial^\beta f_k(x)| = \lim_{j \to +\infty} |x^n| |\partial^\beta f_k(x) - \partial^\beta f_j(x)|$$

$$\leq \lim_{j \to +\infty} \rho(\alpha, \beta)(f_k - f_j)$$

$$\leq \varepsilon,$$

for $j, k \geq k_0$. Then $\rho(\alpha, \beta)(f_k - g_0) \to 0$ as $k \to +\infty$. \qed

We define the space $\mathcal{S}'$ of tempered distributions as the dual space of $\mathcal{S}$. We endow $\mathcal{S}'$ with the weak* topology, that is, the weakest topology that makes the elements of $\mathcal{S}'$ continuous.
There are many noticeable examples of tempered distributions. We now see some examples. We begin by observing that for every positive integer $N$ there exists a constant $C = C_N > 0$ such that for all $x \in \mathbb{R}^n$ we have

\[
\frac{1}{C} \sum_{|\alpha| \leq N} |x^\alpha| \leq (1 + |x|)^N \leq C \sum_{|\alpha| \leq N} |x^\alpha|.
\]

(We leave the simple proof as an exercise.) Hence, $f \in \mathcal{S}$ if and only if $f$ is $C^\infty$ and for all non-negative integers $N$ and multi-indices $\beta$

\[
\sup_{x \in \mathbb{R}^n} (1 + |x|)^N|\partial_\beta^\alpha f(x)| < \infty.
\]

**Examples 1.5.** (i) Functions in $\mathcal{S}$, in any $L^p$ class, $1 \leq p \leq \infty$ give rise to bounded linear functionals on $\mathcal{S}$, by setting:

\[
L_f(\varphi) = \int f \varphi \, dx \quad \varphi \in \mathcal{S}, \ f \in L^p \ (\text{or } \mathcal{S}).
\]

For,

\[
|L_f(\varphi)| \leq \int |f| \varphi \, dx \leq \|f\|_{L^p} \|\varphi\|_{L^{p'}}
\]

\[
= \|f\|_{L^p} \left( \int_{\mathbb{R}^n} (1 + |x|)^{-Np'}(1 + |x|)^{Np'}|\varphi(x)|^{p'} \, dx \right)^{1/p'}
\]

\[
\leq C \|f\|_{L^p} \sup_x \{(1 + |x|)^N|\varphi(x)|\}
\]

\[
\leq C \sum_{|\alpha| \leq N} \rho(\alpha,0)(\varphi),
\]

dove $N$ è scelto $\geq n + 1$.

(ii) A function $f$ is called *tempered* if there exists $N > 0$ such that $(1 + |x|)^{-N} f \in L^1$. Then, the formula

\[
L_f(\varphi) = \int \varphi f \, dx
\]

defines a tempered distribution, since

\[
| \int \varphi f \, dx | \leq \sup_x \{(1 + |x|)^N|\varphi(x)|\} \int (1 + |x|)^{-N} |f| \, dx \leq C \sum_{|\alpha| \leq N} \rho(\alpha,0)(\varphi).
\]

Analogously, a Borel measure $\mu$ is called *tempered* if there exists $N > 0$ such that $\int (1 + |x|)^{-N} d|\mu| < \infty$. Then, the pairing

\[
L_\mu(\varphi) = \int \varphi d|\mu|
\]

defines a tempered distribution. Then, Dirac deltas are examples of tempered distributions.

Notice however, not every $C^\infty$ function defines a tempered distribution; for instance one with exponential growth.
Remark 1.6. On $S'$ we can introduce a few operations, besides the ones that define the vector space structure.

(i) Differentiation. For $u \in S'$ and $\alpha$ a multi-index we set

$$\partial^{\alpha} u(\phi) = u((-1)^{|\alpha|} \partial^{\alpha} \phi).$$

Notice that, if $u \in S \subset S'$, then $\partial^{\alpha} u \in S$ and therefore it defines again an element of $S'$ by the integral pairing, and integrating by parts the boundary terms equal 0 (since both $u$ and $\psi$ are Schwartz functions)

$$\int (\partial^{\alpha} u) \psi \, dx = - \int (\partial^{\alpha-\epsilon_1} u)(\partial_{x-1} \psi) \, dx = (-1)^{|\alpha|} \int u \partial^{\alpha} \psi \, dx.$$

Hence, the definition of derivative of a distribution coincides with the classical definition if the distribution admits classical derivatives.

(ii) Convolution with a Schwartz function $\varphi$. We first define the operator $\hat{\cdot}$ on functions by setting $\hat{\varphi}(x) = \varphi(-x)$.

$$(u \ast \varphi)(\psi) = u(\hat{\varphi} \ast \psi).$$

Notice that the latter is well defined since $\varphi \ast \psi \in S$ for $\varphi, \psi \in S$, and that this definition extends the case when also $u \in S$ that we obtain by switching the integration order:

$$(u \ast \varphi)(\psi) = \int \left( \int u(y) \varphi(x - y) \, dy \right) \psi(x) \, dx = \int u(y) \left( \int \varphi(x - y) \psi(y) \, dy \right) \, dx = \int u(y) (\hat{\varphi} \ast \psi)(y) \, dy.$$

(iii) Given $u \in S'$ we define the tempered distribution $\hat{u}$ by setting

$$\hat{u}(\psi) = u(\hat{\psi}),$$

for any $\psi \in S$. Observe that, using identity (1.3), this definition extends the definition of the Fourier transform given on $S$; that is, if $u$ above is in fact an element of $S$, then the equality above is justified by (1.3).

(iv) Let $f \in C^\infty$. We say that $f$ is of moderate growth if for each multi-index $\alpha$ there exist an integer $N$ and a positive constant $C$ such that

$$|\partial_x^{\alpha} f(x)| \leq C(1 + |x|)^N.$$ 

Clearly such a function defines an element of $S'$, and if $\varphi \in S$, then $f \varphi \in S$. Moreover, if $u \in S'$, we can define another element $fu$ of $S'$ by setting

$$fu(\varphi) = u(f \varphi).$$

(v) Thus, it follows from (i) and (iv) that if $u \in S'$

$$(\partial^{\alpha} u)(\varphi) = \partial^{\alpha} \hat{u} = (-1)^{|\alpha|} u(\partial^{\alpha} \hat{\varphi}) = (-1)^{|\alpha|} u((2\pi \xi)^{\alpha} \hat{\varphi}).$$
A consequence of these facts is that $S$ is dense in $S'$.

**Proposition 1.7.** If $\psi \in S$, $\int \psi = 1$ and $u \in S'$, then $u * \psi_t \to u$ in $S'$, as $t \to 0$. In particular, $S$ is dense in $S'$ in its topology. Moreover, $C_0^\infty$ is dense in $S$ and in $S'$, in the respective topologies.

**Proof.** One first shows that, for $\varphi \in S$, $\psi \varepsilon \varphi \to \varphi$ in $S$, as $\varepsilon \to 0$, where $\psi \varepsilon(x) = \psi(\varepsilon x)$. Therefore, $\psi_t * \varphi \to \varphi$ in $S$, as $t \to 0$. From these facts, the first statement follows.

For the density of $C_0^\infty$ in $S$, take $\psi$ above in $C_0^\infty$, and for the density in $S'$ take $(u * \psi_t) \theta_\varepsilon$, with $\theta \in C_0^\infty$, $\theta(0) = 1$. We leave the simple details to the reader. □

1.3. The Marcinkiewicz interpolation theorem. We recall that, given a measurable function $f$ on $\mathbb{R}^n$, its distribution function $\alpha_f$ is the function, defined for $\lambda \geq 0$

$$\alpha_f(\lambda) = \left| \left\{ x \in \mathbb{R}^n : |f(x)| > \lambda \right\} \right| .$$

(1.4)

Using Fubini’s theorem, it is easy to see that for $p > 0$

$$\|f\|_{L^p}^p = p \int_0^{+\infty} \lambda^{p-1} \alpha_f(\lambda) \, d\lambda .$$

(1.5)

We also observed that

$$\alpha_{f_1+f_2}(\lambda) \leq \alpha_{f_1}(\lambda/2) + \alpha_{f_2}(\lambda/2) ,$$

since

$$\{ x \in \mathbb{R} : |(f_1 + f_2)(x)| > \lambda \} \subseteq \{ x \in \mathbb{R} : |f_1(x)| > \lambda/2 \} \cup \{ x \in \mathbb{R} : |f_2(x)| > \lambda/2 \} .$$

An operator $T$ defined on measurable functions on $\mathbb{R}^n$ is said to be **sublinear** if

- $|T(f_0 + f_1)(x)| \leq |T(f_0)(x)| + |T(f_1)(x)|$;
- $|T(\lambda f)(x)| = |\lambda||Tf(x)|$;

for all $x \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$. We remark that, if $T$ is a linear operator, $|T|$ is sublinear. Moreover, a typical example of an operator that is genuinely sublinear is the Hardy–Littlewood maximal function $Mf(x) = \sup_{B(x,r),r>0} \frac{1}{|B|} \int_B |f(x)| \, dx$.

**Definition 1.8.** Given a sublinear operator $T$ defined on measurable functions on $\mathbb{R}^n$ is of weak-type $(p,q)$ if there exists a constant $C > 0$ such that

$$\alpha_{Tf}(\lambda) = \left| \left\{ x \in \mathbb{R}^n : |Tf(x)| > \lambda \right\} \right| \leq \left( \frac{C}{\lambda} \|f\|_{L^p} \right)^q ,$$

while it is said to be strong-type $(p,q)$ if

$$T : L^p \to L^q$$

is bounded.
It is easy to see that if $T$ is of strong-type $(p,q)$, then it is also of weak-type $(p,q)$. For,

$$
\alpha_T f (\lambda) = \int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}} dx \\
\leq \int_{\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}} \left( \frac{|Tf(x)|}{\lambda} \right)^q dx \\
\leq \frac{1}{\lambda^q} \|Tf\|^q_{L^q} \\
\leq \left( \frac{C}{\lambda} \|f\|_{L^p} \right)^q.
$$

**Theorem 1.9.** (Marcinkiewicz interpolation theorem) Let $T$ be a sublinear operator defined on $L^{p_0}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n)$, $1 \leq p_j, q_j \leq +\infty$, $j = 0, 1$. Suppose that $T$ is of weak-type $(p_0, q_0)$ and of weak-type $(p_1, q_1)$. Then, $T$ is of strong-type $(p,q)$, where $p$ and $q$ are given by the relations

$$
\frac{1}{p} = \frac{\theta}{p_0} + \frac{1 - \theta}{p_1}, \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{q_0} + \frac{1 - \theta}{q_1},
$$

and $0 < \theta < 1$.

**Proof.** We will prove the theorem only in the case $q_0 = p_0$ and $q_1 = p_1$. Then it suffices to assume that $p_0 < p < p_1$, without any reference to $\theta$. For the proof in the general case (that we need in Subsection 2.5) we refer the reader to [St].

Let $f \in L^p$ and $\lambda > 0$ be given. For a constant $c > 0$ to be selected later, we decompose $f$ as $f = f_0 + f_1$, where

$$f_0 = f \chi_{\{x : |f(x)| > c \lambda\}}, \quad \text{and} \quad f_1 = f \chi_{\{x : |f(x)| \leq c \lambda\}}.$$

Notice that $f_0 \in L^{p_0}$ and $f_1 \in L^{p_1}$, since $p_0 < p < p_1$. For,

$$
|f_0(x)|^{p_0} = |f(x)|^{p_0-p} |f(x)|^{p} \chi_{\{x : |f(x)| > c \lambda\}}(x) \\
< (c \lambda)^{p_0-p} |f(x)|^{p} \chi_{\{x : |f(x)| > c \lambda\}}(x),
$$

so that $f_0 \in L^{p_0}$. Analogous reasoning shows that $f_1 \in L^{p_1}$:

$$
|f_1(x)|^{p_1} \leq (c \lambda)^{p_1-p} |f(x)|^{p} \chi_{\{x : |f(x)| \leq c \lambda\}}(x).
$$

Then,

$$
|T(f_0 + f_1)(x)| \leq |T(f_0)(x)| + |T(f_1)(x)|
$$

and

$$
\alpha_{T(f_0 + f_1)}(\lambda) \leq \alpha_{Tf_0}(\lambda/2) + \alpha_{Tf_1}(\lambda/2).
$$
We distinguish two different cases. First assume that \( p_1 < +\infty \). Then,

\[
\| Tf \|_{L^p}^p = p \int_0^{+\infty} \lambda^{p-1} \alpha_{TF}(\lambda) \, d\lambda \\
\leq p \int_0^{+\infty} \lambda^{p-1} \alpha_{TF_0}(\lambda) \, d\lambda + p \int_0^{+\infty} \lambda^{p-1} \alpha_{TF_1}(\lambda/2) \, d\lambda \\
\leq p \int_0^{+\infty} \lambda^{p-1} \left( \frac{2A_0}{\lambda} \| f_0 \|_{L^{p_0}} \right)^{p_0} \, d\lambda + p \int_0^{+\infty} \lambda^{p-1} \left( \frac{2A_1}{\lambda} \| f_1 \|_{L^{p_1}} \right)^{p_1} \, d\lambda \\
\leq p(2A_0)^{p_0} \int_0^{+\infty} \lambda^{p-p_0-1} \int_{\{x : |f(x)| > c\lambda\}} |f(x)|^{p_0} \, dx \, d\lambda \\
+ p(2A_1)^{p_1} \int_0^{+\infty} \lambda^{p-p_1-1} \int_{\{x : |f(x)| \leq c\lambda\}} |f(x)|^{p_1} \, dx \, d\lambda \\
\leq p(2A_0)^{p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0} \int_{\{ f(x) |/c \}} \lambda^{p-p_0-1} \, d\lambda \, dx \\
+ p(2A_1)^{p_1} \int_{\mathbb{R}^n} |f(x)|^{p_1} \int_{\{ f(x) |/c \}} \lambda^{p-p_1-1} \, d\lambda \, dx \\
= \frac{(2A_0)^{p_0}}{(p-p_0)(c^{p-p_0})} + \frac{(2A_1)^{p_1}}{(p_1-p)(c^{p-p_1})} \| f \|_{L^p}^p.
\]

In the second case, that is, \( p_1 = +\infty \), then \( T : L^\infty \to L^\infty \) is bounded. We may choose \( c = 1/(2A_1) \), where \( A_1 \) is the \((L^\infty, L^\infty)\)-norm of \( T \), so that \( \alpha_{TF_1}(\lambda/2) = 0 \). Then, we have that

\[
\| Tf \|_{L^p}^p \leq p \int_0^{+\infty} \lambda^{p-1} \left( \frac{2A_0}{\lambda} \| f_0 \|_{L^{p_0}} \right)^{p_0} \, d\lambda \\
= p(2A_0)^{p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0} \int_{\{ f(x) |/c \}} \lambda^{p-p_0-1} \, d\lambda \, dx \\
= \frac{p}{p-p_0} (2A_0)^{p_0} (2A_1)^{p-p_1} \| f \|_{L^p}^p.
\]

This proves the theorem. □

1.4. The Calderón–Zygmund decomposition of an \( L^1 \)-function. We now introduce a fundamental decomposition of the whole space \( \mathbb{R}^n \), the dyadic decomposition. We define the unit cube to be the set \([0, 1)^n\) and the set \( Q_0 \) to be the family of sets obtained by translating by \( k e_j, k \in \mathbb{Z}, j = 1, \ldots, n \). Now we dilate this family by a factor \( 2^{-k} \), with \( k \in \mathbb{Z} \) and obtain a family \( Q_k \). The union of the families \( Q_k, k \in \mathbb{Z} \), is called the family of dyadic cubes in \( \mathbb{R}^n \).

It is immediate to see that these sets are cubes with sides parallel to the axes, that a cube in \( Q_k \) have vertices at adjacent points in the lattices \((2^{-k}\mathbb{Z})^n\). Moreover, the following properties are also easily checked.

1. For each \( k \) fixed, the cubes in \( Q_k \) are (mutually) disjoint and their union is all of \( \mathbb{R}^n \).
2. Given any two dyadic cubes, they are either disjoint, or one is contained in the other one.
3. Given \( j, k \in \mathbb{Z}, \) with \( j < k \), then each cube in \( Q_k \) is contained in a unique cube in \( Q_j \) and contains \( 2^{k+1} \) cubes in \( Q_{k+1} \).
The proof of Thm. 1.12 uses an important and far-reaching result, called the Calderón–Zygmund decomposition, that here we state and prove in the setting of $\mathbb{R}^n$.

**Theorem 1.10.** Let $f \in L^1(\mathbb{R}^n)$ and non-negative. Given $\lambda > 0$, there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes such that

(i) $f(x) \leq \lambda$ for almost all $x \not\in \cup_j Q_j$;

(ii) $\left| \cup_j Q_j \right| \leq \frac{1}{\lambda} \|f\|_{L^1}$;

(iii) $\lambda < \frac{1}{|Q_j|} \int_{Q_j} f(x) \, dx \leq 2^n \lambda$.

**Proof.** Let $\lambda > 0$ be fixed. Since $f \in L^1$, there exists $k_0 \in \mathbb{Z}$ so that

$$\frac{1}{|Q|} \int_Q f \, dx \leq \lambda$$

for all $Q \in Q_{k_0}$.

Now we consider the cubes of the next generation, that are obtained from the cubes in $Q_{k_0}$ by bisecting each cube of the collection $Q_{k_0}$. We say that a cube $Q \in Q_{k_0+1}$ belongs to the sequence we seek if

$$\frac{1}{|Q|} \int_Q f \, dx > \lambda$$

while (by construction),

$$\frac{1}{|Q'|} \int_{Q'} f \, dx \leq \lambda \text{ for (the unique) } Q' \in Q_{k_0} \text{ such that } Q' \supseteq Q_{k_0}.$$

If a cube is not chosen, that is, $\frac{1}{|Q|} \int_Q f \, dx \leq \lambda$, we iterate this process and we bisect it by considering the dyadic cubes of the next generation that are contained in it. Then, we say that a cube $Q_j \in Q_j$, $j > k_0$ is in the sequence if

$$\frac{1}{|Q_j|} \int_{Q_j} f \, dx > \lambda,$$

while,

$$\frac{1}{|Q'|} \int_{Q'} f \, dx \leq \lambda \text{ for } Q' \in Q_{j-1} \text{ such that } Q' \supseteq Q_j.$$

We need to check that the sequence $\{Q_j\}$ so constructed satisfies the required conditions. By definition, if $Q_j \in Q_j$ is a cube in the sequence and $Q \in Q_{j-1}$ contains $Q_j$ then

$$\lambda < \frac{1}{|Q_j|} \int_{Q_j} f(x) \, dx \leq \frac{1}{2^{-n}|Q|} \int_Q f(x) \, dx \leq 2^n \lambda,$$

so that (iii) is satisfied.

Next,

$$|\cup_j Q_j| = \sum_j |Q_j| \leq \frac{1}{\lambda} \sum_j \int_{Q_j} f(x) \, dx$$

$$\leq \frac{1}{\lambda} \|f\|_{L^1},$$

so that (ii) also holds.
Finally, (i) follows from the Lebesgue differentiation theorem. If \( x \in c(\cup jQ_j) \), then for all dyadic cubes \( Q \) containing \( x \) then
\[
\left| \frac{Q}{Q} \right| \int_{Q} f \, dx \leq \lambda.
\]
Letting the side length of the cube containing \( x \) tend to 0, by the Lebesgue differentiation theorem we obtain
\[
f(x) = \lim_{\ell(Q) \to 0} \frac{1}{|Q|} \int_{Q} f \, dx \leq \lambda,
\]
for a.a. \( x \in c(\cup jQ_j) \). This proves the theorem. \( \square \)

1.5. The \( L^p \)-boundedness of the Hilbert transform. Let \( n = 1 \) and consider the real line \( \mathbb{R} \). We recall that the principal value of \( 1/x \) is defined as the tempered distribution
\[
p.v. \frac{1}{x}(\varphi) = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{\varphi(x)}{x} \, dx.
\]
We have also defined \( Q(x) = \frac{1}{\pi} \frac{x}{x^2 + 1} \) and \( Q_t(x) = \frac{1}{t} Q(x/t) \).

We collect in the next statement the definition and main properties of the Hilbert transform.

**Theorem 1.11.** The following properties hold:

(i) \( |p.v. \frac{1}{x}(\varphi)| \leq C(\|\varphi'\|_\infty + \|x\varphi\|_\infty) \);

(ii) in \( S' \), \( \lim_{t \to 0} Q_t = \frac{1}{\pi} p.v. \frac{1}{x} \);

(iii) \( \mathcal{F} \left( \frac{1}{\pi} p.v. \frac{1}{x} \right)(\xi) = -i \text{sgn}(\xi) \).

The following expressions are equivalent and define the Hilbert transform \( Hf \) of a function \( f \in S' \):

(i') \( Hf(x) = \frac{1}{\pi} p.v. \frac{1}{x} * f(x) \);

(ii') \( Hf(x) = \lim_{t \to 0} Q_t * f(x) \);

(iii') \( \mathcal{F}(Hf)(\xi) = -i \text{sgn}(\xi) \hat{f}(\xi) \).

**Proof.** For the proofs of the statements above we use the notations and results in [Soa]. Using the fact that \( \int_{|x| \leq 1} 1/x \, dx = 0 \) we write
\[
p.v. \frac{1}{x}(\varphi) = \lim_{\epsilon \to 0} \int_{\epsilon \leq |x| \leq 1} \frac{\varphi(x)}{x} \, dx + \int_{1 \leq |x|} \frac{\varphi(x)}{x} \, dx
\]
\[
= \lim_{\epsilon \to 0} \int_{\epsilon \leq |x| \leq 1} \frac{\varphi(x) - \varphi(0)}{x} \, dx + \int_{1 \leq |x|} \frac{\varphi(x)}{x} \, dx
\]
\[
= \int_{|x| \leq 1} \frac{\varphi(x) - \varphi(0)}{x} \, dx + \int_{1 \leq |x|} \frac{\varphi(x)}{x} \, dx.
\]

Now (i) follows using the mean value theorem.

Next, Thm. 3.19 of Ch. 2 in [Soa] in particular implies (ii). Now, the Thm. 3.25 of Ch. 2 in [Soa] gives (iii).

Finally, (i')-(iii') follow from the first part and Remark 1.6. \( \square \)
The regularity result for the Hilbert transform is the following. 

**Theorem 1.12.** The Hilbert transform $H$, initially defined on Schwartz functions, can be extended to an operator, still denoted by $H$, which is weak-type $(1,1)$ and strong-type $(p,p)$ when $1 < p < \infty$. More precisely, there exists $C > 0$ such that 

$$
\left| \{ x \in \mathbb{R} : |Hf(x)| > \lambda \} \right| \leq \frac{C}{\lambda} \|f\|_{L^1(\mathbb{R})},
$$

and

$$
\|Hf\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^p(\mathbb{R})}
$$

when $1 < p < \infty$.

**Proof.** We first observe that $H$ is bounded on $L^2$, since by Plancherel’s theorem 

$$
\|Hf\|_{L^2} = \|(Hf)'\|_{L^2} = \|\operatorname{sgn}(\xi)\hat{f}\|_{L^2} = \|f\|_{L^2}.
$$

Next, suppose that we have shown that $H$ is weak-type $(1,1)$. Then, by Marcinkiewicz it follows that $H$ is strong-type $(p,p)$, for $1 < p \leq 2$. If $2 < p < \infty$ we use duality: 

$$
\|Hf\|_{L^p} = \sup_{\|g\|_{L^{p'}} \leq 1} \left| \int_{\mathbb{R}} Hf(x)g(x) \, dx \right|
$$

$$
\leq \sup_{\|g\|_{L^{p'}} \leq 1} \left| \int_{\mathbb{R}} f(x)Hg(x) \, dx \right|
$$

$$
\leq \sup_{\|g\|_{L^{p'}} \leq 1} \|f(x)\|_{L^p} \|Hg\|_{L^{p'}}
$$

$$
\leq C \|f\|_{L^p},
$$

that is, $H$ is bounded on $L^p$ also when $2 < p < \infty$.

Hence, we have reduced ourselves to show the weak-type $(1,1)$ inequality. Fix $\lambda > 0$ and we form the Calderón–Zygmund decomposition at height $\lambda$. We obtain a sequence of disjoint interval $\{I_j\}$ such that:

(i) $f(x) \leq \lambda$ for almost all $x \not\in \bigcup_j I_j$;

(ii) $|\bigcup_j I_j| \leq \frac{1}{\lambda} \|f\|_{L^1}$;

(iii) $\lambda < \frac{1}{|I_j|} \int_{I_j} f(x) \, dx \leq 2\lambda$.

Now we decompose $f$ as sum $f = g + b$, where 

$$
g(x) = \begin{cases} 
  f(x) & \text{if } x \not\in \bigcup_j I_j \\
  \frac{1}{|I_j|} \int_{I_j} f(x) \, dx & \text{if } x \in I_j,
\end{cases}
$$

and

$$
b(x) = \sum_j b_j(x) = \sum_j \left( f(x) - \frac{1}{|I_j|} \int_{I_j} f \right) \chi_{I_j}(x).
$$

(The decomposition $f = g + b$ is what is called the Calderón–Zygmund decomposition of the non-negative $L^1$ function $f$.)

We observe that $0 \leq g(x) \leq \lambda$, if $x \not\in \bigcup_j I_j$, while $0 \leq g(x) = \frac{1}{|I_j|} \int_{I_j} f(x) \leq 2\lambda$ if $x \in I_j$. Hence, $g(x) \leq 2\lambda$, and also $g \in L^1$ since it is obviously integrable on $c(\bigcup_j I_j)$ and on each $I_j$. 


(that are disjoint) \( \int_{I_j} g(x) dx = \int_{I_j} f(x) dx < \infty \). Notice in particular that \( \int_{\mathbb{R}} g = \int_{\mathbb{R}} f \). On the other hand, \( b \) has mean equal to 0, since each \( b_j \) does.

We now proceed. Since \( |Hf(x)| \leq |Hg(x)| + |Hb(x)| \), we have

\[
|\{x : |Hf(x)| > \lambda\}| \leq |\{x : |Hg(x)| > \lambda/2\}| + |\{x : |Hb(x)| > \lambda/2\}|
\]

Now,

\[
|\{x : |Hg(x)| > \lambda/2\}| = \int_{\{x : |Hg(x)| > \lambda/2\}} g(x) dx \leq \int_{\{x : |Hg(x)| > \lambda/2\}} \frac{|Hg(x)|^2}{(\lambda/2)^2} dx
\]

\[
\leq \frac{4}{\lambda^2} \int_{\mathbb{R}} |Hg(x)|^2 dx = \frac{4}{\lambda^2} \int_{\mathbb{R}} g(x)^2 dx
\]

\[
\leq \frac{8}{\lambda} \int_{\mathbb{R}} g(x) dx = \frac{8}{\lambda} \int_{\mathbb{R}} f(x) dx
\]

Next, let \( I_j^* \) denote the interval with the same center \( c_j \) as \( I_j \) with twice the length. Then we estimate,

\[
|\{x : |Hb(x)| > \lambda/2\}| = |\{x \in \bigcup \{I_j^*\} : |Hb(x)| > \lambda/2\}| + |\{x \in \bigcap \{I_j^*\} : |Hb(x)| > \lambda/2\}|
\]

\[
\leq \bigcup \{I_j^*\} + |\{x \in \bigcap \{I_j^*\} : |Hb(x)| > \lambda/2\}|
\]

\[
\leq \frac{2}{\lambda} \|f\|_{L^1} + \frac{2}{\lambda} \int_{\mathbb{R} \setminus \bigcup \{I_j^*\}} |Hb(x)| dx
\]

Since \( |Hb(x)| \leq \sum_j |Hb_j(x)| \) a.e., it will suffice to prove that

\[
\sum_j \int_{\mathbb{R} \setminus I_j^*} |Hb_j(x)| dx \leq C\|f\|_{L^1} \quad (1.8)
\]

Now, \( b_j \not\in \mathcal{S} \), nonetheless, for \( x \not\in I_j^* \) we have the formula

\( Hb_j(x) = \int_{I_j} \frac{b_j(y)}{x - y} dy \).

Recall that \( b_j \) has mean 0 and that we denote by \( c_j \) the center of the interval \( I_j \). Notice that \( y \in I_j \) implies that \( |y - c_j| \leq |I_j|/2 \) and that \( x \not\in I_j^* \) and \( y \in I_j \) imply that \( |x - y| > |x - c_j|/2 \).

Then,

\[
\int_{\mathbb{R} \setminus I_j^*} |Hb_j(x)| dx = \int_{\mathbb{R} \setminus I_j^*} \left| \int_{I_j} \frac{b_j(y)}{x - y} dy \right| dx
\]

\[
= \int_{\mathbb{R} \setminus I_j^*} \left| \int_{I_j} b_j(y) \left( \frac{1}{x - y} - \frac{1}{x - c_j} \right) dy \right| dx
\]

\[
\leq \int_{I_j} |b_j(y)| \int_{\mathbb{R} \setminus I_j^*} \frac{|y - c_j|}{|x - y||x - c_j|} dx dy
\]

\[
\leq \int_{I_j} |b_j(y)| \int_{\mathbb{R} \setminus I_j^*} \frac{|I_j|}{|x - c_j|^2} dx dy
\]

\[
= 2 \int_{I_j} |b_j(y)| dy
\]
Therefore,
\[ \sum_j \int_{\mathbb{R} \setminus I_j^*} |Hb_j(x)| \, dx \leq 2 \sum_j \int_{I_j} |b_j(y)| \, dy \leq 4\|f\|_{L^1}, \]
which gives (2.21). This concludes the proof. \(\square\)

1.6. Further properties of the Hilbert transform. We now make a few observations.

1. We have shown that, when \(f \in \mathcal{S}\), then \(Hf\) satisfies a weak-type \((1,1)\) bound and a strong-type \((p,p)\) bound. We now extend the definition of \(Hf\) to all of \(L^p\) and all of \(L^1\), with the same bounds.

If \(f \in L^p, 1 < p < \infty\), then there exists a sequence \(\{f_n\}\) of Schwartz functions converging in \(L^p\) to \(f\):
\[ \lim_{n \to +\infty} \|f_n - f\|_{L^p} = 0 \]
Then, in order to define \(Hf\), notice that \(\|Hf_n - Hf_m\|_{L^p} \leq C\|f_n - f_m\|_{L^p}\), so that \(\{Hf_n\}\) is a Cauchy sequence in \(L^p\) and converges to \(g \in L^p\). We set then \(Hf = g\). It is immediate to see that this definition is well-posed, in the sense that does not depend on the choice of the sequence \(\{f_n\}\) and that \(H\) then satisfies the bound \(\|Hf\|_{L^p} \leq C\|f\|_{L^p}\), with the same constant \(C\) as in Thm. 1.12.

2. If \(f \in L^1\), then there exists a sequence \(\{f_n\}\) of Schwartz functions converging in \(L^1\) to \(f\). The weak \((1,1)\) inequality gives that, for all \(\varepsilon > 0\) fixed,
\[ \lim_{n,m \to +\infty} \left| \{x \in \mathbb{R} : |(Hf_n - Hf_m)(x)| > \varepsilon \} \right| \leq \frac{C}{\varepsilon} \|f_n - f_m\|_{L^1} = 0. \]
Then, \(\{Hf_n\}\) is a Cauchy sequence in measure, so it converges to a measurable function \(g\) a.e.. We set, \(Hf = g\). It is easy to check that \(H\) satisfies the same weak-type \((1,1)\) bound on all of \(L^1\).

3. When \(p = 1\), the strong-type inequality fails. For instance, if we take \(f = \chi_{[0,1]}\), then
\[ Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(y)}{x - y} \, dy = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon}^{1} \frac{1}{x - y} \, dy = \frac{1}{\pi} \log \left| \frac{x}{x - 1} \right|. \]
It is clear that \(Hf \notin L^1\), while \(f \in L^1\).

4. Having defined the Hilbert transform of a function \(f \in L^p\), for \(1 \leq p < \infty\) as limit in norm (when \(1 < p < \infty\)) or in measure (when \(p = 1\)), we now wish to defined it pointwise as well. In this part we will omit the proofs and we refer the reader to [Du]. Consider the truncated integrals
\[ H_{\varepsilon}f(x) = \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x - y)}{y} \, dy. \]

**Lemma 1.13.** Let \(1 < p < \infty\). Then, if \(f \in L^p\), \(Hf = \lim_{\varepsilon \to 0} H_{\varepsilon}f\) in the \(L^p\)-norm.

When \(p = 1\) and \(f \in L^1\), then \(Hf = \lim_{\varepsilon \to 0} H_{\varepsilon}f\) in measure.
Proof. Notice that the function \( \frac{1}{y} \chi_{|y|>\varepsilon} \) is in \( L^q \) for all \( 1 < q \leq \infty \). Then, the function

\[
\frac{1}{\pi} \int y \chi_{|y|>\varepsilon}(y)f(x-y)\,dy = H_\varepsilon f(x),
\]

is well defined for all \( f \in L^p \), \( 1 \leq p < \infty \).

Now,

\[
\mathcal{F}\left( \frac{1}{y} \chi_{|y|>\varepsilon} \right)(\xi) = \lim_{N \to +\infty} \int_{\varepsilon < |y| < N} \frac{e^{-2\pi iy\xi}}{y} \, dy = \lim_{N \to +\infty} \int_{\varepsilon < |y| < N} \frac{-i \sin(2\pi y\xi)}{y} \, dy = -2i \text{sgn}(\xi) \lim_{N \to +\infty} \int_{\varepsilon < |y| < N} \frac{\sin x}{x} \, dx.
\]

Therefore, \( |\mathcal{F}\left( \frac{1}{y} \chi_{|y|>\varepsilon} \right)(\xi)| \leq 1 \), for all \( \varepsilon > 0 \). This implies that \( H_\varepsilon \) is of strong-type \((2,2)\), with constants uniform in \( \varepsilon \).

The weak-type \((1,1)\) also follows with uniform constant, so the strong-type \((p,p)\) follows by interpolation and duality.

We consider now the case \( 1 < p < \infty \); the case \( p = 1 \) begin analogous. If \( f \in L^p \), then let \( \{f_n\} \) be in \( L^p \) converging to \( f \). Then, using the uniform bound for the \( L^p \)-boundedness of \( H_\varepsilon \), we have

\[
Hf = \lim_{n \to +\infty} Hf_n = \lim_{n \to +\infty} \lim_{\varepsilon \to 0} H_\varepsilon f_n = \lim_{\varepsilon \to 0} \lim_{n \to +\infty} H_\varepsilon f_n = \lim_{\varepsilon \to 0} H_\varepsilon f.
\]

This proves the lemma.

We conclude this section by stating the following result concerning the pointwise definition of the Hilbert transform.

**Theorem 1.14.** Let \( 1 \leq p < \infty \), and \( f \in L^p \). Then

\[
Hf(x) = \lim_{\varepsilon \to 0} H_\varepsilon f(x) \quad a.e. \, x \in \mathbb{R}.
\]

We only mention that, similarly to the case of the Hardy–Littlewood maximal function and the Lebesgue differentiation theorem, the proof relies on the boundedness of a maximal function.

Set

\[
H^* f(x) = \sup_{\varepsilon > 0} |H_\varepsilon f(x)|.
\]

Then the following result holds true.

**Theorem 1.15.** The maximal Hilbert transform \( H^* \) is weak-type \((1,1)\) and strong-type \((p,p)\) for \( 1 < p < \infty \).
2. Singular integrals

2.1. The Calderón–Zygmund theorem. Main result of this section is the following theorem. Given a function \( K \) that is locally integrable in \( \mathbb{R}^n \setminus \{0\} \), and it is also a tempered distribution, it makes sense to consider the convolution \( K \ast f \) with a Schwartz function \( f \). Therefore, we define the operator

\[
Tf(x) = K \ast f(x),
\]

initially defined on Schwartz functions.

The next result generalizes the theorem on the boundedness of the Hilbert transform.

**Theorem 2.1.** Let \( K \in C^1(\mathbb{R}^n \setminus \{0\}) \) be a locally integrable function and a tempered distribution. Suppose that

\[
|\hat{K}(\xi)| \leq A,
\]

(2.1)

and furthermore

\[
\int_{|x| > 2|y|} |K(x - y) - K(x)| \, dx \leq B.
\]

(2.2)

Then, the operator \( T \) is bounded from \( L^p \) into itself, i.e., it is of strong-type \((p,p)\) for \( 1 < p < \infty \) and it is of weak-type \((1,1)\), that is,

\[
|\{ x : |Tf(x)| > \lambda \}| \leq \frac{C}{\lambda} \| f \|_{L^1}.
\]

**Corollary 2.2.** Let \( K \in C^1(\mathbb{R}^n \setminus \{0\}) \) be a locally integrable function and a tempered distribution. Suppose that \( K \) satisfies (2.1) and also

\[
|\nabla K(x)| \leq \frac{B'}{|x|^{n+1}}.
\]

(2.3)

Then, for the operator \( T \) the same conclusions hold as in the previous theorem.

**Proof.** In order to prove the corollary, assuming the validity of the theorem, it suffices to show that (2.3) implies (2.2). But this follows from the mean value theorem. Let \( \sigma(x, y) \) denote the segment having endpoints in \( x - y \) and \( x \). Then, using the fact that \( |x| > 2|y| \), it holds that if \( z \in \sigma(x, y) \) then

\[
|z| \geq \min\{|x - y|, |x|\} \geq |x - y| \geq |x| - \frac{|x|}{2} = \frac{|x|}{2}.
\]

Therefore,

\[
|K(x - y) - K(x)| \leq |y| \sup_{z \in \sigma(x, y)} |\nabla K(z)| \leq B' |y| \sup_{z \in \sigma(x, y)} \frac{1}{|z|^{n+1}} \leq B' |y| \frac{2^{n+1}}{|x|^{n+1}} = C \frac{|y|}{|x|^{n+1}}.
\]

Hence,

\[
\int_{|x| > 2|y|} |K(x - y) - K(x)| \, dx \leq C |y| \int_{|x| > 2|y|} \frac{1}{|x|^{n+1}} \, dx = C |y| \int_{2|y|}^{\infty} r^{-2} \, dr = C'.
\]
Thus, (2.3) implies (2.2) and the conclusion follows from Thm. 2.1.

Proof of Thm. 2.1. The proof goes along the same lines of the proof of Thm. 1.12.

By the assumption (2.1) and the Plancherel theorem it immediately follows that $T$ is bounded on $L^2$:

$$\|Tf\|_{L^2}^2 = \int_{\mathbb{R}^n} |\mathcal{F}(K \ast f)(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |\hat{K}(\xi)\hat{f}(\xi)|^2 d\xi \leq A^2\|f\|_{L^2}^2.$$  

We now observe that the adjoint operator $T^*$ has kernel $K^*(x) = K(-x)$, so that it also satisfies hypotheses (2.1) and (2.2). Thus, if we prove the weak-(1, 1) inequality for $T$, by interpolation it would follow that $T$ is $L^p$-bounded for $1 < p \leq 2$, and then, by duality is $L^p$-bounded also for $2 \leq p < \infty$.

Hence, it suffices to prove the weak-(1, 1) inequality. To do this, we may assume that $f \geq 0$ and let $\lambda > 0$ be fixed. We decompose $f$ as sum $f = g + b$, where $g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_j Q_j \\ \frac{1}{|Q_j|} \int_{Q_j} f(x) dx & \text{if } x \in Q_j, \end{cases}$

and $b(x) = \sum_j b_j(x) = \sum_j \left( f(x) - \left( \frac{1}{|Q_j|} \int_{Q_j} f(x) \right) \chi_{Q_j}(x) \right)$.

Arguing as in the proof of Thm. 1.12, we have

$$|\{x : |Tf(x)| > \lambda\}| \leq \left| \{x : |Tg(x)| > \lambda/2\} \right| + \left| \{x : |Tb(x)| > \lambda/2\} \right|,$$

and then

$$|\{x : |Tg(x)| > \lambda/2\}| \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} f(x) dx.$$

Next,

$$|\{x : |Tb(x)| > \lambda/2\}| \leq \frac{C}{\lambda} \|f\|_{L^1} + \frac{2}{\lambda} \int_{\mathbb{R} \setminus (\bigcup_j Q_j^*)} |Tb(x)| dx,$$

where $Q_j^*$ denote the cube with the same center $c_j$ as $Q_j$ with side length $2\sqrt{n}$-times longer. Then we reduce ourselves to show that

$$\sum_j \int_{\mathbb{R} \setminus Q_j^*} |Tb_j(x)| dx \leq C\|f\|_{L^1},$$

\footnote{This boundedness is equivalent to the $L^p$-boundedness of $T^*$ for $1 < p \leq 2$.}
which, in turn, is implied by the inequality
\[
\int_{\mathbb{R}\setminus Q_j^*} |Tb_j(x)| \, dx \leq C \|b_j\|_{L^1}. \tag{2.4}
\]

We are going to use the Hörmander condition (2.2) and the fact that \(b_j\) has integral equal to 0. For \(x \notin Q_j^*\),
\[
Tb_j(x) = \int_{Q_j} K(x - y)b_j(y) \, dy = \int_{Q_j} [K(x - y) - K(x - c_j)] b_j(y) \, dy.
\]
Therefore,
\[
\int_{\mathbb{R}^n\setminus Q_j^*} |Tb_j(x)| \, dx \leq \int_{Q_j} |b_j(y)| \int_{\mathbb{R}^n\setminus Q_j^*} |K(x - y) - K(x - c_j)| \, dx \, dy
\]
\[
\leq B \int_{Q_j} |b_j(y)| \, dy,
\]
since
\[
\mathbb{R}^n \setminus Q_j^* \subseteq \{ x \in \mathbb{R}^n : |x - c_j| > 2|y - c_j| \},
\]
so that
\[
\int_{\mathbb{R}^n\setminus Q_j^*} |K(x - y) - K(x - c_j)| \, dx \leq \int_{|x - c_j| > 2|y - c_j|} |K(x - y) - K(x - c_j)| \, dx
\]
\[
= \int_{|x'| > 2|y - c_j|} |K(x' - (y - c_j)) - K(x')| \, dx'
\]
\[
\leq B.
\]
Thus, (2.4) follows and we are done. \(\square\)

2.2. Homogeneous distributions. In order to describe some classical and fundamental examples of singular integrals in \(\mathbb{R}^n\), examples that generalize the case of the Hilbert transform in the case of the real line, we need a preliminary discussion of the so-called homogeneous distributions.

We recall that a function \(f\) is said to be homogeneous of degree \(a > 0\) if for all \(\lambda > 0\) and all \(x \in \mathbb{R}^n\)
\[
f(\lambda x) = \lambda^a f(x).
\]

Given another function \(\varphi\), we see that
\[
\int_{\mathbb{R}^n} f(x) \varphi(x) \, dx = \lambda^{-n} \int_{\mathbb{R}^n} f(\lambda x') \varphi(x') \, dx' = \lambda^a \int_{\mathbb{R}^n} f(x') \varphi(x') \, dx'.
\]
Therefore, we may extend the definition of homogeneity to tempered distributions as follows.

**Definition 2.3.** Given a tempered distribution \(u\), we say that \(u\) is homogeneous of degree \(a > 0\) if for all \(\varphi \in \mathcal{S}\) and \(\lambda > 0\) we have
\[
u(\varphi) = \lambda^a u(\varphi).
\]

**Example 2.4.** It is immediate to see that the distribution p.v. \(\frac{1}{x}\) on \(\mathcal{S}(\mathbb{R})\) is homogeneous of degree \(-1\), while \(m(\xi) = -i \text{sgn}(\xi)\) is a homogeneous (function) of degree 0.
We now wish to generalize the tempered distribution p.v. $\frac{1}{x}$ and the Hilbert transform to higher dimensions. A typical situation will be the following.

Let $\Omega(x)$ be a function in $\mathbb{R}^n \setminus \{0\}$ homogeneous of degree 0, that is, such that $\Omega(\lambda x) = \Omega(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Notice that $\Omega$, as all homogeneous functions, is uniquely determined by its values on the unit sphere $S^{n-1}$. Suppose that $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x') \, d\sigma(x') = 0,$$

where $d\sigma$ denotes the Lebesgue surface measure on $S^{n-1}$. We define the tempered distribution

$$u(\varphi) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x)}{|x|^n} \varphi(x) \, dx$$

$$= \text{p.v.} \int_0^{+\infty} \int_{S^{n-1}} \Omega(x') \varphi(rx') \, d\sigma(x') r^{-1} \, dr$$

$$= \int_0^{+\infty} \int_{S^{n-1}} \Omega(x') (\varphi(rx') - \varphi(0)) \, d\sigma(x') r^{-1} \, dr,$$

where we have used the fact that $\Omega$ has zero integral on the unit sphere. Now, using the mean value theorem, as in the case of p.v. $\frac{1}{x}$, it is easy to see that the last integral above converges absolutely, for $\varphi \in S(\mathbb{R}^n)$.

**Lemma 2.5.** If $u \in S'$ is a tempered distribution homogeneous of degree $a$, then its Fourier transform is a tempered distribution homogeneous of degree $-n - a$.

**Proof.** Recall the identities (1.2). For $\varphi \in S$ and $\lambda > 0$ we have

$$\hat{u}(\varphi_\lambda) = u(\hat{\varphi}_\lambda) = u((\hat{\varphi})^\lambda) = \lambda^n u((\hat{\varphi})_{\lambda^{-1}})$$

$$= \lambda^{-n-a} u(\hat{\varphi}) = \lambda^{-n-a} \hat{u}(\varphi),$$

as we wished to show. \(\square\)

**Lemma 2.6.** Let $0 < a < n$. Then the function $|x|^{-a}$ is locally integrable and defines a tempered distribution. The Fourier transform, as an element of $S'$ satisfies the equality

$$\left(|x|^{-a}\right)(\xi) = c_{n,a} |\xi|^{a-n},$$

where

$$c_{n,a} = \frac{\pi^{n-1/2} \Gamma(\frac{n-a}{2})}{\Gamma(\frac{n}{2})}. \quad (2.5)$$

**Proof.** We recall that the Fourier transform of a radial function is also radial.\(^3\) Suppose first that $n/2 < a < n$, so that

$$|x|^{-a} = |x|^{-a} \chi_{\{|x| \leq 1\}} + |x|^{-a} \chi_{\{|x| > 1\}} = f_1(x) + f_2(x),$$

where $f_1 \in L^1$ and $f_2 \in L^2$. Then we can compute the Fourier transform of $|x|^{-a}$ as a function. Its Fourier transform is a radial function, and by the previous lemma, homogeneous of degree $-n + a$. Hence, it is a constant multiple of $|\xi|^{-a}$, i.e.,

$$\left(|x|^{-a}\right)(\xi) = c_{n,a} |\xi|^{a-n}.$$

\(^3\)This is immediate to check, using the fact that a function $f$ is radial if and only if $f(Ox) = f(x)$ for all $x \in \mathbb{R}^n$ and all orthogonal transformations $O$. 

We now compute the constant $c_{n,a}$. Using the Parseval formula (1.3) and the identity 
\[ (e^{-\pi|x|^2})(\xi) = e^{-\pi|\xi|^2} \] (see Thm. 2.17 in [So]), we have
\[
\int_{\mathbb{R}^n} e^{-\pi|x|^2} |x|^{-a} \, dx = c_{n,a} \int_{\mathbb{R}^n} e^{-\pi|\xi|^2} |\xi|^{a-n} \, d\xi.
\] (2.6)

Now, setting $\pi r^2 = s$ we see that
\[
\int_0^{+\infty} e^{-\pi r^2} r^b \, dr = \frac{1}{2\sqrt{\pi}} \int_0^{+\infty} e^{-s} \left( \frac{s}{\pi} \right)^{b/2} \frac{1}{\sqrt{s}} \, ds
\]
\[
= \frac{1}{2\pi(1+b)/2} \int_0^{+\infty} e^{-s} s^{(b-1)/2} \, ds
\]
\[
= \frac{1}{2\pi(1+b)/2} \Gamma\left((1+b)/2\right).
\]

From (2.6) it then follows
\[
c_{n,a} = \frac{1}{2\pi(n-a)/2} \Gamma\left((n-a)/2\right) \left[ \frac{1}{2\pi^{a/2}} \Gamma\left(a/2\right) \right]^{-1}
\]
\[
= \frac{\pi^{a-n/2} \Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}.
\]

This proves the proposition in the case $n/2 < a < n$.

In the case $0 < a < n/2$ we use the first part, since in this case $n/2 < n - a < n$, and the inversion formula, valid also in $S'$. The case $a = n/2$ follows by passing to the limit, and using the continuity of the Fourier transform in $S'$. \(\square\)

2.3. The Riesz transforms. We now define the main generalizations of the Hilbert transform, the Riesz transforms $R_j$, $j = 1, \ldots, n$.

Given $f \in S$, we set
\[
R_j f(x) = c_n \text{ p.v.} \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x - y) \, dy, \tag{2.7}
\]

where the constant $c_n (= c_{n,-1})$ is given by\footnote{Incidentally we recall that the volume of the unit ball $B_n$ and of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ are, resp., $|B_n| = \pi^{n/2}/\Gamma((n/2) + 1)$ and $\sigma(S^{n-1}) = 2\pi^{n/2}/\Gamma(n/2)$, resp.}
\[
c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}.
\]

Notice that for $n = 1$ we have $c_1 = 1/\pi$ and we recover the definition of the Hilbert transform.

Lemma 2.7. For $f \in L^2(\mathbb{R}^n)$ and $j = 1, \ldots, n$ we have that
\[
(R_j f)^\wedge(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi).
\]

Hence,
\[
\sum_{j=1}^n R_j^2 = -I.
\]
Proof. We wish to show that

\[ F \left( c_n \, \text{p.v.} \, \frac{y_j}{|y|^{n+1}} \right)(\xi) = -i \frac{\xi_j}{|\xi|}, \]

as tempered distributions.

We claim that, again as tempered distributions,

\[ \partial_{x_j} |x|^{-n+1} = (1 - n) \, \text{p.v.} \, \frac{x_j}{|x|^{n+1}}, \tag{2.8} \]

Assuming the claim we finish the proof. Using Lemma 2.6 and Remark 1.6 (v) we have

\[ F \left( \text{p.v.} \, \frac{x_j}{|x|^{n+1}} \right)(\xi) = \frac{1}{1 - n} F \left( \partial_{x_j} |x|^{-n+1} \right)(\xi) \]

\[ = \frac{2\pi i \xi_j}{1 - n} F(|x|^{-n+1})(\xi) \]

\[ = \frac{2\pi i \xi_j}{1 - n} \int \frac{\pi^{n-1} \Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} |\xi|^{-1} \]

\[ = -i \frac{\pi^{(n+1)/2}}{\Gamma \left( \left( n + 1 \right)/2 \right)} \xi_j |\xi|^{-1} \]

\[ = -i \frac{\xi_j}{c_n |\xi|}, \]

as we wish to prove.

It only remains to prove the claim. For \( \varphi \in S \) we have

\[ (1 - n) \, \text{p.v.} \, \frac{x_j}{|x|^{n+1}}(\varphi) = \lim_{\varepsilon \to 0^+} (1 - n) \int_{\mathbb{R}^n} \frac{x_j}{\varepsilon^2 + |x|^2} \varphi(x) \, dx \]

\[ = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \frac{\partial_{x_j}}{\varepsilon^2 + |x|^2} \frac{1}{(n+1)/2} \varphi(x) \, dx \]

\[ = - \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} \frac{\partial_{x_j}}{\varepsilon^2 + |x|^2} \frac{1}{(n-1)/2} \partial_{x_j} \varphi(x) \, dx \]

\[ = - \int_{\mathbb{R}^n} \frac{1}{|x|^{n-1}} \partial_{x_j} \varphi(x) \, dx \]

\[ = (\partial_{x_j} |x|^{1-n})(\varphi). \]

This proves the claims and therefore the lemma. \( \square \)

**Theorem 2.8.** The Riesz transforms \( R_j, j = 1, \ldots, n \), are of weak-type \( (1,1) \) and of strong-type \( (p,p) \), for \( 1 < p < \infty \).

**Proof.** It suffices to show that the tempered distributions \( K_j(x) = c_n \, \text{p.v.} \, (x_j/|x|^{n+1}) \) satisfy the hypotheses of Thm. 2.1, for \( j = 1, \ldots, n \), resp.

Clearly \( K_j \) coincides with a locally integrable function on \( \mathbb{R}^n \setminus \{0\} \). Moreover, by the lemma,

\[ \hat{K}_j(\xi) = -i \frac{\xi_j}{|\xi|}, \]

which is clearly bounded.
Finally, on $\mathbb{R}^n \setminus \{0\}$

$$|\nabla K_j(x)| \leq c_n \frac{1}{|x|^{n+1}} + c_n \frac{n+1}{2} \frac{|x_j||x|}{|x|^{n+2}}$$

$$\leq C \frac{1}{|x|^{n+1}}.$$ 

Thus, (2.2) is satisfied and we are done. $\square$

2.4. **Solution of the Laplace equation.** In this section we consider the Laplace operator

$$\Delta = \sum_{j=1}^{n} \partial_j^2,$$

and prove a regularity result for the solution of the equation

$$\Delta g = f.$$

Given a partial differential operator $P$ with constant coefficients, we say that a tempered distribution $E$ is a fundamental solution for $P$ if $PE = \delta_0$, the Dirac delta at the origin, that is, if

$$P(f * E) = f * PE = f * \delta_0 = f,$$

for all $f \in \mathcal{S}$. Notice in particular that, for a given $f \in \mathcal{S}$, the tempered distribution $f * E$ solves the equation $Pg = f$.

We now compute the fundamental solution for $\Delta$.

**Proposition 2.9.** Let $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ denote the volume of the unit sphere in $\mathbb{R}^n$ and define

$$E(x) = \begin{cases} 
\frac{1}{(2-n)\omega_n} \frac{1}{|x|^{n-2}} & \text{for } n > 2, \\
\frac{1}{4\pi} \log |x| & \text{for } n = 2.
\end{cases}$$

Then $E$ is a fundamental solution for $\Delta$.

**Proof.** We consider a family of regularized tempered distributions, namely $E^\varepsilon$, given by

$$E^\varepsilon(x) = \begin{cases} 
\frac{1}{(2-n)\omega_n} \frac{1}{(\varepsilon^2 + |x|^2)^{(n-2)/2}} & \text{for } n > 2, \\
\frac{1}{4\pi} \log(\varepsilon^2 + |x|^2) & \text{for } n = 2.
\end{cases}$$

It is easy to see that $E^\varepsilon \to E$ pointwise as $\varepsilon \to 0$, and that there exists $g \in L^1_{\text{loc}}$ such that $|E^\varepsilon| \leq g$ for all $\varepsilon \leq 1$, in both cases $n = 2$ and $n > 2$. (It suffices to take $|E|$ when $n > 2$ and $|\log |x|| + 1$ when $n = 2$.) Hence, $E^\varepsilon \to E$ in $\mathcal{S}'$ and therefore, also $\Delta E^\varepsilon \to \Delta E$ in $\mathcal{S}'$. 
Let $\varphi \in S$, we wish to show that $\Delta E^\varepsilon(\varphi) \to \varphi(0)$ as $\varepsilon \to 0$. We provide the details for $n > 2$. We have

$$\partial_j E^\varepsilon(x) = \frac{1}{(2 - n)\omega_n} \cdot (2 - n)x_j \left(\frac{1}{|x|^2 + \varepsilon^2} \right)^{n/2}$$

and

$$\partial^2_{jk} E^\varepsilon(x) = \frac{1}{\omega_n} \left(\frac{1}{|x|^2 + \varepsilon^2} \right)^{n/2} - \frac{nx^2_j}{\left(\frac{1}{|x|^2 + \varepsilon^2} \right)^{(n+2)/2}},$$

so that

$$\Delta E^\varepsilon(x) = \frac{n}{\omega_n} \sum_{j=1}^n \left(\frac{1}{|x|^2 + \varepsilon^2} \right)^{n/2} - \frac{nx^2_j}{\left(\frac{1}{|x|^2 + \varepsilon^2} \right)^{(n+2)/2}}$$

$$= \frac{n}{\omega_n} \varepsilon^2 \left(\frac{1}{|x|^2 + 1} \right)^{(n+2)/2}.$$ 

Therefore,

$$\Delta E^\varepsilon(x) = \psi_\varepsilon(x),$$

where, as usual, $\psi_\varepsilon(x) = \varepsilon^{-n} \psi(x/\varepsilon)$ and

$$\psi = \Delta E^1 = \frac{n}{\omega_n} \left(\frac{1}{|x|^2 + 1} \right)^{(n+2)/2}.$$

Next we use the fact that $\psi(x) = \psi(-x)$ to see that

$$\Delta E^\varepsilon(\varphi) = \int \Delta E^\varepsilon(-x) \varphi(x) \, dx = \psi_\varepsilon * \varphi(0)$$

$$- (f \psi) \varphi(0),$$

by the proprieties of the summability kernels.

Thus, we only need to compute $\int \psi$. But, by passing to polar coordinates and making the change of variables $s = r^2/(1 + r^2)$, so that $ds = 2(1 + r^2)^{-2} r dr$, and

$$\frac{r(n-1)}{(1 + r^2)^{(n+2)/2}} = \left(\frac{r^2}{1 + r^2}\right)^{(n-2)/2} \frac{r}{(1 + r^2)^2}.$$

We then find

$$\int \psi(x) \, dx = \frac{n}{\omega_n} \int \frac{1}{\left(\frac{1}{|x|^2 + 1} \right)^{(n+2)/2}} \, dx$$

$$= n \int_0^{+\infty} \frac{1}{\left(1 + r^2\right)^{(n+2)/2}} r^{n-1} \, dr$$

$$= \frac{n}{2} \int_0^1 s^{(n-2)/2} \, ds = 1.$$ 

This completes the proof (at least in the case $n > 2$, the case $n = 2$ being left to the reader).

We now have the regularity result for the solutions of the Laplace equation.

**Theorem 2.10.** Let $f \in S$ be given. Define the operator

$$Tf(x) = \partial^2_{jk} E * f = f * \partial^2_{jk} E,$$

$j, k = 1, \ldots, n$. Then, $T$ extends to an operator that is of weak-type $(1, 1)$ and of strong-type $(p, p)$, for $1 < p < \infty$. 
Remark 2.11. Notice that the result shows that, for a given $f \in L^p$, it is possible to define the solution $u = f * E$ of the equation $\Delta u = f$ is such a way that the map $T(f) = \partial^2_{x_j x_k} (f * E)$ is bounded on $L^p$.

On the other hand, the map $T$ defined as $f \mapsto u = f * E$ is bounded, when $n \geq 3$,

$$T : L^p \to L^q,$$

as a consequence of Thm. 2.15 of the next section, where

$$\frac{1}{q} = 1 - \frac{2}{n}.$$

**Proof of Thm. 2.10.** It suffices to show that the kernel $K = \partial^2_{x_j x_k} E$ satisfies the hypotheses of Thm. 2.1. It is clear that, by Lemma 2.6,

$$F(\partial^2_{x_j x_k} E)(\xi) = -(2\pi)^2 \xi_j \xi_k \hat{E}(\xi) = C_n \xi_j \xi_k |\xi|^{2-n},$$

so that (2.1) is satisfied.

On the other hand, also the condition (2.3) is also easily seen to be satisfied:

$$|\nabla (\partial^2_{x_j x_k} E)(x)| \leq C \frac{1}{|x|^{n+1}},$$

The result now follows from 2.1. □

2.5. The Riesz potentials. We now discuss some operators $I_\alpha$, called the Riesz potential that arise in a natural way in connection with the Laplacian. The operators $I_\alpha$ are integral operators whose kernel is homogenous of degree $-(n-\alpha)$, then locally integrable when $0 < \alpha < n$, case to which we restrict ourselves.

**Definition 2.12.** Let $0 < \alpha < n$ and define the Riesz potential of order $\alpha$ the integral operator

$$I_\alpha(f)(x) = \frac{(2\pi)^\alpha}{c_{n,n-\alpha}} \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-\alpha}} f(y) \, dy,$$

where $c_{n,n-\alpha}$ is the constant defined in (2.5). (It will be sometimes convenient to set $a = n - \alpha$. Notice that $0 < a < n$ exactly when $0 < \alpha < n$.)

We remark that the Laplacian satisfies the identity

$$(-\Delta f)^\alpha(\xi) = (2\pi |\xi|)^2 \hat{f}(\xi),$$

for $f \in S$. By Lemma 2.6, it follows that

$$(I_\alpha f)^\alpha(\xi) = \frac{(2\pi)^\alpha}{c_{n,n-\alpha}} |x|^{-(n-\alpha)} \ast \hat{f}(\xi) = (2\pi)^\alpha |\xi|^\alpha \hat{f}(\xi).$$

Hence, it is natural to define the fractional powers of $-\Delta$ by setting

$$(-\Delta)^{\alpha/2} f)(\xi) = (2\pi |\xi|)^\alpha \hat{f}(\xi). \quad (2.9)$$

We now study the $(L^p, L^q)$ boundedness of the operators $I_\alpha$. We begin with a simple result that will give us a necessary condition on $p, q$ and $\alpha$. We set $D_\lambda f(x) = f^\lambda(x) = f(\lambda x)$, for $\lambda > 0$. 

Lemma 2.13. For \( \lambda > 0 \) we have the following identities:

(i) \( D_{\lambda^{-1}} I_\alpha D_\lambda = \lambda^{-\alpha} I_\alpha \);

(ii) \( \|D_\lambda f\|_{L^p} = \lambda^{-n/p} \|f\|_{L^p} \);

(iii) \( \|D_{\lambda^{-1}} I_\alpha (f)\|_{L^q} = \lambda^{n/q} \|I_\alpha (f)\|_{L^q} \).

Proof. These are elementary. For a given function \( f \) sufficiently regular we have

\[
D_{\lambda^{-1}} I_\alpha D_\lambda (f)(x) = \frac{(2\pi)^\alpha}{c_{n,n-\alpha}} \int_{\mathbb{R}^n} \frac{1}{|\lambda^{-1} x - y|^{n-\alpha}} f(\lambda y) \, dy = \frac{(2\pi)^\alpha}{c_{n,n-\alpha}} \lambda^n \int_{\mathbb{R}^n} \frac{1}{|\lambda^{-1}(x - z)|^{n-\alpha}} f(z) \, dz = \lambda^{-\alpha} I_\alpha (f)(x),
\]

as we claimed.

(ii) and (iii) are simply a change of variables. \( \Box \)

Lemma 2.14. Let \( 0 < \alpha < n, \ 1 < p, q < \infty \) be given. Then, if \( I_\alpha : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \) is bounded, then necessarily

\[
\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}.
\]

Proof. If \( I_\alpha \) is bounded, there exists a constant \( C > 0 \) such that

\[
\|I_\alpha (f)\|_{L^q} \leq C \|f\|_{L^p}
\]

for all \( f \in L^p \). Then, for all \( \lambda > 0 \) it must hold that

\[
\|I_\alpha (D_\lambda f)\|_{L^q} \leq C \|D_\lambda f\|_{L^p} .
\]

Now, by Lemma 2.13,

\[
\|I_\alpha (D_\lambda f)\|_{L^q} = \|D_\lambda D_{\lambda^{-1}} I_\alpha (D_\lambda f)\|_{L^q} = \lambda^{-n/q} \lambda^{-\alpha} \|I_\alpha (f)\|_{L^q} ,
\]

Then, (2.10) is equivalent to

\[
\|I_\alpha (f)\|_{L^q} \leq C \lambda^{\frac{n}{q} + \frac{\alpha}{p}} \|f\|_{L^p} .
\]

Since this must hold for all \( \lambda > 0 \), if the exponent of \( \lambda \) in the inequality above is \( \neq 0 \), by letting \( \lambda \to 0 \), or \( \lambda \to +\infty \) we get a contradiction. The conclusion now follows. \( \Box \)

We now prove the positive result, valid in the case \( 1 < p, q < \infty \).

Theorem 2.15. Let \( 0 < \alpha < n, \ 1 \leq p, q < \infty \) satisfy

\[
\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} ,
\]

and consider the fractional integral \( I_\alpha \). For \( f \in L^p \), the integral defining \( I_\alpha (f) \) converges absolutely. Moreover,

(i) if \( 1 < p, q < \infty \)

\[
I_\alpha : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)
\]

is strong-type \( (p, q) \);

(ii) if \( p = 1 \), then \( I_\alpha \) is weak-type \( (1, q) \).
Proof. Clearly, it suffices to consider the convolution operator \( T : f \mapsto K * f \), where \( K(x) = |x|^{-n+\alpha} \) and prove (i) and (ii) for this operator.

By the general form of the Marcinkiewicz interpolation theorem, if we show that \( T \) is weak-type \((p_1, q_1)\) and also weak-type \((1, q_0)\), both pairs satisfying the condition (2.11), then it follows that it is strong-type \((p, q)\), where

\[
\frac{1}{p} = \frac{\theta}{p_1} + 1 - \theta, \quad \text{and} \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1 - \theta}{q_0}.
\]

Notice that

\[
\frac{1}{q} = \theta \left( \frac{1}{p_1} - \frac{\alpha}{n} \right) + (1 - \theta) \left( 1 - \frac{\alpha}{n} \right) = \frac{\theta}{p_1} + 1 - \theta - \frac{\alpha}{n} = 1 - \frac{\alpha}{n}.
\]

We write \( K = K_1 + K_\infty \), where

\[
K_1(x) = K(x)\chi_{\{|x| \leq R\}}, \quad \text{and} \quad K_\infty(x) = K(x)\chi_{\{|x| > R\}},
\]

and where \( R > 0 \) is a positive constant to be selected later.

We begin by observing that \( K_1 \in L^1(\mathbb{R}^n) \), since

\[
\int_{\mathbb{R}^n} |K_1(x)| dx = \int_{\{|x| \leq R\}} |x|^{-n+\alpha} dx = \omega_n \int_0^R r^{\alpha - 1} dr = \frac{\omega_n R^\alpha}{\alpha}.
\]

(2.12)

On the other hand, \( K_\infty \in L^{p'} \), where \( p' \) is the exponent conjugate to \( p \). For,

\[
\int_{\mathbb{R}^n} |K_\infty(x)|^{p'} dx = \int_{\{|x| > R\}} |x|^{(-n+\alpha)p'} dx = \omega_n \int_R^{+\infty} r^{n(1-p')-\alpha p'-1} dr = \frac{\omega_n}{n(p' - 1 - \alpha p')} R^{n(1-p') + \alpha p'},
\]

(2.13)

and the integral is convergent since \( n(1-p') + \alpha p' < 0 \), that is, \( -\frac{n}{p} + \alpha < 0 \), which is equivalent to \( q < \infty \).

Now, let \( f \in L^p, 1 \leq p < \infty \). Then, the integral defining \( K * f \) converges absolutely since \( K_1 \in L^1 \) and \( K_\infty \in L^{p'} \). (Make sure you agree with this assertion.) This proves the first part of the statement.

In order to prove (i) and (ii), it suffices to show that, for \( 1 \leq p < q < \infty \) such that \( 1/q = 1/p - \alpha/n \), \( T \) is weak-type \((p, q)\), that is,

\[
\left| \left\{ x : |K * f(x)| > \lambda \right\} \right| \leq \left( C \frac{\|f\|_{L^p}}{\lambda} \right)^q.
\]

(2.14)

Notice that we may assume that \( \|f\|_{L^p} = 1 \) (as it easy to check, and we invite you to do so). Then we have,

\[
\left| \left\{ x : |K * f(x)| > \lambda \right\} \right| \leq \left| \left\{ x : |K_1 * f(x)| > \lambda/2 \right\} \right| + \left| \left\{ x : |K_\infty * f(x)| > \lambda/2 \right\} \right|.
\]
Now, using (2.12) and the assumption $\|f\|_{L^p} = 1$, we have
\[\left| \left\{ x : |K_1 * f(x)| > \lambda/2 \right\} \right| = \int_{\{x : |K_1 * f(x)| > \lambda/2\}} dx \leq \int \left( \frac{|K_1 * f(x)|}{\lambda/2} \right)^p dx \leq \frac{2p\|K_1 * f\|_{L^p}^p}{\lambda^p} \leq \frac{2p\|K_1\|_{L^p}^p \|f\|_{L^p}^p}{\lambda^p} \leq C \left( \frac{R^n}{\lambda} \right)^p. \] (2.15)

Next, using (2.13) we see that
\[\|K_\infty * f\|_{L^\infty} \leq \|K_\infty\|_{L^p'} \|f\|_{L^p} = \|K_\infty\|_{L^p'} \leq CR^{\alpha-n/p}. \] (2.16)

We now choose $R$ such that
\[CR^{\alpha-n/p} = \frac{\lambda}{2}, \quad \text{that is,} \quad R = C' \lambda^{1/(\alpha-n/p)} = C' \lambda^{-q/n}. \]

With choice of $R$, by (2.16) it follows that
\[\left| \left\{ x : |K_\infty * f(x)| > \lambda/2 \right\} \right| = 0. \]

Therefore, by (2.16) (recalling that $\|f\|_{L^p} = 1$),
\[\left| \left\{ x : |K * f(x)| > \lambda \right\} \right| \leq C \left( \frac{R^n}{\lambda} \right)^p \leq C \left( \lambda^{q(\alpha/n+1/q)} \right)^p = C\lambda^{-q} = C \left( \frac{\|f\|_{L^p}}{\lambda} \right)^q. \]

This concludes the proof of the theorem. \(\square\)

### 2.6. Vector-valued singular integrals.

We conclude this part by extending Thm. 2.1 to the case of vector-valued functions. This extension will be used in the next section, when developing the Littlewood–Paley theory.

Although this part of the theory works in the case of functions taking values in a separable Banach space, we restrict our attention to the case of functions with values in a separable Hilbert space $H$.

We say that a function $f$ from $\mathbb{R}^n$ to $H$ is measurable if for every $v \in H$, the mapping
\[\mathbb{R}^n \ni x \mapsto \langle f(x), v \rangle_H \in \mathbb{C}\]
is measurable.

For $1 \leq p \leq \infty$ we define the space $L^p(\mathbb{R}^n, H)$ to be the space of measurable functions $f : \mathbb{R}^n \to H$ such that
\[\left( \int_{\mathbb{R}^n} \|f(x)\|_H^p \, dx \right)^{1/p} < \infty. \]
Then, $L^p(\mathbb{R}^n) = L^p(\mathbb{R}^n, \mathbb{C})$, but we will still denote it by $L^p$ for simplicity.

A typical example of $L^p(\mathbb{R}^n, H)$ is a function of the form $f(x) = f_s(x)v$, where $f_s$ is a scalar-valued function in $L^p(\mathbb{R}^n)$ and $v$ is a fixed element of $H$. The subset
\[L^p \otimes H = \{ f : f = \sum_{j=1}^N f_j v_j \text{ where } f_j \in L^p, \ v_j \in H \},\]
of finite linear combination of functions of the form above, is dense in $L^p(\mathbb{R}^n, H)$. (The proof of this fact is simple, for this and other facts about this topic, see [Ru].)
Given $f = \sum_{j=1}^{N} f_jv_j \in L^1 \otimes \mathcal{H}$, we define its integral to be the element of $\mathcal{H}$ given by
\[
\int_{\mathbb{R}^n} f(x) \, dx = \sum_{j=1}^{N} \left( \int_{\mathbb{R}^n} f_j(x) \, dx \right) v_j.
\]
This map $f \mapsto \int f(x) \, dx$ extends to all of $L^1(\mathbb{R}^n, \mathcal{H})$ by density. Then, for $f \in L^1(\mathbb{R}^n, \mathcal{H})$, $\int f(x) \, dx$ is the unique element of $\mathcal{H}$ such that
\[
\left\langle \int f(x) \, dx, v \right\rangle_{\mathcal{H}} = \int_{\mathbb{R}^n} \langle f(x), v \rangle_{\mathcal{H}} \, dx
\]
for all $v \in \mathcal{H}$.

Similarly reasoning works also in the case of $f \in L^p(\mathbb{R}^n, \mathcal{H})$ and $g \in L^{p'}(\mathbb{R}^n, \mathcal{H})$: notice that by applying the standard Hölder’s inequality it follows that
\[
\int_{\mathbb{R}^n} |\langle f(x), g(x) \rangle_{\mathcal{H}}| \, dx \leq \left( \int_{\mathbb{R}^n} \|g(x)\|_{\mathcal{H}}^{p'} \, dx \right)^{1/p'} \left( \int_{\mathbb{R}^n} \|f(x)\|_{\mathcal{H}}^p \, dx \right)^{1/p} = \|g\|_{L^{p'}(\mathbb{R}^n, \mathcal{H})} \|f\|_{L^p(\mathbb{R}^n, \mathcal{H})}.
\]
Therefore $\int_{\mathbb{R}^n} \langle f(x), g(x) \rangle_{\mathcal{H}} \, dx$ converges and it turns out that
\[
\|g\|_{L^{p'}(\mathbb{R}^n, \mathcal{H})} = \sup \left\{ \left| \int_{\mathbb{R}^n} \langle f(x), g(x) \rangle \, dx \right| : \|f\|_{L^p(\mathbb{R}^n, \mathcal{H})} = 1 \right\}.
\]
This implies that $L^{p'}(\mathbb{R}^n, \mathcal{H}) \subseteq \left( L^p(\mathbb{R}^n, \mathcal{H}) \right)^*$. In fact equality holds (in the Hilbert case), that is,
\[
\left( L^p(\mathbb{R}^n, \mathcal{H}) \right)^* \equiv L^{p'}(\mathbb{R}^n, \mathcal{H})
\]

We conclude this preliminary discussion introducing the singular integral in the vector-value case.

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Suppose $f : \mathbb{R}^n \to \mathcal{H}_1$ and $\tilde{K} : \mathbb{R}^n \setminus \{0\} \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ are measurable functions. We may consider the integral operator
\[
\tilde{T}f(x) = \int_{\mathbb{R}^n \setminus \{0\}} K(y) f(x - y) \, dy
\]
Here clearly the expression $\tilde{K}(y) f(x - y)$ can only be interpreted as the element $\tilde{K}(y)$ of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ acting on $f(x - y) \in \mathcal{H}_1$; hence, it $\tilde{T}f(x)$ is well defined, it is an element of $\mathcal{H}_2$.

**Theorem 2.16.** Suppose $\tilde{T}$ is a bounded linear operator from $L^r(\mathbb{R}^n, \mathcal{H}_1)$ to $L^r(\mathbb{R}^n, \mathcal{H}_2)$ for some $r$, $1 < r < \infty$, defined by the integral operator with kernel $K$ as in (2.19). Further, assume that $K$ satisfies the (vector-valued) Hörmander condition
\[
\int_{|x| + |y| > 2|x|} \|\tilde{K}(x - y) - \tilde{K}(x)\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)} \, dx \leq B.
\]
Then $\tilde{T}$ is bounded from $L^p(\mathbb{R}^n, \mathcal{H}_1)$ to $L^p(\mathbb{R}^n, \mathcal{H}_2)$ when $1 < p < \infty$ and it is weak-type $(1, 1)$, that is,
\[
\left| \left\{ x \in \mathbb{R}^n : \|\tilde{T}f(x)\|_{\mathcal{H}_2} > \lambda \right\} \right| \leq \frac{C}{\lambda} \|f\|_{L^p(\mathbb{R}^n, \mathcal{H}_1)}.
\]
Proof. For simplicity, we are going to drop the "\(-\)"-notation.

The proof does not follow from the scalar case, but, it follows from the same proof. We begin by observing that the adjoint of $T$ has kernel $K^*(x)$, since

$$
\langle Tf, g \rangle = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x - y)f(y)dy \right)g(x) dx
$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle K(x - y)f(y), g(x) \rangle_{\mathcal{H}_2} dy dx
$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle f(y), K^*(x - y)g(x) \rangle_{\mathcal{H}_2} dx dy
$$

$$= \int_{\mathbb{R}^n} \langle f(y), \int_{\mathbb{R}^n} K^*(x - y)g(x) dx \rangle_{\mathcal{H}_1} dy.
$$

If we show that $T$ is weak-type $(1, 1)$, by Marcinkiewicz interpolation theorem it would follow that $T$ is of strong-type $(p, p)$ for $1 < p \leq r$. The operator $T^*$ satisfies the same assumption as $T$, with the assumption that $T^*$ is a bounded linear operator from $L^r'(\mathbb{R}^n, \mathcal{H}_2)$ to $L^r'(\mathbb{R}^n, \mathcal{H}_1)$, so that it is also of strong-type $(p, p)$ for $1 < p \leq r'$. Thus, by duality, $T$ is of strong-type $(p, p)$ for $r < p < \infty$, so it follows that $T$ is of strong-type $(p, p)$ for $1 < p < \infty$.

Thus, (as in the scalar case) it suffices to show that $T$ is weak-type $(1, 1)$.

We now proceed as in the proof of Thm. 2.1. Hence, we only sketch the argument, indicating the main differences.

The main point is to generalize the Calderón–Zygmund decomposition of an $L^1$-function. In the proof of Thm. 1.12, as well as in Thm. 1.10, we assumed that $f$ is non-negative (since we wrote an arbitrary complex-valued function as $f = (\Re f)_+ - (\Re f)_- + i[(\Im f)_+ - (\Im f)_-])$.

We provide the details for sake of completeness.

Given $\lambda > 0$, by applying the Calderón–Zygmund decomposition to the function $\|f(x)\|_{\mathcal{H}_1}$ we find that there exists a sequence of disjoint cubes $\{Q_j\}$ such that:

(i) $0 \leq \|f(x)\|_{\mathcal{H}_1} \leq \lambda$ for almost all $x \not\in \cup_j Q_j$;

(ii) $|\cup_j Q_j| \leq \frac{1}{\lambda} \|f\|_{L^1(\mathbb{R}^n, \mathcal{H}_1)}$;

(iii) $\lambda < \frac{1}{|Q_j|} \int_{Q_j} \|f(x)\|_{\mathcal{H}_1} dx \leq 2^n \lambda$.

Now we decompose $f$ as the sum $f = g + b$, where

$$g(x) = f(x)\chi_{\{\cup_j Q_j\}}(x) + \sum_j \left( \frac{1}{|Q_j|} \int_{Q_j} f(y) dy \right) \chi_{Q_j}(x),$$

and

$$b(x) = \sum_j b_j(x) = \sum_j \left( f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(y) dy \right) \chi_{Q_j}(x).$$

Notice that, as $f(x)$, $g(x)$ and $b(x)$ are vectors in $\mathcal{H}_1$, for a.a. $x \in \mathbb{R}^n$, and that $\frac{1}{|Q_j|} \int_{Q_j} f(y) dy$ is also a well-defined element of $\mathcal{H}_1$, since $f \in L^1(\mathbb{R}^n, \mathcal{H}_1)$ and by the identity (2.17).
Notice that, similarly to the proof of Thm.'s 1.12 and 2.1, we have that
\[\|g(x)\|_{\mathcal{H}_1} \leq \|f(x)\|_{\mathcal{H}_1} \lambda \chi_{\bigcup_j Q_j}(x) + \sum_j \left| \int_{Q_j} f(y) \, dy \right|_{\mathcal{H}_1} \chi_{Q_j}(x)\]
and also that
\[\int_{\mathbb{R}^n} |g(x)|_{\mathcal{H}_1} \, dx \leq \int_{\mathbb{R}^n} |f(x)|_{\mathcal{H}_1} \, dx\].
(This last inequality follows since \(g(x) = f(x)\) on \(\mathcal{C}(\bigcup_j Q_j)\), while \(\int_{Q_j} g(x) \, dx = \int_{Q_j} f(x) \, dx\) for all \(Q_j\).)

On the other hand, \(\int_{\mathbb{R}^n} b(x) \, dx = 0_{\mathcal{H}_1}\), the zero element in \(\mathcal{H}_1\).

We now proceed and in the proof of Thm. 2.1. Since
\[\|T f(x)\|_{\mathcal{H}_2} \leq \|T g(x)\|_{\mathcal{H}_2} + \|T b(x)\|_{\mathcal{H}_2}\],
we have
\[\left\{ x : \|T f(x)\|_{\mathcal{H}_2} > \lambda \right\} \leq \left\{ x : \|T g(x)\|_{\mathcal{H}_2} > \lambda/2 \right\} + \left\{ x : \|T b(x)\|_{\mathcal{H}_2} > \lambda/2 \right\}\].

Now, using the boundedness of \(T : L^r(\mathbb{R}^n, \mathcal{H}_1) \to L^r(\mathbb{R}^n, \mathcal{H}_2)\),
\[\left\{ x : \|T g(x)\|_{\mathcal{H}_2} > \lambda/2 \right\} = \int_{\left\{ x : \|T g(x)\|_{\mathcal{H}_2} > \lambda/2 \right\}} \frac{\|T g(x)\|_{\mathcal{H}_2}}{(\lambda/2)^r} \, dx \leq C \int_{\left\{ x : \|T g(x)\|_{\mathcal{H}_2} > \lambda/2 \right\}} \frac{\|T g(x)\|_{\mathcal{H}_2}}{(\lambda/2)^r} \, dx \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} \|g(x)\|_{\mathcal{H}_1} \, dx \]
\[= \frac{C}{\lambda} \int_{\mathbb{R}^n} \|f(x)\|_{\mathcal{H}_1} \, dx\].

Next, let \(Q_j^*\) denote the cube with the same center \(c_j\) as \(Q_j\) with side length \(\sqrt{2n} \) times longer. Then we estimate,
\[\left\{ x : \|T b(x)\|_{\mathcal{H}_2} > \lambda/2 \right\} = \left\{ x : \|T b(x)\|_{\mathcal{H}_2} > \lambda/2 \right\} + \left\{ x : \|T b(x)\|_{\mathcal{H}_2} > \lambda/2 \right\} \]
\[\leq \left| \bigcup_j Q_j^* \right| + \left\{ x : \|T b(x)\|_{\mathcal{H}_2} > \lambda/2 \right\} \]
\[\leq \frac{2}{\lambda} \|f\|_{L^1(\mathbb{R}^n, \mathcal{H}_1)} + \frac{2}{\lambda} \int_{\mathbb{R}^n \setminus \left( \bigcup_j Q_j^* \right)} \|T b(x)\|_{\mathcal{H}_2} \, dx\].

Since \(\|T b(x)\|_{\mathcal{H}_2} \leq \sum_j \|T b_j(x)\|_{\mathcal{H}_2} \) a.e., it will suffice to prove that
\[\sum_j \int_{\mathbb{R}^n \setminus Q_j^*} \|T b_j(x)\|_{\mathcal{H}_2} \, dx \leq C \|f\|_{L^1(\mathbb{R}^n, \mathcal{H}_1)}\].
(2.21)
Notice that, since \( \int_{Q_j} b_j(y) dy = 0_{H_1} \), also
\[
\int_{Q_j} K(x - c_j)b_j(y) dy = K(x - c_j) \int_{Q_j} b_j(y) dy = 0_{H_2},
\]
so that
\[
\int_{R \setminus Q_j^*} \| T b_j(x) \|_{H_2} dx = \int_{R \setminus Q_j^*} \left\| \int_{Q_j} K(x - y)b_j(y) dy \right\|_{H_2} dx
\]
\[
= \int_{R \setminus Q_j^*} \left\| \int_{Q_j} (K(x - y)b_j(y) \right. - K(x - c_j)b_j(y)) dy \right\|_{H_2} dx
\]
\[
\leq \int_{R \setminus Q_j^*} \int_{Q_j} \left\| (K(x - y) - K(x - c_j))b_j(y) \right\|_{H_2} dy dx
\]
\[
\leq \int_{R \setminus Q_j^*} \int_{Q_j} \left\| K(x - y) - K(x - c_j) \right\|_{L(H_1, H_2)} \| b_j(y) \|_{H_1} dy dx
\]
\[
\leq \int_{Q_j} \| b_j(y) \|_{H_1} \int_{R \setminus Q_j^*} \left\| K(x - y) - K(x - c_j) \right\|_{L(H_1, H_2)} dx dy
\]
\[
\leq B \int_{Q_j} \| b_j(y) \|_{H_1} dy.
\]
Therefore,
\[
\sum_j \int_{R \setminus Q_j^*} \| T b_j(x) \|_{H_2} dx \leq B \sum_j \int_{Q_j} \| b_j(y) \|_{H_1} dy \leq 2B \| f \|_{L^1(R^n, H_1)},
\]
estimates that completes the proof. \( \square \)
3. Littlewood–Paley theory

A simple application of the theory of vector-valued singular integrals is the following result. In this theorem the vector-valued kernel $\vec{K}$ is a constant sequence. In this case it is easy to prove the boundedness of the corresponding vector-valued singular integral $\vec{T}$.

**Theorem 3.1.** Let $T$ be a convolution operator which is bounded $L^2(\mathbb{R}^n)$ and whose integral kernel satisfy the Hörmander condition (2.2). Let $1 < p, r < \infty$. Then we have the strong $(p,p)$-bound

$$\left\| \left( \sum_j |Tf_j|^r \right)^{1/r} \right\|_p \leq C_{p,r} \left\| \left( \sum_j |f_j|^r \right)^{1/r} \right\|_p,$$

and the weak $(1,1)$-bound

$$\left\{ x \in \mathbb{R}^n : \left( \sum_j |Tf_j|^r \right)^{1/r} > \lambda \right\} \leq \frac{C_r}{\lambda} \left\| \left( \sum_j |f_j|^r \right)^{1/r} \right\|_{L^1}.$$

**Proof.** We have stated the theorem in the case of the $\ell^r$-norm, $1 < r < \infty$. Since we stated and proved Thm. 2.16 in the case of functions taking values in a Hilbert space (rather than in a separable, reflexive, Banach space).

Thus, we consider only the case $r = 2$. We only have to check that the hypotheses of Thm. 2.16 are satisfied.

In the current situation we consider functions $f : \mathbb{R}^n \to \ell^2$, that is, $f(x) = \{f_j(x)\}$, where $f_j$ are scalar valued functions. Then, we have that

$$\vec{T}(f) = \vec{T}(\{f_j\}) = \{K * f_j\}.$$

Then $\vec{T}(f) = \vec{K} * f$, where

$$\vec{K}(x) : \ell^2 \to \ell^2$$

$$\{a_j\} \mapsto \{K(x) a_j\}.$$

Hence, $\vec{T} : L^2(\mathbb{R}^n, \ell^2) \to L^2(\mathbb{R}^n, \ell^2)$ is clearly bounded since, (as in the discussion prior to the theorem) if $f \in L^2(\mathbb{R}^n, \ell^2)$,

$$\|\vec{T}(f)\|_{L^2(\mathbb{R}^n, \ell^2)}^2 = \int_{\mathbb{R}^n} \sum_j |K * f_j(x)|^2 \, dx = \int_{\mathbb{R}^n} \sum_j |\hat{K}(\xi)|^2 |\hat{f}_j(\xi)|^2 \, d\xi$$

$$\leq A \|f\|_{L^2(\mathbb{R}^n, \ell^2)}^2.$$

Next,

$$\int_{|x| > 2|y|} \|K(x) - K(y)\|_{\ell^2(\ell^2)} \, dx = \int_{|x| > 2|y|} \|K(x) - K(y)\| \, dx \leq B.$$

The result now follows from Thm. 2.16. $\square$

In order to illustrate the Littlewood–Paley theorem we consider a collection $\{E_j\}$ of mutually disjoint measurable sets in $\mathbb{R}^n$ and the operator $S$ initially defined on Schwartz functions,

$$Sf = \mathcal{F}^{-1} \left( \sum_j (\chi_{E_j} \hat{f}) \right). \quad (3.1)$$
Then, it is immediate to see that $S$ is bounded on $L^2$, since, by Plancherel’s theorem
\[ \|Sf\|_{L^2}^2 = \| \sum_j (\chi_{E_j} \hat{f})(\xi) \|_{L^2}^2 = \sum_j \| \hat{f}(\xi) \|_{L^2}^2 = \|f\|_{L^2}^2. \] (3.2)

Starting from this simple observation we now state the following result. Notice that in this theorem, we restrict ourselves to the 1-dimensional case.

**Theorem 3.2.** Let $E_j = (-2^{j+1}, -2^j] \cup [2^j, 2^{j+1}) \subset \mathbb{R}$ and let $S_j$ be defined as
\[ (S_j f)(\xi) = \chi_{E_j} \hat{f}(\xi). \]
Then, for $1 < p < \infty$ there exists $C > 0$ such that, for all $f \in L^p(\mathbb{R})$
\[ \frac{1}{C} \|f\|_{L^p} \leq \left( \sum_j \|S_j f\|_{L^p}^2 \right)^{1/2} \leq C \|f\|_{L^p}. \]

We will prove this theorem as a consequence of the Littlewood–Paley theorem, Thm. 3.6 in the next section, that holds true also in $\mathbb{R}^n$. In that theorem we will deal with an operator as the one in (3.1). Notice that
\[ \sum_j \chi_{E_j}(\xi) = \sum_j \chi_{E_0}(2^{-j} \xi) = 1 \]
for all $\xi \in \mathbb{R} \setminus \{0\}$. We will need a smooth decomposition of the function identically 1.

**Lemma 3.3.** There exists a function $\varphi \in C_0^\infty$ such that
\[ (i) \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1 \quad \text{for all } \xi \neq 0. \]
Moreover, there exists another function $\varphi_0 \in C_0^\infty$ such that
\[ (ii) \varphi_0(\xi) + \sum_{j=1}^{+\infty} \varphi(2^{-j} \xi) = 1 \quad \text{for all } \xi. \]

**Proof.** Let $\varphi_0 \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi \geq 0$, $\varphi_0(\xi) = 1$ if $|\xi| \leq 1$ and $\varphi_0(\xi) = 0$ if $|\xi| \geq 2$. (Such function exists by the $C^\infty$-Urysohn’s lemma, see [Fo], for instance.) We set
\[ \varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi). \]

Notice that $\varphi(\xi) = 0$ if $|\xi| \leq 1/2$ or $|\xi| \geq 2$, that is, $\text{supp } \varphi \subseteq \{ \xi : 1/2 \leq |\xi| \leq 2 \}$. By induction, it is elementary to prove that, for $\ell \geq 1$ and all $\xi$,
\[ \varphi_0(\xi) + \sum_{j=1}^{\ell} \varphi(2^{-j} \xi) = \varphi_0(2^{-\ell} \xi). \] (3.3)

Letting $\ell \to +\infty$ we obtain (ii).

Next, again by induction on $\ell'$, with $\ell' \leq 0$, it is easy to show that
\[ \sum_{j=\ell'}^{\ell} \varphi(2^{-j} \xi) = \varphi_0(2^{\ell'} \xi) - \varphi_0(2^{-\ell'+1} \xi). \] (3.4)
Letting $\ell \to +\infty$ and $\ell' \to -\infty$ we obtain (i). (Notice that $\varphi_0(2^{-\ell}\xi) - \varphi_0(2^{-\ell'+1}\xi) \to 1$ only when $\xi \neq 0$.) \hfill \Box

**Corollary 3.4.** There exists $\psi \in S(\mathbb{R}^n)$ such that $\text{supp } \psi \subseteq \{\xi : 1/2 \leq |\xi| \leq 2\}$ and

$$\sum_{j \in \mathbb{Z}} |\psi(2^{-j}\xi)|^2 = 1 \quad \text{ for } \xi \neq 0.$$  

(3.5)

**Proof.** With $\varphi$ as in the Lemma, we define $\Phi$ by setting $(\hat{\Phi})^2 = \varphi$ so that

$$(\mathcal{F}(\Phi_{2^{-j}}))^2(\xi) = (\hat{\Phi}(2^{-j}\xi))^2 = \varphi(2^{-j}\xi),$$

and therefore,

$$\sum_{j \in \mathbb{Z}} |\mathcal{F}(\Phi_{2^{-j}})(\xi)|^2 = \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1$$

for $\xi \neq 0$.

Then, it suffices to set $\psi = \hat{\Phi}$. \hfill \Box

The next result is the first part of the Littlewood–Paley Thm. In a sense, it is the simplest, and least interesting, part. It shows that a certain decomposition on the Fourier transform side of a function gives rise to an operator that is bounded in $L^p$. It is more interesting to show that, under an additional hypothesis, such an operator is also bounded from below-- as we shall see in Thm. 3.6.

**Theorem 3.5. (Littlewood–Paley Thm., part 1)** Let $\hat{\Phi} \in S(\mathbb{R}^n)$ be such that

$$\sum_{j \in \mathbb{Z}} |\hat{\Phi}(2^{-j}\xi)|^2 \leq C$$

for all $\xi \neq 0$, and define $\tilde{\Delta}_j f = f \ast \hat{\Phi}_{2^{-j}}$. Then, for $1 < p < \infty$, there exists $C > 0$ such that

$$\left\| \left( \sum_{j \in \mathbb{Z}} |\tilde{\Delta}_j f|^2 \right)^{1/2} \right\|_{L^p} \leq C \|f\|_{L^p}.$$  

(3.6)

**Proof.** If we set

$$\bar{T}(f) = \{\tilde{\Delta}_j f\}_{j \in \mathbb{Z}}$$

if suffices to show that $\bar{T}$ satisfies the hypotheses of Thm. 2.16, when

$$\bar{T} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n, \ell^2).$$

As we have seen already, $\bar{T}$ is bounded when $p = 2$. For,

$$\|\bar{T} f\|_{L^2(\mathbb{R}^n, \ell^2)}^2 = \int_{\mathbb{R}^n} \|\bar{T}(f)(x)\|_{\ell^2}^2 \, dx = \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |\Delta_j(f)(x)|^2 \, dx$$

$$= \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |\hat{\Phi}(2^{-j}\xi)|^2 |\hat{f}(\xi)|^2 \, d\xi$$

$$\leq C \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \, d\xi$$

$$= C \|f\|_{L^2}^2.$$
Thus, it will suffice to show that
\[
\|\nabla \tilde{K}(x)\|_{L^\infty(C,\ell^2)} \leq \frac{C}{|x|^{n+1}}.
\]
Now,
\[
\nabla \tilde{K}(x) = \{ \nabla \tilde{\Phi}_{2^{-j}}(x) \} = \{ \nabla (2^{nj} \tilde{\Phi}(2^j x)) \} = \{ 2^{(n+1)j} (\nabla \tilde{\Phi})(2^j x) \}.
\]
Then, using the fact that \( \tilde{\Phi} \) is a Schwartz function, for any integer \( N > 0 \) there exists a constant \( C_N > 0 \) such that
\[
\|\nabla \tilde{K}(x)\|_{\ell^2} = \left( \sum_{j \in \mathbb{Z}} |2^{(n+1)j} (\nabla \tilde{\Phi})(2^j x)|^2 \right)^{1/2} \leq C_N \left( \sum_{j \in \mathbb{Z}} 2^{(n+1)j} \min(1, |2^j x|^{-N}) \right) \leq C_N \left( \sum_{j \leq j_0} 2^{(n+1)j} + \sum_{j > j_0} 2^{(n+1)j} |2^j x|^{-N} \right) = C_N \left( \sum_{j \leq j_0} 2^{(n+1)j} + \frac{1}{|x|^N} \sum_{j > j_0} 2^{(n+1-N)j} \right),
\]
with \( j_0 \) to be selected. Choosing \( j_0 \) so that \( 2^{j_0} \simeq |x|^{-1} \), and \( N > n+1 \) we have\(^5\),
\[
\|\nabla \tilde{K}(x)\|_{\ell^2} \leq C_N \left( 2^{(n+1)j_0} + \frac{2^{(n+1-N)j_0}}{|x|^N} \right) \leq \frac{C}{|x|^{n+1}},
\]
as we wished to show. \( \square \)

3.1. The Littlewood–Paley theorem.

**Theorem 3.6.** (Littlewood–Paley theorem) Let \( \psi \in \mathcal{S}(\mathbb{R}^n) \) be as in (3.5). Let \( \Phi = F^{-1}\psi \) and set

\[
\Delta_j f = f \ast \Phi_{2^{-j}}.
\]
Then, for \( 1 < p < \infty \), there exists \( C > 0 \) such that
\[
\frac{1}{C} \|f\|_{L^p} \leq \left( \sum_{j} |\Delta_j f|^2 \right)^{1/2} \left\|\ell^2 \right\|_{L^p} \leq C \|f\|_{L^p}.
\]

**Proof.** The bound from above follows directly from Thm. 3.5, where we use only the estimate
\[
\sum_{j \in \mathbb{Z}} |\psi(2^{-j} \xi)|^2 \leq C.
\]
\(^5\) Here we use the elementary estimates \( \sum_{j \geq j_0} q^j \simeq q^{j_0} \) if \( 0 < q < 1 \), and \( \sum_{j \leq j_0} q^j \simeq q^{j_0} \) if \( q > 1 \).
For the bound from below, we notice that
\[ \|f\|_{L^2} = \left( \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \sum_{j \in \mathbb{Z}} |\psi(2^{-j}\xi)|^2 \, d\xi \right)^{1/2} \]
\[ = \left( \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}} |\hat{f}(\xi)|^2 \psi(2^{-j}\xi) \, d\xi \right)^{1/2} \]
\[ = \left\| \left( \sum_j |\Delta_j f|^2 \right)^{1/2} \right\|_{L^2}. \]

Having the identity \( \|f\|_{L^2} = \|\mathcal{T}(f)\|_{L^2(\mathbb{R}^n,\ell^2)} \), we can polarized it\(^6\), to obtain that for all \( f, g \in L^2(\mathbb{R}^n) \)
\[ \int f \bar{g} = \int \sum_j \Delta_j f \overline{\Delta_j g}. \]

Therefore,
\[
\|f\|_{L^p} = \sup_{\|g\|_{L^p'} = 1} \left| \int f(x) \overline{g(x)} \, dx \right| = \sup_{\|g\|_{L^p'} = 1} \left| \int \sum_j \Delta_j f(x) \overline{\Delta_j g(x)} \, dx \right| \\
\leq \sup_{\|g\|_{L^p'} = 1} \left( \int \left( \sum_j |\Delta_j f(x)|^2 \right)^{1/2} \left( \sum_j |\Delta_j g(\xi)|^2 \right)^{1/2} \, dx \right) \\
\leq \sup_{\|g\|_{L^p'} = 1} \left( \int \left( \sum_j |\Delta_j f(x)|^2 \right)^{p/2} \, dx \right)^{1/p} \left( \int \left( \sum_j |\Delta_j g(\xi)|^2 \right)^{p'/2} \, dx \right)^{1/p'} \\
\leq C \left( \int \left( \sum_j |\Delta_j f(x)|^2 \right)^{p/2} \, dx \right)^{1/p} \tag{3.7},
\]
where we have used the estimate from above. This proves the theorem. □

We are now ready to prove Thm. 3.2. We recall that in this situation the space dimension is \( n = 1 \).

**Proof of Thm. 3.2.** Let \( \psi \in C_0^{\infty}(\mathbb{R}) \), supp \( \psi \subseteq \{ \xi : 1/2 \leq |\xi| \leq 4 \} \) and such that \( \psi(\xi) = 1 \) if \( 1 \leq |\xi| \leq 2 \), that is, if \( \xi \in E_0 \). Therefore,
\[ \psi^{2^{-j}}(\xi) \chi_{E_0}(\xi) = \psi(2^{-j}\xi) \chi_{E_0}(2^{-j}\xi) = \chi_{E_0}(2^{-j}\xi) \]
for all \( \xi \). This implies that, if we define \( \tilde{\Delta}_j f \) by setting
\[ (\tilde{\Delta}_j f)(\xi) = \psi(2^{-j}\xi) \hat{f}(\xi), \]
then
\[ \tilde{\Delta}_j S_j f = S_j f. \]

Of course we can write \( \tilde{\Delta}_j f = \hat{\Phi}_{2^{-j}} * f \), where \( \hat{\Phi} = \psi. \)

---

\(^6\)Recall the identity \( 4\langle f, g \rangle_{H} = \|f + g\|_{L^2}^2 - \|f - g\|_{L^2}^2 + i\|f + ig\|_{L^2}^2 - i\|f - ig\|_{L^2}^2 \) valid on any Hilbert space, and notice that in our case \( \langle f, g \rangle_{H} = \int \sum_j \Delta_j f \overline{\Delta_j g}. \)
Notice that for any given \( \xi \), since \( \text{supp} \psi \subseteq \{ \xi : 1/4 \leq |\xi| \leq 4 \} \), there at most 3 indices \( j_1, j_2, j_3 \) such that \( \psi_j(\xi) \neq 0 \). Therefore,

\[
\sum_j |\psi(2^{-j} \xi)|^2 \leq C,
\]

for all \( \xi \neq 0 \).

We now make the following claim: For \( 1 < p < \infty \), there exists \( C > 0 \) such that for all \( g = \{g_j\} \in L^p(\mathbb{R}^n, \ell^2) \)

\[
\left\| \left( \sum_j |S_j g_j|^2 \right)^{1/2} \right\|_{L^p} \leq C \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_{L^p} = C \|g\|_{L^p(\mathbb{R}^n, \ell^2)}. \tag{3.8}
\]

Assuming the claim we obtain the estimate from above, since

\[
\left\| \left( \sum_j |S_j f_j|^2 \right)^{1/2} \right\|_{L^p} \leq C \left\| \left( \sum_j |\tilde{\Delta}_j f_j|^2 \right)^{1/2} \right\|_{L^p}
\]

\[
\leq C \|f\|_{L^p},
\]

by Thm. 3.5.

The estimate from below follows by polarizing the identity (3.2) and using duality, as in the proof of Thm. 3.6, inequality (3.7), by replacing \( \Delta_j \) with \( S_j \).

Thus, we only need to prove the claim.

We now consider the vector-valued operator \( \vec{S} : L^p(\mathbb{R}^n, \ell^2) \rightarrow L^p(\mathbb{R}^n, \ell^2) \) defined as

\[
\vec{S}(g) = \{ S_j(g_j) \} \equiv \{ K_j * g_j \}
\]

where \( K_j = \mathcal{F}^{-1}(\chi_{E_j}) \). For this operator we wish to prove the bound (3.8).

Instead of proving that the operator \( \vec{S} \) and its kernel satisfy the hypotheses of Thm. 2.16, we proceed in a direct way.

Consider for the moment the (bounded) interval \((a,b)\). Notice that

\[
\chi_{(a,b)}(\xi) = \frac{1}{2} \left[ \text{sgn}(\xi - a) - \text{sgn}(\xi - b) \right].
\]

Let \( a \in \mathbb{R} \) and observe that the operator (called the modulation operator)

\[
M_a f(x) = e^{2\pi i a x} f(x)
\]

is bounded on \( L^p \) for all \( p \), and also that \( (M_{-a}f)'(\xi) = \hat{f}(\xi + a) = (\tau_a \hat{f})(\xi) \). Then notice that, for \( f \in S(\mathbb{R}) \),

\[
\chi_{(a,b)}(\xi) \hat{f}(\xi) = \frac{1}{2} \left[ \text{sgn}(\xi - a) - \text{sgn}(\xi - b) \right] \hat{f}(\xi)
\]

\[
= \frac{1}{2} \left[ \tau_{-a} \left( \text{sgn}(\xi) \tau_a(\hat{f}) \right)(\xi) - \tau_{-b} \left( \text{sgn}(\xi) \tau_b(\hat{f}) \right)(\xi) \right]
\]

\[
= \frac{i}{2} \mathcal{F} \left[ \left( M_a H M_{-a} - M_b H M_{-b} \right)(\hat{f}) \right](\xi),
\]

where \( \mathcal{F} \) denotes the Fourier transform.
where $H$ denotes the Hilbert transform. (This last equality can be easily be checked by computing the Fourier transform on the right hand side, recalling that $(Hf)\hat{\cdot}(\xi) = -i\text{sgn}(\xi)\hat{f}(\xi)$.)

Therefore,

$$\mathcal{F}^{-1}(\chi_{(a,b)}) * f = \frac{i}{2} \left[ M_aHM_{-a} - M_bHM_{-b} \right](f).$$ (3.9)

Finally we go back to the operator $\tilde{S}$. By (3.9) above we have that

$$S_j(g_j)(x) = \mathcal{F}^{-1}\left(\chi_{(-2^j+1,-2^j]} + \chi_{[2^j,2^j+1)}\right) * g_j(x)$$

$$= \frac{i}{2} \left[ M_{-2^j+1}HM_{2^j+1} - M_{-2^j}HM_{2^j} + M_{2^j}HM_{-2^j} - M_{2^j+1}HM_{-2^j+1} \right](g_j),$$

so that

$$|S_j(g_j)(x)| \leq |H(M_{2^j+1}g_j)(x)| + |H(M_{2^j}g_j)(x)| + |H(M_{-2^j}g_j)(x)| + |H(M_{-2^j+1}g_j)(x)| .$$

Therefore, in order to prove the claim it suffices to show that each of the four operators on the right hand side above satisfies (3.8), that is,

$$\left\| \left( \sum_j |H(M_{a_j}g_j)|^2 \right)^{1/2} \right\|_{L^p} \leq C \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_{L^p},$$

where $\{a_j\}$ is a sequence of real numbers.

Applying Thm. 3.1 we have that

$$\left\| \left( \sum_j |H(M_{a_j}g_j)|^2 \right)^{1/2} \right\|_{L^p} \leq C \left\| \left( \sum_j |M_{a_j}g_j|^2 \right)^{1/2} \right\|_{L^p}$$

$$= C \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_{L^p} .$$

This proves the claim and therefore the theorem. ☐
4. Fourier multipliers

In this chapter we can to the analysis of the so-called Fourier multipliers, that is, of the operators of the form, initially defined on Schwartz functions,

\[ T_m(f)(x) = \mathcal{F}^{-1}(m \hat{f})(x), \]

where \( m \in S' \) is a measurable function on \( \mathbb{R}^n \). Moreover, observe that \( m \in S' \) implies that \( m = \hat{K} \), for some \( K \in S' \), so that

\[ T_m(f) = K * f. \]

Therefore, the singular integrals (of convolution type) that we have seen in the previous sections, are particular cases of Fourier multipliers.

Notice that \( \hat{f} \in S \), so that \( m \hat{f} \in S' \) and it makes sense to define its inverse Fourier transform.

We mention, without proving, the following result, due to Hörmander, that shows that every bounded linear operator between \( L^p \)-spaces that commutes with translations is given by the convolution with a tempered distribution \( K \); hence it is a Fourier multiplier.

Recall that, if \( a \in \mathbb{R}^n \) we define \( \tau_a f(x) = f(x + a) \). An operator \( T \) is said to commute with translations if

\[ \tau_a[T(f)](x) = T[\tau_a f](x), \]

for all \( a \in \mathbb{R}^n \), and a.a. \( x \).

**Theorem 4.1.** Let \( 1 \leq p, q \leq \infty \). Suppose that \( T : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \) is a bounded operator that commutes with translations. Then there exists a unique \( u \in S' \) such that

\[ T(f) = u * f \]

for all \( f \in S \).

For the proof we refer to [StWe], Thm. 3.16.

We will first show a few basic properties of these operators, and then describe some sufficient conditions on the function \( m \) to ensure that \( T_m \) is a bounded operator on \( L^p \).

4.1. The space \( M_p \) of \( L^p \)-bounded Fourier multipliers. We define the space

\[ M_p = \{ m \in S' : T_m : L^p \to L^p \text{ is bounded} \}, \]

and we set

\[ \| m \|_{M_p} = \| T_m \|_{(L^p,L^p)} = \sup_{\| f \|_{L^p} = 1} \| T_m(f) \|_{L^p}. \]

By Plancherel’s theorem we know that if \( m \) is bounded then \( T_m \) is bounded on \( L^2 \). The converse of this assertion is also true, as consequence of the following result.

**Lemma 4.2.** If \( T_m \) is bounded on \( L^p \) then \( m \) is bounded. In particular, \( T_m \) is bounded on \( L^2 \) and only if \( m \in L^\infty \) and, in this case, \( \| T_m \|_{(L^2,L^2)} = \| m \|_{L^\infty} \). Finally, \( M_p = M_{p'} \), if \( p \) and \( p' \) are conjugate exponents.

**Proof.** If \( m \) is not bounded, then it is easy to see that \( T_m \) cannot be bounded on \( L^2 \) (exercise). Now, it is easy to see that \( \| T_m \|_{(L^2,L^2)} = \| m \|_{L^\infty} \). For, by Plancherel theorem,

\[ \| T_m f \|_{L^2} = \| m \hat{f} \|_{L^2} \leq \| m \|_{L^\infty} \| \hat{f} \|_{L^2} = \| m \|_{L^\infty} \| f \|_{L^2}, \]

so that \( \| T_m \|_{(L^2,L^2)} \leq \| m \|_{L^\infty} \).
Next, if $\varepsilon > 0$ is fixed, let

$$E_\varepsilon = \{x : |m(x)| > \|m\|_{L^\infty} - \varepsilon\}. $$

Then $E_\varepsilon$ has positive and finite measure, and let $f \in L^2$ be such that $\hat{f} = \chi_{E_\varepsilon}$. Hence,

$$\|T_m f\|^2_{L^2} = \|m\|^2_{L^2} = \int |m(\xi)\hat{f}(\xi)|^2 d\xi$$

$$> (\|m\|_{L^\infty} - \varepsilon)^2 \int |\hat{f}(\xi)|^2 d\xi$$

$$= (\|m\|_{L^\infty} - \varepsilon)^2 \|f\|^2_{L^2}. $$

It follows that $\|T_m\|_{(L^2,L^2)} \geq \|m\|_{L^\infty}$, and then equality holds.

Next we show that if $T_m$ is bounded on $L^p$, $1 \leq p < \infty$, then $m$ is bounded. For, notice that the adjoint operator $T_m^*$ is given by convolution with $\hat{K}$. Thus, $T_m^*$ is also bounded on $L^p$. Since it is bounded on $L^{p'}$, by the Riesz-Thorin interpolation theorem is also bounded on $L^r$ for $r$ between $p$ and $p'$; hence in particular on $L^2$. This implies that $m \in L^\infty$ and we are done. □

**Proposition 4.3.** For $1 < p < q < 2$ we have the inclusion

$$\mathcal{M}_1 \subset \mathcal{M}_p \subset \mathcal{M}_q \subset \mathcal{M}_2 \quad (4.1)$$

and moreover,

$$\|m\|_{L^\infty} \leq \|m\|_{\mathcal{M}_q} \leq \|m\|_{\mathcal{M}_p}. $$

**Proof.** This follows easily from the previous lemma and the Riesz-Thorin theorem, if $m \in \mathcal{M}_1$, then $m$ is also in $\mathcal{M}_2$ and therefore in $\mathcal{M}_r$ for all $1 < r < 2$. The same argument proves the other inclusions in (4.1).

Using again the Riesz-Thorin theorem, for $1 < p < 2$ we have that

$$\|T_m\|_{(L^2,L^2)} \leq \|T_m\|_{(L^p,L^p)}^{1/2} \|T_m\|_{(L^{q'},L^{q'})}^{1/2} = \|T_m\|_{(L^p,L^p)},$$

since

$$\frac{1}{2} = \frac{1}{p} + \frac{1}{p'}.$$ A similar argument shows that, for $1 < p < q < 2$, $\|T_m\|_{(L^q,L^q)} \leq \|T_m\|_{(L^p,L^p)}$ and we are done. □

**Proposition 4.4.** For $1 \leq p < \infty$ the space $\mathcal{M}_p$ is a Banach space with respect to the norm $\|\cdot\|_{\mathcal{M}_p}$. Moreover, $\mathcal{M}_p$ is closed under pointwise multiplication and it is a Banach algebra.\(^7\)

**Proof.** It suffices to consider the case $1 \leq p \leq 2$.

It is obvious that $\mathcal{M}_p$ is a linear space and that the equality $\|m\|_{\mathcal{M}_p} = \|T_m\|_{(L^p,L^p)}$ defines a norm on $\mathcal{M}_p$.

Next, if $m_1, m_2 \in \mathcal{M}_p$, then

$$(T_{m_1,m_2}f)(\xi) = m_1(\xi)m_2(\xi)\hat{f}(\xi) = (T_{m_1}(T_{m_2})f)(\xi),$$

\(^7\)See the first part of the course, or [So].

\(^8\)An algebra $\mathcal{A}$ which is a Banach space w.r.t. the norm $\|\cdot\|_{\mathcal{A}}$ is called a *Banach algebra* if the product satisfies the inequality $\|xy\|_{\mathcal{A}} \leq \|x\|_{\mathcal{A}}\|y\|_{\mathcal{A}}$ for all $x, y \in \mathcal{A}$. \(^8\)
so that \( T_{m_1m_2} = T_{m_1} T_{m_2} \). Moreover,
\[
\|T_{m_1m_2} f\|_{L^p} \leq \|T_{m_1}\|_{(L^p,L^p)} \|T_{m_2} f\|_{L^p} \\
\leq \|T_{m_1}\|_{(L^p,L^p)} \|T_{m_2}\|_{(L^p,L^p)} \|f\|_{L^p},
\]
that is,
\[
\|m_1 m_2\|_{\mathcal{M}_p} = \|T_{m_1m_2}\|_{(L^p,L^p)} \leq \|T_{m_1}\|_{(L^p,L^p)} \|T_{m_2}\|_{(L^p,L^p)} = \|m_1\|_{\mathcal{M}_p} \|m_2\|_{\mathcal{M}_p}.
\]
This proves that \( \mathcal{M}_p \) is a Banach algebra w.r.t. the pointwise product.

We now show that \( \mathcal{M}_p \) is complete. Let \( \{m_j\} \) be a Cauchy sequence in \( \mathcal{M}_p \). By Prop. 4.3 it follows that \( \{m_j\} \) is a Cauchy sequence in \( L^\infty \), so it converges, in the \( L^\infty \)-norm, hence a.e., to a function \( m \). We wish to show that \( m_j \to m \) in \( \mathcal{M}_p \).

Let \( f \in \mathcal{S} \) be fixed. Then, by Lebesgue’s dominated convergence theorem we have that
\[
T_{m_j}(f)(x) = \int_{\mathbb{R}^n} f(\xi)m_j(\xi)e^{2\pi i x \xi} d\xi \to \int_{\mathbb{R}^n} f(\xi)m(\xi)e^{2\pi i x \xi} d\xi = T_m(f)(x)
\]
a.e. By Fatou’s lemma it follows that
\[
\int_{\mathbb{R}^n} |T_m(f)(x)|^p dx \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} |T_{m_j}(f)(x)|^p dx \\
\leq \liminf_{j \to \infty} \|m_j\|_{\mathcal{M}_p} \|f\|_{L^p},
\]
that is,
\[
\|m\|_{\mathcal{M}_p} \leq \liminf_{j \to \infty} \|m_j\|_{\mathcal{M}_p} \leq C^p
\]
since \( \{m_j\} \) is a Cauchy sequence, hence the sequence of the norms is bounded.

From the last inequality it also follows that
\[
\|m - m_k\|_{\mathcal{M}_p} \leq \liminf_{j \to \infty} \|m_j - m_k\|_{\mathcal{M}_p} < \varepsilon
\]
if \( k \geq k_0 \), since \( \{m_j\} \) is a Cauchy sequence. This proves the proposition. \( \square \)

**Examples 4.5.**

(1) The first example we consider is \( m(\xi) = -i \operatorname{sgn}(\xi) \). Then \( T_m \) is the Hilbert transform, and we know that \( T_m \) is bounded on \( L^p \) for \( 1 < p < \infty \).

Now, as we saw in the proof of Thm. 3.2, and more precisely in (3.9),
\[
T_{\chi_{(a,b)}}(f) = \frac{i}{2} \left[ M_a HM_{-a} - M_b HM_{-b} \right](f).
\]
This identity has an obvious extension to the case in which the interval is unbounded.

Therefore, for all \( a, b \in \mathbb{R} \cup \{\pm \infty\} \), \( m = \chi_{(a,b)} \) is in \( \mathcal{M}_p(\mathbb{R}) \) for \( 1 < p < \infty \), with operator norms uniformly bounded in \( a \) and \( b \).

(2) Let \( m = \chi_{(|\xi|<1)} \) be the characteristic function of the unit ball in \( \mathbb{R}^n \). If \( n = 1 \) it follows from the previous discussion that \( m \in \mathcal{M}_p \) for all \( p, 1 < p < \infty \). On the other hand, if \( n > 1 \) it follows from Fefferman’s theorem on the ball multiplier\(^9\) that \( m \in \mathcal{M}_p \) if and only if \( p = 2 \).

\(^9\)See the first part of the course, or [So].
We conclude this section by stating the characterization of $\mathcal{M}_1$, a significant result, for whose proof we refer to [StWe] or [Gr]. We recall that a finite complex-valued Borel measure $\mu$ defines a tempered distribution by setting $\mu(f) = \int f(x)d\mu(x)$.

**Theorem 4.6.** A measurable function $m \in \mathcal{M}_1$ if and only if $m$ is the Fourier transform of a finite complex-valued Borel measure $\mu$.

4.2. The Sobolev spaces $H^s$. We have remarked that the characteristic function of the unit ball in $\mathbb{R}^n$, with $n > 1$, defines a Fourier multiplier that is bounded on $L^p$ only for $p = 2$. It is clear that, besides the boundedness, the function $m$ needs to possess some regularity. In order to measure regularity we introduce a family of spaces, called the Sobolev spaces, whose elements have derivatives (in some weak sense) in $L^2$.

**Definition 4.7.** We define the Sobolev space $H^s$ of order $s \in \mathbb{R}$ as the space of distributions $f \in S'$ such that $\hat{f}(\xi) \in L^2((1 + |\xi|^2)^s d\xi)$, that is,

$$H^s = \{ f \in S' : \int |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty \}.$$

Notice that $s < t$ implies that $H^t \subseteq H^s$.

Let $k$ be a positive integer. We observe that, by Plancherel theorem, $f \in H^k$ if and only if $(2\pi)^\alpha \hat{f} \in L^2$ for all $\alpha$, $|\alpha| \leq k$. We claim that there exist positive constants $c_1, c_2$ such that, for all $\xi \in \mathbb{R}^n$,

$$c_1(1 + |\xi|^2)^k \leq \sum_{|\alpha| \leq k} |(2\pi)^\alpha|^2 \leq c_2(1 + |\xi|^2)^k.$$

For, if $\alpha$ is a multi-index with $|\alpha| \leq k$, if $|\xi| \geq 1$ we have

$$|(2\pi)^\alpha|^2 \leq C|\xi|^{2k} \leq C(1 + |\xi|^2)^k,$$

while, if $|\xi| \leq 1$,

$$|(2\pi)^\alpha|^2 \leq C \leq C(1 + |\xi|^2)^k.$$ 

On the other hand, since $|\xi|^{2k}$ and $\sum_{j=1}^k |\xi_j|^{2k}$ are both homogeneous of degree $2k$ non-vanishing for $\xi \neq 0$, we have

$$(1 + |\xi|^2)^k \leq c_0(1 + |\xi|^{2k}) \leq c_0 \left(1 + c_0 \sum_{j=1}^k |\xi_j|^{2k}\right) \leq C \sum_{|\alpha| \leq k} |(2\pi)^\alpha|^2.$$

By the claim, it follows that $f \in H^k$ if and only if $(1 + |\xi|^2)^{k/2} \hat{f} \in L^2$. Moreover,

$$c_1 \int (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 d\xi \leq \sum_{|\alpha| \leq k} \int |(2\pi)^\alpha|^2 |\hat{f}(\xi)|^2 d\xi \leq \sum_{|\alpha| \leq k} \int |\partial_x^\alpha f(x)|^2 d\xi \leq c_2 \int (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 d\xi,$$
so that the two norms
\[
\left( \sum_{|\alpha| \leq k} |\partial^\alpha f(x)|^2 \, dx \right)^{1/2} \quad \text{and} \quad \left( \int (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}
\]
are equivalent, and the latter one is defined for all \( k \) real, not just non-negative integers.

We conclude this part with the Sobolev immersion theorem. We set \( C^k_{(0)} = \{ f \in C^k(\mathbb{R}^n) : \partial^\alpha f \in C(0), \text{ for } |\alpha| \leq k \} \).

**Lemma 4.8.** Let \( f \) be such that \( \hat{f} \in H^s \), where \( s > n/2 \). Then \( f \in L^1 \) and for all \( \varepsilon < s - n/2 \) we have
\[
\int_{\mathbb{R}^n} |f(x)|(1 + |x|)^\varepsilon \, dx \leq C_\varepsilon \| \hat{f} \|_{H^s}.
\]

**Proof.** Using the Cauchy-Schwarz inequality we have
\[
\int |f(x)|(1 + |x|)^\varepsilon \, dx \leq 2^\varepsilon \int |f(x)|(1 + |x|^2)^{\varepsilon/2} \, dx \\
\leq 2^\varepsilon \left( \int |f(x)|^2(1 + |x|^2)^s \, dx \right)^{1/2} \left( \int (1 + |x|^2)^{\varepsilon-s} \, dx \right)^{1/2} \tag{4.2}
\]
where \( C_\varepsilon = 2^\varepsilon \left( \int (1 + |x|^2)^{\varepsilon-s} \, dx \right)^{1/2} \) is finite since \( \varepsilon < s - n/2 \). \( \Box \)

**Theorem 4.9.** (Sobolev Embedding Theorem) Let \( t > k + (n/2) \). Then, \( H^t \) embeds continuously in \( C^k_{(0)} \).

**Proof.** Let \( g \in H^t \) and \( |\alpha| \leq k \). By the previous lemma, \( \mathcal{F}^{-1}(\partial^\alpha g) \in L^1 \), since \( \partial^\alpha g \in H^s \), where \( s = t - |\alpha| > n/2 \), and
\[
\int_{\mathbb{R}^n} |\mathcal{F}^{-1}(\partial^\alpha_x g)(x)|(1 + |x|)^\varepsilon \, dx \leq C_\varepsilon \| \partial^\alpha g \|_{H^s} \leq C_\varepsilon \| g \|_{H^t}.
\]
By the Riemann-Lebesgue lemma, \( \partial^\alpha_x g \in C(0) \), for \( |\alpha| \leq k \). \( \Box \)

For sake of clarity, we isolate the result that we need in the remaining of this part.

**Corollary 4.10.** Let \( s > n/2 \). Then, if \( f \in H^s \) it follows that \( f \) is continuous and
\[
\|f\|_{L^\infty} \leq C \|f\|_{H^s}.
\]

We conclude this part with a lemma that we are going to need later on.

**Lemma 4.11.** Let \( f \in H^s \) and \( \varphi \in \mathcal{S} \). Then \( f \varphi \in H^s \) and
\[
\|f \varphi\|_{H^s} \leq C_{\varphi} \|f\|_{H^s}.
\]

\(^{10}\)We recall that we denote by \( C(0) \) the space of continuous functions that vanishes at infinity.
Proof. We begin with the preliminary observation that
\[ 1 + |\xi + \xi'|^2 \leq 1 + 2(|\xi|^2 + |\xi'|^2) \leq 2(1 + |\xi|^2)(1 + |\xi'|^2). \]
Next, since \((\hat{f}\phi)^*(\xi) = \hat{f} \ast \hat{\phi}^*(\xi)\), using the Cauchy-Schwarz inequality, we have
\[
\int_{\mathbb{R}^n} |(\hat{f}\phi)(\xi)|^2 (1 + |\xi|^2)^s d\xi = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \hat{f}(\xi - \xi')\hat{\phi}(\xi') d\xi' \right|^2 (1 + |\xi|^2)^s d\xi
\leq \|\hat{\phi}\|_{L^1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\hat{f}(\xi - \xi')|^2 |\hat{\phi}(\xi')| d\xi' (1 + |\xi|^2)^s d\xi
\leq \|\hat{\phi}\|_{L^1} \int_{\mathbb{R}^n} |\hat{\phi}(\xi')| \left( \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi + \xi'|^2)^s d\xi \right) d\xi'
\leq 2^s \|\hat{\phi}\|_{L^1} \int_{\mathbb{R}^n} |\hat{\phi}(\xi')|(1 + |\xi'|^2)^s \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi d\xi'
\leq C_{\phi} \|f\|^2_{H^s},
\]
where we have used the fact that \(\varphi \in \mathcal{S}\). This proves the lemma. □

4.3. The Mihlin–Hörmander multiplier theorem. In this section we prove the Mihlin–Hörmander multiplier theorem. The Mihlin–Hörmander condition that guarantees that a function gives rise to Fourier multiplier that is bounded on \(L^p(\mathbb{R}^n)\) has the following features:

(i) it is invariant under the dilations \(m \mapsto m(r \cdot)\) for \(r > 0\), in the sense that \(m\) satisfies this condition then also \(m(r \cdot)\) does;
(ii) a certain vector-valued singular integral defined in terms of \(m\) satisfies the (vector-valued) Hörmander condition (2.20); hence it is bounded on \(L^p(\mathbb{R}^n)\), \(1 < p < \infty\) and weak-type \((1,1)\).

We are now ready to define the space of Mihlin-Hörmander multipliers.

Definition 4.12. Let \(0 < a_0 < a < b < b_0\) and let \(\psi \in C^\infty_0\) be such that

(i) supp \(\psi \subseteq \{\xi : a_0 \leq |\xi| \leq b_0\}\);
(ii) \(\psi \geq 0\) and \(\psi(\xi) = 1\) for \(a \leq |\xi| \leq b\).

We call Mihlin-Hörmander multiplier a function \(m\) such that
\[
\sup_{j \in \mathbb{Z}} \|m(2^j \cdot)\psi\|_{H^s} = \|m\|_{MH^s} < \infty,
\]
for \(s > n/2\). We denote the space of such multipliers by \(MH^s\).

Remark 4.13. 

(1) Although we are not going to prove this assertion, and the one below, it is important to notice that the above definition is independent on the choice of \(\psi\) and that, a different \(\psi\) just gives rise to an equivalent norm.

In particular, we choose \(\psi\) such that
\[
\sum_{j \in \mathbb{Z}} |\psi(2^{-j} \xi)|^2 = 1 \quad \text{for } \xi \neq 0, \tag{4.3}
\]
as in condition (3.5).
(2) An equivalent definition can be obtained if we replace the dilations by $2^j$, $j \in \mathbb{Z}$, with the dilations by $r$, with $r > 0$, that is, if we set

$$\|m\|_{M^p} = \sup_{r > 0} \|m(r \cdot)\|_{H^s}.$$  

**Lemma 4.14.** For $s > n/2$, if $m \in MH^s$, then $m$ is bounded and

$$\|m\|_{L^\infty} \leq C\|m\|_{MH^s}.$$  

**Proof.** By assumption, the functions $m(2^j \cdot)\psi$ are in $H^s$ with $s > n/2$ and norms uniformly bounded. By Cor. 4.10, it follows that

$$\|m(2^j \cdot)\|_{L^\infty} = \|m\|_{L^\infty} \leq C\|m\|_{MH^s},$$

with $C$ independent of $j$.

Hence,

$$\sup_{\xi} |m(\xi)|^2 = \sup_j \sup_{2^{-j} \leq |\xi| \leq 2^{-j+1}} |m(\xi)|^2$$

$$= \sup_j \sup_{2^{-j} \leq |\xi| \leq 2^{-j+1}} \sum_k |m(\xi)\psi(2^{-k}\xi)|^2$$

$$= \sup_j \sup_{2^{-j} \leq |\xi| \leq 2^{-j+1}} \sum_{k \in \{j-1, \ldots, j+2\}} |m(\xi)\psi(2^{-k}\xi)|^2$$

$$\leq 4\sup_j \sup_{|\xi| \in \mathbb{R}^n} |m(\xi)\psi(2^{-k}\xi)|^2$$

This proves the lemma. □

**Theorem 4.15.** Let $\psi$ be as in (3.5), and let $m \in MH^s$, with $s > n/2$. Then, the multiplier operator $T_m$ is bounded on $L^p$, $1 < p < \infty$.

Before proving the theorem, we see a corollary.

**Corollary 4.16.** Let $m \in C^k \setminus \{0\}$, with $k > \lfloor n/2 \rfloor + 1$, be such that

$$\sup_{r > 0} r^{n\alpha} \left( \int_{r/2 < |\xi| < 2r} |\partial^\alpha m(\xi)|^2 \, d\xi \right)^{1/2} < \infty,$$  

(4.4)

for all $|\alpha| \leq k$. Then, $m \in M_p$, for $1 < p < \infty$.

In particular, $m \in M_p$, $1 < p < \infty$, if

$$|\partial^\alpha m(\xi)| \leq C|\xi|^{-|\alpha|},$$  

(4.5)

for all $|\alpha| \leq k$.

**Proof.** We make the change of variables $\xi = \xi/r$ in (4.4) to obtain

$$\sup_{r > 0} r^{n\alpha} \left( \int_{r/2 < |\xi| < 2r} |\partial^\alpha m(\xi)|^2 \, d\xi \right)^{1/2} = \sup_{r > 0} \left( \int_{1/2 < |\xi| < 2} r^{2\alpha} |(\partial^\alpha m)(r\cdot)|^2 \, d\xi \right)^{1/2}$$

$$= \sup_{r > 0} \left( \int_{1/2 < |\xi| < 2} |(\partial^\alpha m(r\cdot))(\xi)|^2 \, d\xi \right)^{1/2}.$$
This quantity is, by the assumption (4.4), finite. We prove that (4.3) is also satisfied. For, if $|\alpha| \leq k$, by the Leibnitz rule, using the facts that $\text{supp} \psi \subseteq \{1/2 \leq |\xi| \leq 2\}$ and that $|\partial^\beta \psi(\xi)| \leq C$,

$$\|\partial^\alpha (m(2^j \cdot) \psi)\|_{L^2} = \left\| \sum_{\beta \leq \alpha} c_{\alpha, \beta} \partial^\beta m(2^j \cdot) \partial^{\alpha-\beta} \psi \right\|_{L^2}$$

$$\leq C \sum_{\beta \leq \alpha} \left( \int_{\mathbb{R}^n} \left| (\partial_\xi^\beta m(2^j \cdot)) (\xi) \partial^{\alpha-\beta} \psi(\xi) \right|^2 d\xi \right)^{1/2}$$

$$\leq C \sum_{|\beta| \leq k} \left( \int_{1/2 < |\xi| < 2} \left| (\partial^\beta m(\cdot)) (\xi) \right|^2 d\xi \right)^{1/2}.$$

Therefore, $m(2^j \cdot) \psi \in H^k$, that is, $m \in MH^k$, with $k > |n/2| + 1$. Applying Thm. 4.15 we obtain that $m \in \mathcal{M}_p$, for $1 < p < \infty$.

Finally, if $m$ satisfies (4.5), then

$$r^{|\alpha|} \left( r^{-n} \int_{\mathbb{R}^n} \left| \partial^\alpha m(\xi) \right|^2 d\xi \right)^{1/2} \leq C r^{|\alpha|} \left( r^{-n} \int_{\mathbb{R}^n} \left| \xi |^{-2|\alpha|} d\xi \right)^{1/2} \leq C,$$

that is, (4.4) is also satisfied, and the corollary is proven. \(\square\)

**Proof of Thm. 4.15.** We are going to use the Littlewood–Paley Thm. and the theory of vector-valued singular integrals.

Let $m$ be as in Def. 4.12. Then,

$$\|T_m f\|_{L^p} \leq \left\| \left( \sum_j |\Delta_j T_m f|^2 \right)^{1/2} \right\|_{L^p},$$

(4.6)

where

$$(\Delta_j T_m f)^\ast(\xi) = \psi(2^{-j} \xi)(T_m f)^\ast(\xi) = \psi(2^{-j} \xi)m(\xi) \hat{f}(\xi) = (K_j * f)^\ast(\xi).$$

(4.7)

Here, $\hat{K}_j(\xi) = \psi(2^{-j} \xi)m(\xi)$, and recalling the relations (1.1) and (1.2) we see that

$$\hat{K}_j(\xi) = (m(2^j \cdot))^{\ast -} = \mathcal{F} \mathcal{F}^{-1}(m(2^j \cdot) \psi)\hat{f}(\xi) = \mathcal{F}\left(\mathcal{F}^{-1}(m(2^j \cdot) \psi)\right)_{2^{-j}}(\xi) = \mathcal{F}(k_j)_{2^{-j}}(\xi),$$

that is,

$$K_j = (k_j)_{2^{-j}}, \quad \text{where} \quad k_j = \mathcal{F}^{-1}(\psi m(2^j \cdot)).$$

(4.8)

We now define

$$\tilde{T} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n, \ell^2)$$

by setting $\tilde{T}(f) = \{K_j * f\} \in \ell^2$ and write $\tilde{K} = \{K_j\}$.

If we show that $\tilde{T} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n, \ell^2)$ is bounded, then, from (4.6) and (4.7) it would follow that $T_m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded and weak-type $(1,1)$, as we wish to show.
We wish to show that $\vec{T}$ and $\vec{K}$ satisfy the hypotheses in Thm. 2.16 (with $\mathcal{H}_1 = \mathbb{C}$, $\mathcal{H}_2 = \ell^2$ and $r = 2$). Notice that,

$$\sum_j |\hat{K}_j(\xi)|^2 = \sum_j |\psi(2^{-j}\xi)m(\xi)|^2 = |m(\xi)|^2,$$

so that, by Plancherel theorem and Lemma 4.14,

$$\|\vec{T}f\|_{L^2(\mathbb{R}^n, \ell^2)}^2 = \int_{\mathbb{R}^n} \sum_j |T_j f(x)|^2 \, dx = \sum_j \int_{\mathbb{R}^n} |K_j * f(x)|^2 \, dx
= \sum_j \int_{\mathbb{R}^n} |\hat{K}_j(\xi)\hat{f}(\xi)|^2 \, d\xi
= \int_{\mathbb{R}^n} |m(\xi)\hat{f}(\xi)|^2 \, d\xi
\leq C\|f\|_{L^2}^2.$$

Hence, $\vec{T} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, \ell^2)$ is bounded.

Next we want to prove that $\vec{K} = \{K_j\} \in \mathcal{L}(\mathbb{C}, \ell^2)$ satisfies the hypothesis (2.20), that is,\(^\text{11}\)

$$\int_{|x| > 2|y|} \left( \sum_j |K_j(x-y) - K_j(x)|^2 \right)^{1/2} \, dx \leq C.$$

In order to conclude the proof, it suffice to prove the following claim.

**Claim.** There exists $C > 0$ such that

$$\sum_j \int_{|x| > 2|y|} |K_j(x-y) - K_j(x)| \, dx \leq C. \quad (4.9)$$

Recall that $K_j$ is defined in terms of $m$, and that the dependence is given in (4.8). With $k_j$ also defined in (4.8), observe that, since $m \in \mathcal{MH}$, $m(2^j \cdot \psi)$ are in $H^s$ with norm uniformly bounded in $j$, so that

$$\int_{\mathbb{R}^n} |k_j(\xi)|^2 (1 + |\xi|^2)^s \, d\xi = \int_{\mathbb{R}^n} |\mathcal{F}^{-1}(m(2^j \cdot \psi)(\xi))(1 + |\xi|^2)^s| \, d\xi
= \int_{\mathbb{R}^n} |(m(2^j \cdot \psi)\hat{\cdot}(-\xi))^2 (1 + |\xi|^2)^s| \, d\xi
= \int_{\mathbb{R}^n} |(m(2^j \cdot \psi)(\xi))^2 (1 + |\xi|^2)^s| \, d\xi
\leq C \|m\|_{\mathcal{MH}^s}, \quad (4.10)$$

(again with $C$ independent of $j$).

\(^{11}\)In fact, it is obvious that, given a sequence $\vec{b} = \{b_j\}$, then $\vec{b}$ can be identified with the operator $\vec{b} : \mathbb{C} \to \ell^2$ given by $\vec{b}(a) = \{ab_j\}$ and

$$\|\vec{b}\|_{\mathcal{L}(\mathbb{C}, \ell^2)} = \sup_{a \in \mathbb{C}, \|a\| = 1} \|\{ab_j\}\|_{\ell^2} = \sup_{a \in \mathbb{C}, \|a\| = 1} |a|\|\{b_j\}\|_{\ell^2} = \|\{b_j\}\|_{\ell^2}.$$
Suppose that $2^\nu \leq |y| < \frac{3}{2}2^\nu = 3 \cdot 2^{\nu-1}$ for a given $\nu \in \mathbb{Z}$. It suffices to prove that

$$
\sum_j \int_{|x|>2^{\nu+1}} |K_j(x - y) - K_j(x)| \, dx \leq C, \quad (4.11)
$$

with $C$ independent of $\nu$. We have

$$
\sum_j \int_{|x|>2^{\nu+1}} |K_j(x - y) - K_j(x)| \, dx = \sum_j \int_{|x|>2^{\nu+j}} |(k_j)_{2^{-j}}(x - y) - (k_j)_{2^{-j}}(x)| \, dx
$$

$$
= \sum_j \int_{|x|>2^{\nu+j+1}} |k_j(z - 2^j y) - k_j(z)| \, dz
$$

$$
= \sum_{j \leq j_0} \int_{|x|>2^{\nu+j+1}} |k_j(z - 2^j y) - k_j(z)| \, dz
$$

$$
+ \sum_{j > j_0} \int_{|x|>2^{\nu+j+1}} |k_j(z - 2^j y) - k_j(z)| \, dz
$$

$$
= I + II,
$$

where $j_0$ is an integer to be selected later.

To estimate $II$, we use the fact that $|z - 2^j y| \geq 2^{\nu+j+1} - 3 \cdot 2^{\nu+j-1} = 2^{\nu+j-1}$, the Cauchy–Schwarz inequality, (4.10) (and footnote 5), to obtain

$$
II \leq \sum_{j > j_0} \int_{|x|>2^{\nu+j+1}} |k_j(z - 2^j y)| + |k_j(z)| \, dz
$$

$$
\leq 2 \sum_{j > j_0} \int_{|x|>2^{\nu+j-1}} |k_j(z)| \, dz
$$

$$
\leq 2 \sum_{j > j_0} \left( \int_{|x|>2^{\nu+j-1}} |k_j(z)|^2 \, dz \right)^{1/2} \left( \int_{|x|>2^{\nu+j-1}} |z|^{-2s} \, dz \right)^{1/2}
$$

$$
\leq C \sum_{j > j_0} \left( \int_{|x|>2^{\nu+j-1}} |k_j(z)|^2 (1 + |z|^2)^s \, dz \right)^{1/2} \left( \int_{2^{\nu+j-1}}^{+\infty} r^{-2s+n-1} \, dr \right)^{1/2}
$$

$$
\leq C \|m\|_{\text{MHF}} \sum_{j > j_0} 2^{(\nu+j-1)(n/2-s)}
$$

$$
\leq C \|m\|_{\text{MHF}},
$$

if we select $j_0 = -\nu$.

Finally, we estimate $I$ with $j_0 = -\nu$. Notice that

$$
|k_j(z - 2^j y) - k_j(z)|^2 = \left| \int_0^1 2^j y \cdot \nabla k_j(z - t^2^j y) \, dt \right|^2
$$

$$
\leq |2^j y|^2 \int_0^1 |\nabla k_j(z - t^j y)|^2 \, dt.
$$

(4.12)
Using the mean value theorem, (4.12) and the facts that $|2^j y| \leq 3 \cdot 2^{j+\nu-1} \leq \frac{3}{2}$ when $j \leq -\nu = j_0$, and $|2^j y| \leq C 2^{j+\nu}$, we have

$$I = \sum_{j \leq -\nu} \int_{|z| > 2^{j+\nu+1}} |k_j(z - 2^j y) - k_j(z)| \, dz$$

$$\leq C \sum_{j \leq -\nu} \left( \int_{|z| > 2^{j+\nu+1}} |k_j(z - 2^j y) - k_j(z)|^2 (1 + |z|^2)^{s} \, dz \right)^{1/2}$$

$$\leq C \sum_{j \leq -\nu} 2^{j+\nu} \left( \int_{|z| > 2^{j+\nu+1}} \left| \nabla k_j(z - t 2^j y) \right|^2 dt (1 + |z|^2)^{s} \, dz \right)^{1/2}$$

$$= C \sum_{j \leq -\nu} 2^{j+\nu} \left( \int_{\mathbb{R}^n} \left| \nabla k_j(x) \right|^2 (1 + |x + t 2^j y|^2)^{s} \, dx dt \right)^{1/2}$$

$$\leq C \sum_{j \leq -\nu} 2^{j+\nu} \left( \int_{\mathbb{R}^n} \left| \nabla k_j(x) \right|^2 (1 + |x|^2)^{s} \, dx \right)^{1/2}.$$

We will be done if we show that

$$\left( \int_{\mathbb{R}^n} |\nabla k_j(x)|^2 dt (1 + |x|^2)^{s} \, dx \right)^{1/2} \leq C \|m\|_{MH^s},$$

because $\sum_{j \leq -\nu} 2^{j+\nu}$ is finite. Clearly it suffices to show that

$$\left( \int_{\mathbb{R}^n} |\partial_{x_k} k_j(x)|^2 dt (1 + |x|^2)^{s} \, dx \right)^{1/2} \leq C \|m\|_{MH^s},$$

for $k = 1, \ldots, n$. But,

$$\partial_{x_k} k_j(x) = \partial_{x_k} F^{-1}(m(2^j \cdot)\psi)(x) = F^{-1}(2\pi \xi_k m(2^j \cdot)\psi)(x),$$

so that, if $\psi_1 \in C_0^\infty$ and is identically 1 on the support of $\psi$, by Lemma 4.11,

$$\left( \int_{\mathbb{R}^n} |\partial_{x_k} k_j(x)|^2 dt (1 + |x|^2)^{s} \, dx \right)^{1/2} = 2\pi \|\xi_k m(2^j \cdot)\psi\|_{H^s} = 2\pi \|\xi_k \psi_1 m(2^j \cdot)\psi\|_{H^s} \leq C \|m(2^j \cdot)\psi\|_{H^s} \leq C \|m\|_{MH^s}.$$

At last, the Mihlin-Hörmander theorem is proven. \qed

4.4. **The Marcinkiewicz multiplier theorem.** We finish with a brief discussion, without proofs, of another multiplier theorem, the *Marcinkiewicz multiplier theorem.*

The simplest form of the Mihlin-Hörmander condition is given by the inequality (4.5) for $k \geq \lfloor n/2 \rfloor + 1$. We now consider the condition

$$|\partial^\alpha m(\xi)| \leq C|\xi_1|^{-\alpha_1} \cdots |\xi_n|^{-\alpha_n}. \quad (4.13)$$
Notice that when $n = 1$ it coincides with the Mihlin-Hörmander condition. But, when $n > 1$ the two conditions are different. In the theorem we give a more general condition for the $L^p$-boundedness of the multiplier operator. It is easy to see that the condition is satisfied if (4.13) holds, for $k \geq n$.

**Theorem 4.17. (Marcinkiewicz multiplier theorem)** Let $m \in L^\infty(\mathbb{R}^n)$ be such that there exists a constant $B > 0$ such that for all $0 < k \leq n$

$$\sup_{I_{j_1}, \ldots, I_{j_k}} \int_{I_{j_1} \times \ldots \times I_{j_k}} \left| \partial^k_{\xi_{j_1} - \xi_{j_k}} m(\xi) \right| d\xi_{j_1} \cdots d\xi_{j_k} \leq B,$$

(4.14)

where $I_{j_i}, i = 1, \ldots, k,$ are dyadic intervals in $\mathbb{R}$.

For a discussion and proof of this theorem we refer to [St], [Du], or [Gr].
REFERENCES


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