1. **The Haar System**

Let \( \psi(x) \) be the function 
\[
\psi(x) = \chi_{[0, \frac{1}{2})}(x) - \chi_{[\frac{1}{2}, 1)}(x),
\]
and for all \( k, j \in \mathbb{Z} \) we set 
\[
\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k).
\]

**Theorem 1.1.** The system \( \{\psi_{j,k} : j, k \in \mathbb{Z}\} \) is a complete orthonormal system in \( L^2(\mathbb{R}) \).

**Proof.** We begin with noticing that the function \( \psi_{j,k} \) has support in the interval
\[
I_{j,k} = [k2^{-j}, (k + 1)2^{-j}),
\]
that has length \( 2^{-j} \), that is \( |I_{j,k}| = 2^{-j} \). Moreover,
\[
\int_{\mathbb{R}} \psi_{j,k} \, dx = \int_{I_{j,k}} \psi_{j,k} \, dx = 0.
\]

We make a few remarks and observations about the intervals \( I_{j,k} \).
- The index \( j \) denotes the “generation”, while \( k \) denotes its “position” in the generation.
- If \( k \neq k' \), \( I_{j,k} \cap I_{j,k'} = \emptyset \).
- Given an index \( j \), the intervals \( \{I_{j+1,k}\}_{k \in \mathbb{Z}} \) are obtained by bisecting the intervals \( \{I_{j,k}\}_{k \in \mathbb{Z}} \). In particular, bisecting the interval \( I_{j,k} \) gives rise to the intervals \( I_{j,k+1} \) and \( I_{j,k+1} \) in the subsequent generation.
- As a consequence, when \( j' > j \) and the intervals \( I_{j,k} \) and \( I_{j',k'} \) are not disjoint, then \( I_{j,k} \supseteq I_{j',k'} \) and in fact \( I_{j',k'} \) is contained in either the first or the second half of \( I_{j,k} \).

We now prove the orthonormality of the Haar system \( \{\psi_{j,k}\} \).

Let \( (j, k), (j', k') \) be two pairs of indices.

Suppose first \( j = j' \). If \( k \neq k' \), so that \( I_{j,k} \cap I_{j,k'} = \emptyset \), then
\[
(\psi_{j,k}, \psi_{j',k'}) = 0.
\]

If \( k = k' \), then
\[
(\psi_{j,k}, \psi_{j,k}) = \| \psi_{j,k} \|^2 = 2^j \int_{I_{j,k}} \psi_{j,k} \, dx = 1.
\]

If \( j < j' \), then, by the last observation above,
\[
| (\psi_{j,k}, \psi_{j',k'}) | = 2^{j/2} | \int_{I_{j',k'}} \psi_{j',k'} \, dx | = 0.
\]

Thus, the orthonormality follows.

Next, we wish to show that the system is complete, that is that \( \text{span} \{\psi_{k,j} : j, k \in \mathbb{Z}\} \) is dense in \( L^2(\mathbb{R}) \).

We define
\[
V_j = \text{span} \{\chi_{I_{j,k}} : k \in \mathbb{Z}\}.
\]

It is well known that simple functions are dense in \( L^2 \). Hence, finite linear combinations of characteristic functions of intervals are also dense in \( L^2 \). It is easy to see that for every interval \( I \), the function \( \chi_I \) can be approximated by functions in \( \bigcup_{j \in \mathbb{Z}} V_j \) (show this). Therefore, the closure in \( L^2 \) of \( \text{span} \{V_j : j \in \mathbb{Z}\} \) is all of \( L^2 \).
Furthermore, we observe that:
(1) for every \( j \in \mathbb{Z} \), \( V_{j-1} \subseteq V_j \);
(2) as a consequence, \( \lim_{N \to +\infty} V_N = L^2(\mathbb{R}) \) (notice that this means that given \( f \in L^2 \) and \( \varepsilon > 0 \) there exist \( N \in \mathbb{N} \) and \( g \in V_N \) such that \( \| f - g \|_{L^2} < \varepsilon \));
(3) moreover, \( \cap_{j \in \mathbb{Z}} V_j = \{ 0 \} \), that is, \( \lim_{M \to -\infty} V_M = \{ 0 \} \), since if \( f \in \cap_{j \in \mathbb{Z}} V_j \), then \( f \) is an \( L^2 \) function that is constant on intervals of arbitrarily long length, hence \( f \) must be 0.

We also notice that
- if we set \( \varphi = \chi_{[0,1)} = \chi_{I_0,0} \), and, as in the case of the \( \psi \)'s, \( \varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k) \), then \( \varphi_{j,k} = 2^{j/2}\chi_{I_{j,k}} \);
- as a consequence, \( V_j = \text{span} \{ \varphi_{j,k} : k \in \mathbb{Z} \} \).

Next, for \( j \in \mathbb{Z} \) we define \( W_j = \text{span} \{ \psi_{j,k} : k \in \mathbb{Z} \} \).

Notice that we have shown that \( W_j \perp W_j' \) for \( j \neq j' \) and that our goal is to show that \( \bigoplus_{j \in \mathbb{Z}} W_j = L^2 \).

Claim. For all \( N \in \mathbb{Z} \) we have the orthogonal decomposition
\( V_N = V_{N-1} \oplus W_{N-1} \).

Assume the claim for now and we finish the proof.
From the claim, by induction it follows that
\[
V_N = V_{N-1} \oplus W_{N-1} = \left( V_{N-2} \oplus W_{N-2} \right) \oplus W_{N-1} = V_M \oplus \left( \bigoplus_{j=M}^{N-1} W_j \right)
\]
for any \( M \leq N - 1 \). Hence, by (3) above, letting \( M \to -\infty \),
\[
V_N = \bigoplus_{j=-\infty}^{N-1} W_j.
\]
Using (2), letting \( N \to +\infty \) we obtain
\[
L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j,
\]
and, modulo the claim, we are done.

We prove the claim. Notice that
\[
\varphi_{0,0}(x) = \chi_{[0,1)}(x) = \frac{1}{2} \chi_{[0,2)}(x) + \frac{1}{2} \left( \chi_{[0,1)}(x) - \chi_{[1,2)}(x) \right)
\]
\[
= \frac{1}{2^{1/2}} \varphi_{-1,0}(x) + \frac{1}{2^{1/2}} \psi_{-1,0}(x)
\]
\[
\in V_{-1} + W_{-1}.
\]
By translation we also have
\[
\varphi_{0,k}(x) = \frac{1}{2^{1/2}} \varphi_{-1,k}(x) + \frac{1}{2^{1/2}} \psi_{-1,k}(x) \in V_{-1} + W_{-1}.
\]
Therefore, $V_0 \subseteq V_{-1} + W_{-1}$. Since $W_{-1} \subseteq V_0$ (as it is easy to check directly, or using the relation above and the inclusion $V_{-1} \subseteq V_0$), it follows that

$$V_0 = V_{-1} + W_{-1}.$$ 

The fact that $V_{-1} \perp W_{-1}$ implies that the above is an orthogonal sum.

From the equality

$$V_0 = V_{-1} \oplus W_{-1},$$

it is easy to complete the proof of the claim, either repeating an analogous argument at the level $N$, or using the fact that the dilation

$$f \mapsto 2^{1/2} f(2 \cdot)$$

maps $V_j$ onto $V_{j+1}$ and $W_j$ onto $W_{j+1}$, with inverse

$$f \mapsto 2^{-1/2} f(2^{-1} \cdot).$$

(These maps preserve the orthogonality of the $V$’s and $W$’s.)

This completes the proof. \hfill \square

2. The Hermite functions

Let $D$ be the differential operator defined on $C^1$ functions by

$$Df(x) = \frac{1}{\sqrt{2}} \left( x f(x) + f'(x) \right),$$

and let $D^*$ its (formal) adjoint with respect to the $L^2$-inner product; that is the differential operator defined by the equality

$$\int (Df)g = \int f(D^*g) \quad (2.1)$$

valid for all $C^1_c(\mathbb{R})$ functions.

We now show that

$$D^* f(x) = \frac{1}{\sqrt{2}} \left( x f(x) - f'(x) \right). \quad (2.2)$$

In fact, if $f, g \in C^1_c(\mathbb{R})$ (that is, they are $C^1$-functions with compact support), integrating by parts

$$\int_{\mathbb{R}} D^* f(x) \overline{g(x)} \, dx = \int_{\mathbb{R}} f(x) D \overline{g(x)} \, dx = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} f(x)(xg(x) + g'(x)) \, dx$$

$$= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} x f(x) \overline{g(x)} \, dx + \frac{1}{\sqrt{2}} \int_{\mathbb{R}} f(x) g'(x) \overline{g(x)} \, dx$$

$$= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} x f(x) \overline{g(x)} \, dx + \frac{1}{\sqrt{2}} \left( f(x)g(x) \right)_{-\infty}^{+\infty} - \int_{\mathbb{R}} f'(x) \overline{g(x)} \, dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2}} \left( x f(x) - f'(x) \right) \overline{g(x)} \, dx,$$

using the fact that $f$ and $g$ vanish at $\pm \infty$ (in fact, the vanishing of one of the two functions only would have sufficed).

For simplicity of notation we write

$$T = D^*.$$
The Hermite operator is, by definition,
\[ H = x^2 - \left( \frac{d}{dx} \right)^2. \] (2.3)

It is immediate to see that
\[ H = 2TD + I = 2DT - I. \] (2.4)

Set
\[ h_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2}, \quad h_k(x) = \frac{1}{\sqrt{(k!)}} T^k h_0 \quad \text{for } k = 1, 2, \ldots. \]

We call \( h_k \) the \( k \)-th normalized Hermite function.

**Theorem 2.1.** The functions \( \{h_k\}, k = 0, 1, 2, \ldots, \) form a complete orthonormal system for \( L^2(\mathbb{R}) \).

In the rest of these notes we prove this theorem.

**Claim.** For all \( N, m \) non-negative integer
\[ \sup_{x \in \mathbb{R}} (1 + |x|)^N |h_k^{(m)}(x)| < \infty, \] (2.5)

where \( h_k \) is any normalized Hermite function, and \( h^{(m)} \) denotes the \( m \)-th derivative of the function \( h \).

Disregarding the constants, it suffices to prove the statement for the functions \( e^{-x^2/2} \) and \( T^k e^{-x^2/2} \).

By induction it is easy to see that
\[ \left( \frac{d}{dx} \right)^m e^{-x^2/2} = p_m(x) e^{-x^2/2}, \]
where \( p_m \) is a polynomial of degree \( m \) in \( x \). Since \( (1 + |x|)^N p_m(x) e^{-x^2/2} \) is \( C^\infty \), it is bounded on all compact intervals, and since
\[ \lim_{x \to \pm \infty} (1 + |x|)^N p_m(x) e^{-x^2/2} = 0, \]
the Claim follows for \( h_0 \). For the general case, by induction it easily follows that \( T^k e^{-x^2/2} = q_k(x) e^{-x^2/2} \), where \( q_k \) is a polynomial of degree \( k \), so that
\[ \left( \frac{d}{dx} \right)^m (T^k e^{-x^2/2}) = p_{m+k}(x) e^{-x^2/2}, \]
for some (other) polynomials \( p_{m+k} \) of degree \( m + k \).

The claim now follows easily.\(^1\)

We remark that, as a consequence of the claim, the equality (2.1) is valid for when \( f \) is such that \( f, f' \in L^2 \) and \( g \) is one of the Hermite functions.\(^2\)

**Lemma 2.2.** The Hermite functions \( \{h_k\}, k = 0, 1, 2, \ldots, \) satisfy the following relations:

1. \( Th_k = \sqrt{k+1} h_{k+1} \);
2. Or, more generally, is a Schwartz function.

\(^1\)We remark that \( C^\infty \) functions satisfying the condition (2.5) for all non-negative integers \( N, m \), are called rapidly decreasing functions, or Schwartz functions.

\(^2\)We remark that \( C^\infty \) functions.
(ii) \( Dh_k = \sqrt{k}h_{k-1} \);
(iii) \( T D h_k = k h_k \) and \( H h_k = (2k + 1)h_k \).

In particular, the Hermite functions \( h_k \) are eigenfunctions of the Hermite operator with eigenvalues \( 2k + 1 \).

**Proof.** It is a simple matter to show, by induction, that (i) holds true.

In order to prove (ii), notice that \( Dh_0 = 0 \); hence (ii) is satisfied for \( k = 0 \). Assume by induction that (ii) is satisfied for \( k - 1 \). Observe that from (2.4) it follows that \( DT - TD = I \). Then, using also (i),
\[
D h_k = \frac{1}{\sqrt{k!}} D T^k h_0 = \frac{1}{\sqrt{k}} D T h_{k-1} = \frac{1}{\sqrt{k}} (TD + I) h_{k-1}
\]
\[
= \frac{\sqrt{k-1}}{\sqrt{k}} T h_{k-2} + \frac{1}{\sqrt{k}} h_{k-1}
\]
\[
= \frac{k-1}{\sqrt{k}} h_{k-1} + \frac{1}{\sqrt{k}} h_{k-1} = \sqrt{k} h_{k-1}.
\]

This proves (ii).

Condition (iii) now follows at once. \( \square \)

**Lemma 2.3.** We have
\[
T^k f(x) = (-1)^k 2^{k/2} e^{x^2/2} \left( \frac{d}{dx} \right)^k \left( e^{-x^2/2} f(x) \right).
\]

In particular,
\[
h_k(x) = \frac{(-1)^k}{(\pi^{1/2} k!)^{1/2}} 2^{k/2} e^{x^2/2} \left( \frac{d}{dx} \right)^k e^{-x^2}.
\]

Moreover, if we set
\[
H_k(x) = e^{x^2/2} h_k(x).
\]
then \( H_k \) is a polynomial, \( k = 0, 1, \ldots \), and the linear span of \( \{ H_0, \ldots, H_m \} \) is the linear space of polynomial of degree less or equal to \( m \).

We mention in passing that the polynomials \( H_k \) are called the Hermite polynomials.

**Proof.** This is easy to prove by induction, and we leave it to the reader. \( \square \)

We are now ready to prove Thm. 2.1.

**Proof of Thm. 2.1.** In order to prove that \( \{ h_k \}_{k=0,1,2,\ldots} \) is an orthonormal system for \( L^2 \), we first notice that \( \|h_0\|_2 = 1 \). Next, we show that, for \( k > 0 \), we have that
\[
[D,T^k] = DT^k - T^k D = k T^{k-1}.
\]
\[\text{The operator } [A,B] = AB - BA \text{ is called the commutator of } A \text{ and } B.\]
When $k = 1$, we have already noticed that $DT − TD = I$. In general, for $k ≥ 1$ we have that
\[
[A, B^k] = \sum_{j=0}^{k-1} B^j [A, B] B^{k-1-j}
\]
the equality (2.6) now follows.

Then, let $k, ℓ ≥ 1$. We have that
\[
\int h_k h_\ell = \frac{1}{k} \int T Dh_k h_\ell
\]
\[
= \frac{1}{k} \int D h_k Dh_\ell
\]
\[
= \frac{\sqrt{\ell}}{\sqrt{k}} \int h_{k-1} h_{\ell-1}.
\]
Now we proceed by the induction. If $k = ℓ$ then $\|h_k\| = 1$.

If $k \neq ℓ$, then the equality above (valid for all $k, ℓ$)
\[
\int h_k h_\ell = \frac{1}{k} \int T Dh_k h_\ell
\]
it follows that if $ℓ = 0$ and $k ≥ 1$, then $\int h_k h_0 = 0$. Now we proceed by induction as before.

This proves that the Hermite functions form an orthonormal system in $L^2(\mathbb{R})$.

In order to finish the proof, we need to show that the Hermite functions form a complete system. Let $f \in L^2$ be orthogonal to $h_k$ for all $k = 0, 1, \ldots$. By Lemma 2.3 $f$ is orthogonal to $x^k e^{-x^2/2}$ for $k = 0, 1, \ldots$. If $f$ has compact support and $I$ is a compact interval containing the support of $f$, we have
\[
0 = (f, h_k) = \int_I f x^k e^{-x^2/2},
\]
that is, the function $f e^{-x^2/2}$ is orthogonal to all the polynomials, that are dense in $L^2(I)$. Hence, $f e^{-x^2/2} = 0$, so that $f = 0$.

Since the $L^2$-functions with compact support are dense in $L^2$, we can easily conclude the proof. $\Box$

\[\text{For,}\]
\[
[A, B^k] = AB^k - B^k A = ABB^{k-1} - B^k A
\]
\[
= BAB^{k-1} + [A, B] B^{k-1} - B^k A = BABB^{k-1} + [A, B] B^{k-1} - B^k A
\]
\[
= B^2 AB^{k-2} + B[A, B] B^{k-2} + [A, B] B^{k-1} - B^k A
\]
\[
= \sum_{j=0}^{k-1} B^j [A, B] B^{k-1-j}.
\]