ON THE CONVERGENCE OF AN ALGORITHM CONSTRUCTING THE NORMAL FORM FOR ELLIPTIC LOWER DIMENSIONAL TORI IN PLANETARY SYSTEMS

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Abstract. We give a constructive proof of the existence of elliptic lower dimensional tori in nearly integrable Hamiltonian systems. In particular we adapt the classical Kolmogorov normalization algorithm to the case of planetary systems, for which elliptic tori may be used as replacements of elliptic Keplerian orbits in Lagrange-Laplace theory. With this paper we support with rigorous convergence estimates the semi-analytic work in our previous article [34], where an explicit calculation of an invariant torus for a planar model of the Sun-Jupiter-Saturn-Uranus system has been made. With respect to previous works on the same subject we exploit the characteristic of Lie series giving a precise control of all terms generated by our algorithm. This allows us to slightly relax the non-resonance conditions on the frequencies.

1. Introduction

The theory of secular motions developed by Lagrange and Laplace is based on small deviations of the orbits from circular ones with zero eccentricities and inclinations. In geometric terms, this means that the circular orbits are considered as lying on a n-dimensional torus, with frequencies coinciding with the observed mean-motions of the planets. However, one may remark that the circular orbits are *not* solutions of Newton's equations. In a previous paper [34] we tried to improve the starting point of the Lagrange-Laplace theory by replacing the circular orbit with another one lying on a *true* n-dimensional invariant torus, namely with a solution of Newton's equations. The aim of the present paper is to provide a full mathematical existence proof of the invariant n-dimensional torus that we have constructed there.

1.1 Elliptic lower dimensional tori for a planetary system

The geometric object that we are looking for is a so-called elliptic lower dimensional torus. This is the natural extension in the light of KAM theory of periodic orbits widely investigated by Poincaré (see [20],[29] and [1]). We recall that an elliptic lower dimensional torus is characterized by a number of frequencies less than the number of degrees of freedom, and admits transverse oscillations with different (typically slower) frequencies. In the case of the planetary system, the frequencies on the torus (that we shall denote by ω) are close to the actual mean-motions of the planets, while the transverse oscillations (with frequencies Ω) describe the motions of the eccentricities, inclinations and conjugated angles, namely, the perihelia and the nodes.

As a general fact, the existence of elliptic lower dimensional invariant tori was first stated by Melnikov [27] and, more than 20 years later, proved independently by Eliasson [7] and Kuksin [21]. Their results have also been extended to Hamiltonian PDEs (see [21], [32], [33] and, for more recent results, [2] and references therein).

Concerning the planetary problem, the existence of lower dimensional tori for the three-body system has been proven by Jefferys and Moser [19] and Lieberman [24]. However the configurations considered in those papers are quite far from the ones in the original Lagrange-Laplace theory. In [19] the case of large mutual inclinations is investigated, so that the lower dimensional tori are partially hyperbolic. In [24] the ratio of the semi-major axes of the planets is assumed to be small enough and the perihelia are locked in phase. An application of Pöschel's approach to the Solar System has been produced by Biasco, Chierchia and Valdinoci in two different cases, namely the spatial three-body planetary problem and a planar system with a central star and n planets (see [3] and [4], respectively).

Our remark is that a practical application of the theory to a realistic system needs an explicit constructive algorithm that can be effectively implemented using computer algebra. On the other hand, as suggested by Poincaré, the constructive method should be based on a rigorous mathematical framework. Now, as often happens in the framework of KAM theory, the usual approach is a deep one from a theoretical point of view, but seems to lack the constructive character, even if one is just interested in finding the location of an elliptic invariant torus.

Our point is that a constructive algorithm for elliptic tori may be produced by suitably adapting the classical method of construction of invariant full-dimensional tori proposed by two of the authors in [11], [12] and applied to the planetary problem of three bodies in [25] and [26]. We emphasize that such a construction produces a Hamiltonian which is holomorphic in the neighborhood of the torus, so that the stability of the orbits in Nekhoroshev sense may be investigated too (see [28], [14] and [35]).

As a matter of fact the constructive algorithm for elliptic tori has been produced by the authors in the previous paper [34], already quoted, where the formal procedure is described in detail. Furthermore an explicit calculation for a planar model of the Sun-Jupiter-Saturn-Uranus system has been performed using algebraic manipulation on a computer, and the resulting orbits on an elliptic torus have been found to be in agreement with those obtained by direct numerical integration. The construction of an elliptic torus is performed by giving the Hamiltonian a suitable normal form using an infinite sequence of near the identity canonical transformations defined by Lie series. However, a rigorous proof of convergence of the whole procedure is still lacking. In the present paper we publish such a proof.

1.2 Outline of the algorithm and of the proof

We recall that the Hamiltonian of a planetary system in Poincaré variables reads

$$H = H_0(\Lambda) + \varepsilon H_1(\Lambda, \lambda, \xi, \eta; \varepsilon) \ , \quad H_0 = -\sum_{j=1}^{n} \frac{\mu_j^2 \beta_j^3}{2\Lambda_j^2} \ ,$$

with $\mu_j = \mathcal{G}(m_0 + m_j)$, $\beta_j = m_0 m_j/(m_0 + m_j)$ where $\mathcal{G}$ is the gravitational constant and m_j is the mass of the j-th body. H_0 is the unperturbed Hamiltonian describing the Keplerian motion and εH_1 is the perturbation due to the mutual attraction of the planets (where $\varepsilon = \max\{m_j/m_0\}$ usually plays the role of a small parameter). Here $\Lambda \sim \sqrt{a}$ and λ are the *fast* variables, while ξ and η are Cartesian variables in a neighborhood of the origin of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ describing the *slow* variables, namely, eccentricities, inclinations and conjugated angles.

We also recall that the Lagrange theory may be formulated as follows. Let Λ^* be the initial value, so that we have $\Lambda(0) = \Lambda^*$ for the Keplerian motion. Expand the Hamiltonian in Taylor series around Λ^* and in Fourier series of the angles λ as

$$H = \omega \cdot (\Lambda - \Lambda^*) + h(\Lambda - \Lambda^*) + \varepsilon \sum_{\alpha,\beta,k} c_{\alpha,\beta,k}(\Lambda - \Lambda^*)\xi^\alpha \eta^\beta e^{ik\cdot\lambda} \ . \tag{1}$$

In Lagrange theory the semi-major axes are kept fixed, thus setting $\Lambda = \Lambda^*$, and the Hamiltonian is replaced with its average over the fast angles λ. The averaged Hamiltonian describes a system around an elliptic equilibrium (this is due to the known d'Alembert properties). Finally, the Hamiltonian is truncated to its quadratic part, and the corresponding frequencies for the secular variables are calculated. By the way, let us recall that the Lagrange-Laplace secular theory has also been extended to the case of extrasolar planetary systems in [23], by using a first-order approximation of an elliptic lower dimensional torus in place of the usual circular approximation.

The search for an elliptic torus proceeds by keeping the full Hamiltonian (1) and applying to it a procedure of Kolmogorov type. That is: we replace the usual average over the fast angles λ with a sequence of canonical transformation that give the Hamiltonian a suitable normal form. The corresponding orbits with zero values of the slow variables *are* solutions of Newton's equations. Let us sketch informally the procedure.

With a change of the symbols for the canonical variables we split the Hamiltonian as

$$\begin{aligned} \mathcal{H}^{(0)}(p,q,x,y;\omega^{(0)}) &= \omega^{(0)} \cdot p + \varepsilon \sum_{j=1}^{n_2} \left[\frac{\Omega_j^{(0)}(\omega^{(0)})}{2} \left(x_j^2 + y_j^2\right) \right] + \varepsilon \mathcal{F}_0(q;\omega^{(0)}) \\ &\quad + \varepsilon \mathcal{F}_1(q,x,y;\omega^{(0)}) + \varepsilon \mathcal{F}_2(p,q,x,y;\omega^{(0)}) + \mathcal{F}_{\text{h.o.t.}}(p,q,x,y;\omega^{(0)}) \ , \end{aligned} \tag{2}$$

where $(p,\, q)$ are action-angle variables (actually corresponding to $\Lambda,\, \lambda$) and $(x,\, y)$ belong to a neighborhood of the origin (corresponding to $\xi,\, \eta$). Here $\omega^{(0)}$ is considered to be a parameter with values in some open domain that will be defined later in the text. As discussed above, fixing the initial frequency vector $\omega^{(0)}$ is equivalent to choosing the semi-major axis (or the value of Λ) around which we perform our expansions, because of the third Kepler law. We collect in $\varepsilon\mathcal{F}_0(q;\omega^{(0)})$ all terms depending just on the angles q, in $\varepsilon\mathcal{F}_1(q,x,y;\omega^{(0)})$ all terms that are linear in (x,y) and independent of p, in $\varepsilon\mathcal{F}_2(p,q,x,y;\omega^{(0)})$ the sum of all (x,y)-independent perturbing terms that are linear in p plus all p-independent perturbing terms quadratic in (x,y) and, finally, in $\mathcal{F}_{\rm h.o.t.}(p,q,x,y;\omega^{(0)})$ all terms of higher order. In the spirit of Kolmogorov theory we look for a canonical transformation that gives the Hamiltonian above the normal form

$$\mathcal{H}^{(\infty)}(P,Q,X,Y;\omega^{(0)}) = \omega^{(\infty)}\cdot P + \varepsilon\sum_{j=1}^{n_2}\frac{\Omega_j^{(\infty)}\left(X_j^2+Y_j^2\right)}{2} + o\big(\|P\|+\|(X,Y)\|^2\big)\ ,$$

where $\omega^{(\infty)} = \omega^{(\infty)}(\omega^{(0)})$ and $\Omega^{(\infty)} = \Omega^{(\infty)}(\omega^{(0)})$. To this end the transformation should kill all terms $\mathcal{F}_0(q;\omega^{(0)})$, $\mathcal{F}_1(q,x,y;\omega^{(0)})$ and $\mathcal{F}_2(p,q,x,y;\omega^{(0)})$. The existence of an elliptic invariant torus is a straightforward consequence of the normal form above. Indeed the torus $P=0$, $X=Y=0$ is clearly invariant and elliptic, and carries a quasi-periodic motion with frequencies $\omega^{(\infty)}$. This is the natural adaptation of the original scheme of Kolmogorov.

The proof is divided in two parts: analytic and geometric. The analytic part requires an infinite sequence of steps that we briefly describe and includes a control of the accumulation of the small divisors.

Starting with $\mathcal{H}^{(0)}$ as in (4) we construct an infinite sequence of Hamiltonians

$$\begin{aligned}\mathcal{H}^{(r)}(p,q,x,y;\omega^{(0)}) &= \omega^{(r)}\cdot p + \varepsilon\sum_{j=1}^{n_2}\left[\frac{\Omega_j^{(r)}(\omega^{(0)})}{2}\left(x_j^2+y_j^2\right)\right] + \varepsilon^{r+1}\mathcal{F}_0^{(r)}(q;\omega^{(0)})\\ &+\varepsilon^{r+1}\mathcal{F}_1^{(r)}(q,x,y;\omega^{(0)}) + \varepsilon^{r+1}\mathcal{F}_2^{(r)}(p,q,x,y;\omega^{(0)}) + \mathcal{F}_{\rm h.o.t.}^{(r)}(p,q,x,y;\omega^{(0)})\ .\end{aligned}\tag{3}$$

At every step the size of $\mathcal{F}_0^{(r)}$, $\mathcal{F}_1^{(r)}$ and $\mathcal{F}_2^{(r)}$ is reduced, as indicated by the factor ε^{r+1}. The crucial point is that the frequencies $\omega^{(r)}$ and $\Omega^{(r)}$ change at every step, this represents a major issue. Moreover, small divisors (i.e., integer combinations of the frequencies) appear at every step, so that a strict control of their contribution is necessary. The accumulation is usually controlled by implementing a perturbation method that exhibits a fast convergence (quadratic method), as in the original paper of Kolmogorov. Here, with the aim of producing an effective constructive algorithm, we follow the classical procedure essentially working step-by-step in powers of the parameter ε. The accumulation of small divisors follows some strict "selection rules" that allow us to control it geometrically. The basis of our method is heuristically described in [13] and has been used for the proof of Kolmogorov theorem in [11] and [12]. The detailed exposition is given here in section 4.1 where we transport in the KAM framework a non-resonance condition that has been introduced in [16] for the Poincaré-Siegel problem and in [15] for the Schröder-Siegel one. Actually that condition turns out to be equivalent to the Bruno one, but it produces better analytic estimates. It is worth mentioning that our method presents some relation with the Lindstedt method that has been used in

KAM theory in the last decades (see, e.g., [6], [8], [5] and [9]). The difference is that with Lindstedt method one constructs just an orbit, namely a particular solution of the Hamiltonian equations. As a consequence, having information on the dynamics in a neighborhood of the elliptic torus requires further work. In contrast, our method of constructing a normal form produces a holomorphic Hamiltonian in the neighborhood of the torus. A further crucial point is that our method of control of the accumulation of small divisors is very direct, and avoids the mechanism of *cancellations* required by the Lindstedt method.

The analytic perturbation procedure is followed by a geometric argument concerning the estimate of the measure of a suitable set of non-resonant frequencies, which is basically an adaptation of the approach described in [32]. The scheme is actually a revisit of the geometric construction of non-resonant domains that can be found, e.g., in [1], [30], [31], [7] and [17].

From a practical view point, the small changes of the frequencies is a weakness of our method, because in an explicit construction one is forced to check at every step that the wanted non-resonance conditions are satisfied. This cannot be avoided, because the change of the frequencies is an effect of the perturbation that is not known at the beginning of the procedure. In the case of a full dimensional torus the frequencies are kept fixed by introducing a small translation of the torus, as suggested by Kolmogorov. In our case however the translation entails a small change of the transverse frequencies Ω, so that the latter cannot be fixed in advance. The geometric construction above shows that with high probability the method will work fine at every order, at least up to the reachable order in an explicit calculation.

1.3 Statement of the result

We come now to a formal statement of our main result. Let us consider a $2(n_1+n_2)$-dimensional phase space endowed with n_1 pairs of action-angle coordinates $(p,q)\in\mathcal{O}_1\times\mathbb{T}^{n_1}$ and n_2 pairwise conjugated canonical variables $(x,y)\in\mathcal{O}_2\subseteq\mathbb{R}^{2n_2}$, where both $\mathcal{O}_1\subseteq\mathbb{R}^{n_1}$ and $\mathcal{O}_2$ are open sets containing the origin. We also introduce an open set $\mathcal{U}\subset\mathbb{R}^{n_1}$ and the frequency vector $\omega^{(0)}\in\mathcal{U}$ which plays the role of a parameter.

Theorem 1: *Consider the following family of real Hamiltonians, parameterized by the n_1-dimensional frequency vector $\omega^{(0)}$,*

$$\begin{aligned}\mathcal{H}^{(0)}(p,q,x,y;\omega^{(0)}) &= \omega^{(0)}\cdot p+\varepsilon\sum_{j=1}^{n_2}\left[\frac{\Omega_j^{(0)}(\omega^{(0)})}{2}\left(x_j^2+y_j^2\right)\right]+\varepsilon\mathcal{F}_0(q;\omega^{(0)})\\ &\quad+\varepsilon\mathcal{F}_1(q,x,y;\omega^{(0)})+\varepsilon\mathcal{F}_2(p,q,x,y;\omega^{(0)})+\mathcal{F}_{\mathrm{h.o.t.}}(p,q,x,y;\omega^{(0)})\ ,\end{aligned}\tag{4}$$

with ε playing the usual role of small parameter. Let us assume that

(a) the frequencies $\Omega_j^{(0)}:\mathcal{U}\to\mathbb{R}$ are analytic functions of $\omega^{(0)}\in\mathcal{U}$; similarly $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_{\mathrm{h.o.t.}}$ are analytic functions of $(p,q,x,y;\omega^{(0)})\in\mathcal{O}_1\times\mathbb{T}^{n_1}\times\mathcal{O}_2\times\mathcal{U}$;

(b) one has $\Omega_i^{(0)}(\omega^{(0)})\neq\Omega_j^{(0)}(\omega^{(0)})$ for $\omega^{(0)}\in\mathcal{U}$ and $1\le i<j\le n_2$;

(c) the function $\mathcal{F}_0$ is independent of p and (x,y) ; $\mathcal{F}_1$ is independent of p and linear in (x,y) ; $\mathcal{F}_2$ is the sum of an (x,y)-independent term linear in p and a p-independent

term quadratic in (x,y); $\mathcal{F}_{\mathrm{h.o.t.}}$ *is of higher order in* p *and* (x,y), *i.e.,* $\mathcal{F}_{\mathrm{h.o.t.}} = o\big(\|p\| + \|(x,y)\|^2\big)$;

(d) $\mathcal{F}_{\mathrm{h.o.t.}}$ *splits into an integrable and non-integrable part as* $\mathcal{F}_{\mathrm{h.o.t.}}(p,q,x,y;\omega^{(0)}) = \mathcal{F}_{\mathrm{int}}(p;\omega^{(0)}) + \varepsilon\mathcal{F}_{\mathrm{n.i.}}(p,q,x,y;\omega^{(0)})$; *moreover, the average of* $\mathcal{F}_2$ *over the angles* q *is equal to zero;*

(e) $\mathcal{H}^{(0)}$ *is invariant with respect to the* ϑ*-family of canonical diffeomorphisms*

$$\begin{aligned}&\big(p_1,\ldots,p_{n_1},q_1,\ldots,q_{n_1},x_1,\ldots,x_{n_2},y_1,\ldots,y_{n_2}\big) \mapsto\\ &\quad\big(p_1,\ldots,p_{n_1},q_1+\vartheta,\ldots,q_{n_1}+\vartheta,x_1\cos\vartheta+y_1\sin\vartheta,\ldots,x_{n_2}\cos\vartheta+y_{n_2}\sin\vartheta,\\ &\qquad y_1\cos\vartheta-x_1\sin\vartheta,\ldots,y_{n_2}\cos\vartheta-x_{n_2}\sin\vartheta\big)\end{aligned}$$

where $\vartheta \in \mathbb{T}$;

(f) *for some* $E>0$ *one has*

$$\begin{aligned}\sup_{(p,q,x,y;\omega^{(0)})\in\mathcal{O}_1\times\mathbb{T}^{n_1}\times\mathcal{O}_2\times\mathcal{U}} \big|\mathcal{F}_j(p,q,x,y;\omega^{(0)})\big| &\le E \qquad \textit{for } j=0,\,1,\,2\ ,\\ \sup_{(p;\omega^{(0)})\in\mathcal{O}_1\times\mathcal{U}} \big|\mathcal{F}_{\mathrm{int}}(p;\omega^{(0)})\big| &\le E\ ,\\ \sup_{(p,q,x,y;\omega^{(0)})\in\mathcal{O}_1\times\mathbb{T}^{n_1}\times\mathcal{O}_2\times\mathcal{U}} \big|\mathcal{F}_{\mathrm{n.i.}}(p,q,x,y;\omega^{(0)})\big| &\le E\ .\end{aligned}$$

Then, there is a positive $\varepsilon^\star$ *such that for* $0 \le \varepsilon < \varepsilon^\star$ *the following statement holds true: there exists a non-resonant set* $\mathcal{U}^{(\infty)} \subset \mathcal{U}$ *of positive Lebesgue measure and with the measure of* $\mathcal{U}\setminus\mathcal{U}^{(\infty)}$ *tending to zero for* $\varepsilon\to 0$ *for bounded* $\mathcal{U}$, *such that for each* $\omega^{(0)}\in\mathcal{U}^{(\infty)}$ *there is an analytic canonical transformation* $(p,q,x,y) = \psi^{(\infty)}_{\omega^{(0)}}(P,Q,X,Y)$ *leading the Hamiltonian in the normal form*

$$\mathcal{H}^{(\infty)}(P,Q,X,Y;\omega^{(0)}) = \omega^{(\infty)}\cdot P + \varepsilon\sum_{j=1}^{n_2}\frac{\Omega_j^{(\infty)}\,\big(X_j^2+Y_j^2\big)}{2} + o\big(\|P\| + \|(X,Y)\|^2\big)\ ,$$

where $\omega^{(\infty)} = \omega^{(\infty)}(\omega^{(0)})$ *and* $\Omega^{(\infty)} = \Omega^{(\infty)}(\omega^{(0)})$.

Let us add some more considerations.

A relevant characteristic of the Hamiltonians (4) is that setting $\varepsilon=0$ one is left with the so-called "Keplerian approximation", depending on the actions p only. Furthermore, the oscillations transverse to an elliptic torus have frequencies $\varepsilon\Omega^{(0)}$, namely of order $\mathcal{O}(\varepsilon)$ with respect to the fast frequencies $\omega^{(0)}$. The distinction between the fast variables (p,q) and the slow *secular variables* is a common one in Celestial Mechanics since the Lagrange time. By the way, the functions $\mathcal{F}_{\ldots}$ can be allowed to depend analytically on ε.

The hypothesis in (d) that $\mathcal{F}_2$ has zero average on the angles q is not restrictive, because the average is already included in the first two terms of $\mathcal{H}_0$. Hypothesis (e) may appear quite strange, but it is fully natural for the planetary system since it leads to the so-called "d'Alembert rules". We emphasize that our hypothesis allows us to deal with both the *spatial* planetary problem, where $\Omega^{(0)}_{n_2}(\omega^{(0)}) = 0$, and the *planar* one. The previous mentioned papers by Biasco, Chierchia and Valdinoci are restricted either to the spatial three-body problem (after the reduction of the angular momentum) or to the planar case, because they also require that $\Omega_j^{(0)}(\omega^{(0)}) \neq 0$ for $j = 1, \ldots, n_2$.

Finally, we stress that we do not need the usual non-degeneracy hypothesis on the p-dependence of the unperturbed Hamiltonian. Actually, in planetary systems we just use the non-degeneracy property of the Keplerian approximation in order to give the Hamiltonian the initial form (4). We should also mention that the non-degeneracy hypothesis was assumed in the statement of Theorem 3.1 in our previous paper [34]. That statement of the Theorem should be replaced with the present one.

The paper is organized as follows. In section 2 we briefly recall the formal algorithm, omitting many details that can be found in the full description in [34]. Section 3 is devoted to the analytic setting of the proof, and section 4 contains the quantitative estimates that lead to the proof of convergence. Most of these estimates are now standard matter, so we skip some calculations that may be easily reconstructed by the reader. Instead, a special emphasis is given to the control of small divisors, since this is new in KAM theory (see subsection 4.1). Section 5 contains the geometric part of the proof where we show that our procedure applies to a set of initial frequencies of large relative measure. Finally, in section 6, we conclude the proof of theorem 1. The technical calculations are deferred to an appendix.

2. Formal algorithm

This section is devoted to the formal algorithm that takes a Hamiltonian (4) of the family $\mathcal{H}^{(0)}$, parameterized by the frequency vector $\omega^{(0)}$, and brings it into a normal form. We include all the (often tedious) formulæ that will be used in order to establish the convergence of the normalization process. We use the formalism of Lie series and Lie transforms (see, e.g., [18] and [10] for a self-consistent introduction). The definition of the Lie series is well-known. We recall that the Lie transform of a generic function g is defined as

$$(5) \qquad \mathcal{T}_X g = \sum_{j=0}^{\infty} \mathcal{E}_j\, g \qquad \text{with} \qquad \mathcal{E}_0\, g = g\ , \qquad \mathcal{E}_j\, g = \sum_{i=1}^{j} \frac{i}{j} \mathcal{L}_{X_i} \mathcal{E}_{j-i}\, g\ ,$$

where $X = \{X_1, X_2, \ldots\}$ is a sequence of generating functions. Let us stress the use of Lie series produces an algorithm that can be effectively implemented with the aid of computer algebra (see [34]). As a very minor difference with respect to [34] we found convenient to use the complex variables $z = (x + \mathbf{i}y)/\sqrt{2}$ in order to deal with the transverse oscillations. The transformation $(p, q, z, \mathbf{i}\bar{z}) \mapsto (p, q, x, y)$ is canonical.

2.1 *Initial settings and strategy of the formal algorithm*

For some fixed positive integer K we introduce the distinguished classes of functions

$\widehat{\mathcal{P}}_{\hat{m},\hat{l},sK}$, with integers $\hat{m}$, $\hat{l}$, $s \ge 0$, which can be written as

$$g(p,q,z,\mathbf{i}\bar{z}) = \sum_{\substack{m\in\mathbb{N}^{n_1}\\ |m|=\hat{m}}} \sum_{\substack{(l,\bar{l})\in\mathbb{N}^{2n_2}\\ |l|+|\bar{l}|=\hat{l}}} \sum_{\substack{k\in\mathbb{Z}^{n_1}\\ |k|\le sK}} c_{m,l,\bar{l},k}\, p^m z^l (\mathbf{i}\bar{z})^{\bar{l}} \exp(\mathbf{i}k\cdot q) \ , \tag{6}$$

with coefficients $c_{m,l,\bar{l},k} \in \mathbb{C}$. Here we denote by $|\cdot|$ the l_1-norm and we adopt the multi-index notation, i.e., $p^m = \prod_{j=1}^{n_1} p_j^{m_j}$. We say that $g \in \mathcal{P}_{\ell,sK}$ in case

$$g \in \bigcup_{\substack{\hat{m}\ge 0\,,\,\hat{l}\ge 0\\ 2\hat{m}+\hat{l}=\ell}} \widehat{\mathcal{P}}_{\hat{m},\hat{l},sK}$$

and the Taylor-Fourier expansion of g satisfies the following property: setting

$$\mathcal{C}_{\mathcal{M}}(l,\bar{l}) = \sum_{j=1}^{n_2} (l_j - \bar{l}_j) \ , \qquad \mathcal{C}_{\mathcal{I}}(k) = \sum_{j=1}^{n_1} k_j \ , \tag{7}$$

one has $c_{m,l,\bar{l},k} = 0$ for $\mathcal{C}_{\mathcal{M}}(l,\bar{l}) \ne \mathcal{C}_{\mathcal{I}}(k)$. We also set $\mathcal{P}_{-2,sK} = \mathcal{P}_{-1,sK} = \{0\}$ for $s \ge 0$, $K > 0$.

The latter definition is equivalent to hypothesis (e) of theorem 1 and includes also the d'Alembert rules, mentioned in the introduction. In Celestial Mechanics these rules are usually stated by saying that all terms appearing in the expansions have the "monomial characteristic" $\mathcal{C}_{\mathcal{M}}(l,\bar{l})$ equal to the "characteristic of the inequality" $\mathcal{C}_{\mathcal{I}}(k)$ (see, e.g., [22]).

Finally we shall denote by $\langle g\rangle_\vartheta = \int_{\mathbb{T}^n} \mathrm{d}\vartheta_1 \ldots \mathrm{d}\vartheta_n\, g/(2\pi)^n$ the average of a function g with respect to the angles $\vartheta \in \mathbb{T}^n$. We shall also omit the dependence of the function from the variables, unless it has some special meaning.

The relevant algebraic property is stated by the following

Lemma 1: *Let $g \in \mathcal{P}_{\ell,sK}$ and $g' \in \mathcal{P}_{\ell',s'K}$ for some ℓ, s, ℓ', $s' \ge 0$ and $K > 0$. Then $\{g,g'\} \in \mathcal{P}_{\ell+\ell'-2,(s+s')K}$.*

The straightforward proof is left to the reader.

We start with the Hamiltonian in the form

$$\begin{aligned} H^{(0)} &= \omega^{(0)}\cdot p + \varepsilon \sum_{j=1}^{n_2} \Omega_j^{(0)} z_j \bar{z}_j + \sum_{\ell>2}\sum_{s\ge 0} \varepsilon^s f_\ell^{(0,s)} \\ &\quad + \sum_{s\ge 1} \varepsilon^s f_0^{(0,s)} + \sum_{s\ge 1} \varepsilon^s f_1^{(0,s)} + \sum_{s\ge 1} \varepsilon^s f_2^{(0,s)} \ , \end{aligned} \tag{8}$$

where $f_\ell^{(0,s)} \in \mathcal{P}_{\ell,sK}$; moreover, as in hypothesis (d) of theorem 1, $f_\ell^{(0,0)} = f_\ell^{(0,0)}(p;\omega^{(0)})$ for $\ell \ge 3$ and $\langle f_2^{(0,1)}\rangle_q = 0$. The Hamiltonian (4) may be written in the form (8) (see section 3).

In the spirit of the original Kolmogorov proof, starting from $H^{(0)}$, we construct an infinite sequence of Hamiltonians $\left\{H^{(r)}\right\}_{r\ge 0}$ with $H^{(r)}$ in normal form up to order r , in a sense to be defined below. We transform $H^{(r-1)}$ into $H^{(r)}$ via a near the identity canonical transformation generated by a composition of four Lie series/transforms of the form

$$\mathcal{T}_{\varepsilon^{r-1}\mathcal{D}_2^{(r)}} \circ \exp\left(\varepsilon^r \mathcal{L}_{\chi_2^{(r)}}\right) \circ \exp\left(\varepsilon^r \mathcal{L}_{\chi_1^{(r)}}\right) \circ \exp\left(\varepsilon^r \mathcal{L}_{\chi_0^{(r)}}\right) \ , \tag{9}$$

where $\mathcal{L}_g \cdot = \{\cdot, g\}$ is the Lie derivative operator and $\chi_0^{(r)}(q) \in \mathcal{P}_{0,rK}$, $\chi_1^{(r)}(q, z, \mathbf{i}\bar{z}) \in \mathcal{P}_{1,rK}$, $\chi_2^{(r)}(p, q, z, \mathbf{i}\bar{z}) \in \mathcal{P}_{2,rK}$. The Lie transform operator $\mathcal{T}_{\varepsilon^{r-1}\mathcal{D}_2^{(r)}}$, with a sequence of functions $\left\{\varepsilon^{j(r-1)}\mathcal{D}_2^{(r;j)}(z, \mathbf{i}\bar{z}) \in \mathcal{P}_{2,0}\right\}_{j\ge1}$ induces a canonical linear change of the coordinates $(z, \mathbf{i}\bar{z})$ (see subsection 2.4.1).

The generating functions $\chi_0^{(r)}$, $\chi_1^{(r)}$, $\chi_2^{(r)}$ and $\mathcal{D}_2^{(r)}$ are determined by homological equations. At every normalization step the frequencies $\omega^{(r)}$ and $\Omega^{(r)}$ may change by a small quantity (see formulæ (38) and (39)).

The small divisors are controlled by introducing two non-resonance conditions up to a finite order rK, namely

$$\min_{\substack{k\in\mathbb{Z}^{n_1},\, 0<|k|\le rK \\ l\in\mathbb{Z}^{n_2},\, 0\le|l|\le 2}} \left|k\cdot\omega^{(r-1)}(\omega^{(0)}) + \varepsilon\, l\cdot\Omega^{(r-1)}(\omega^{(0)})\right| \ge a_r \ , \tag{10}$$

and

$$\min_{1\le i<j\le n_2} \left|\Omega_i^{(r-1)}(\omega^{(0)}) - \Omega_j^{(r-1)}(\omega^{(0)})\right| \ge b_r \ , \tag{11}$$

where $\{a_r\}_{r\ge1}$ and $\{b_r\}_{r\ge1}$ are two monotonically non-increasing sequences with $a_r \to 0$ and $b_r \to b_\infty > 0$ for $r \to +\infty$. For $|l| = 0$ condition (10) is the usual condition of strong non-resonance, while for $|l| = 1, 2$ it is usually referred to as the first and second Melnikov condition, respectively.

We come to the description of the generic r-th normalization step. Let us write the Hamiltonian $H^{(r-1)}$ as

$$\begin{aligned} H^{(r-1)} = \omega^{(r-1)}\cdot p + \varepsilon\sum_{j=1}^{n_2}\Omega_j^{(r-1)} z_j\bar{z}_j + \sum_{\ell>2}\sum_{s\ge0}\varepsilon^s f_\ell^{(r-1,s)} \\ + \sum_{s\ge r}\varepsilon^s f_0^{(r-1,s)} + \sum_{s\ge r}\varepsilon^s f_1^{(r-1,s)} + \sum_{s\ge r}\varepsilon^s f_2^{(r-1,s)} \ , \end{aligned} \tag{12}$$

where $f_\ell^{(r-1,s)} \in \mathcal{P}_{\ell,sK}$; moreover, we have $f_\ell^{(r-1,0)} = f_\ell^{(r-1,0)}(p;\omega^{(0)})$ for $\ell \ge 3$ and, just for $r = 1$, $\langle f_2^{(0,1)}\rangle_q = 0$. In the expansion above, the functions $f_\ell^{(r-1,s)}$ may depend analytically on ε, the relevant information being that they carry a common factor ε^s. Such an expansion is clearly not unique, but this is harmless.

2.2 *First stage of the normalization step*

Our aim is to remove the term $f_0^{(r-1,r)}$. We determine the generating function $\chi_0^{(r)}$ by solving the homological equation

$$\mathcal{L}_{\chi_0^{(r)}}\left(\omega^{(r-1)}\cdot p\right) + f_0^{(r-1,r)} - \langle f_0^{(r-1,r)}\rangle_q = 0 \ . \tag{13}$$

Considering the Taylor-Fourier expansion

$$f_0^{(r-1,r)}(q) = \sum_{\substack{|k|\le rK \\ \mathcal{C}_{\mathcal{I}}(k)=0}} c_{0,0,0,k}^{(r-1)} \exp(\mathbf{i}k\cdot q) \ , \tag{14}$$

we readily get

$$\chi_0^{(r)}(q) = \sum_{\substack{0<|k|\le rK \\ \mathcal{C}_{\mathcal{I}}(k)=0}} \frac{c_{0,0,0,k}^{(r-1)}}{\mathbf{i}k\cdot\omega^{(r-1)}} \exp(\mathbf{i}k\cdot q) \ . \tag{15}$$

The denominators are not zero in view of the non-resonance condition (10) with $|l|=0$. The new Hamiltonian is determined as the Lie series with generating function $\varepsilon^r\chi_0^{(r)}$, namely

$$\begin{aligned} H^{(\mathrm{I};r)} &= \exp\left(\varepsilon^r \mathcal{L}_{\chi_0^{(r)}}\right) H^{(r-1)} \\ &= \omega^{(r-1)}\cdot p + \varepsilon\sum_{j=1}^{n_2}\Omega_j^{(r-1)} z_j\bar{z}_j + \sum_{\ell>2}\sum_{s\ge 0}\varepsilon^s f_\ell^{(\mathrm{I};r,s)} \\ &\quad + \sum_{s\ge r}\varepsilon^s f_0^{(\mathrm{I};r,s)} + \sum_{s\ge r}\varepsilon^s f_1^{(\mathrm{I};r,s)} + \sum_{s\ge r}\varepsilon^s f_2^{(\mathrm{I};r,s)} \ . \end{aligned} \tag{16}$$

The functions $f_l^{(\mathrm{I};r,s)}$ are recursively defined as

$$\begin{aligned} f_0^{(\mathrm{I};r,r)} &= 0 \ , \\ f_0^{(\mathrm{I};r,r+m)} &= f_0^{(r-1,r+m)} && \text{for } 0<m<r \, , \\ f_\ell^{(\mathrm{I};r,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{j!}\mathcal{L}^j_{\chi_0^{(r)}} f_{\ell+2j}^{(r-1,s-jr)} && \text{for } \ell=0\,,\ s\ge 2r \ \text{ or } \ \ell=1,\,2\,,\ s\ge r \\ & && \quad \text{or } \ \ell\ge 3\,,\ s\ge 0 \ , \end{aligned} \tag{17}$$

with $f_\ell^{(\mathrm{I};r,s)} \in \mathcal{P}_{\ell,sK}$. The constant term $c_{0,0,0,0}^{(r-1)} = \langle f_0^{(r-1,r)}\rangle_q$ has been omitted.

2.3 Second stage of the normalization step

We remove $f_1^{(\mathrm{I};r,r)}$ from (16). We determine a second generating function $\chi_1^{(r)}$ by solving the homological equation

$$\mathcal{L}_{\chi_1^{(r)}}\left(\omega^{(r-1)}\cdot p + \varepsilon\sum_{j=1}^{n_2}\Omega_j^{(r-1)} z_j\bar{z}_j\right) + f_1^{(\mathrm{I};r,r)} = 0 \ . \tag{18}$$

Again, we consider the Taylor-Fourier expansion

$$f_1^{(\mathrm{I};r,r)}(q,z,\mathbf{i}\bar{z}) = \sum_{|l|+|\bar{l}|=1} \ \sum_{\substack{0<|k|\le rK \\ \mathcal{C}_{\mathcal{I}}(k)=\mathcal{C}_{\mathcal{M}}(l,\bar{l})}} c_{0,l,\bar{l},k}^{(\mathrm{I};r)} z^l (\mathbf{i}\bar{z})^{\bar{l}} \exp(\mathbf{i}k\cdot q) \ , \tag{19}$$

where the average $\langle f_1^{(\mathrm{I};r,r)}\rangle_q$ is zero in view of the d'Alembert rules. More precisely, since each term in (19) must satisfy $\mathcal{C}_{\mathcal{I}}(k)=\mathcal{C}_{\mathcal{M}}(l,\bar{l})=\pm 1$ (see (7)), then all the coefficients $c_{0,l,\bar{l},k}^{(\mathrm{I};r)}$ with even $|k|$ must be zero. The solution is

$$\chi_1^{(r)}(q,z,\mathbf{i}\bar{z}) = \sum_{|l|+|\bar{l}|=1} \sum_{\substack{0<|k|\le rK\\ \mathcal{C}_{\mathcal{I}}(k)=\mathcal{C}_{\mathcal{M}}(l,\bar{l})}} \frac{c_{0,l,\bar{l},k}^{(\mathrm{I};r)}\, z^l (\mathbf{i}\bar{z})^{\bar{l}} \exp(\mathbf{i}k\cdot q)}{\mathbf{i}\big[k\cdot\omega^{(r-1)} + \varepsilon(l-\bar{l})\cdot\Omega^{(r-1)}\big]}\ , \tag{20}$$

where the divisors cannot vanish in view of condition (10) with $|l|=1$. We emphasize that the condition $\Omega_j^{(0)}\neq 0$ for $j=1,\ldots,n_2$ is not requested here thanks to the d'Alembert rules, at difference with previous theorems.

The new Hamiltonian is calculated as

$$H^{(\mathrm{II};r)} = \exp\Big(\varepsilon^r \mathcal{L}_{\chi_1^{(r)}}\Big) H^{(\mathrm{I};r)}\ , \tag{21}$$

and is given the form (16), replacing the upper index I by II, with

$$\begin{aligned}
f_\ell^{(\mathrm{II};r,r)} &= 0 && \text{for } \ell=0,1\,,\\
f_\ell^{(\mathrm{II};r,r+m)} &= f_\ell^{(\mathrm{I};r,r+m)} && \text{for } \ell=0,1\,,\ 0<m<r\,,\\
f_0^{(\mathrm{II};r,2r)} &= f_0^{(\mathrm{I};r,2r)} + \frac{1}{2}\mathcal{L}_{\chi_1^{(r)}} f_1^{(\mathrm{I};r,r)}\ , &&\\
f_0^{(\mathrm{II};r,2r+m)} &= f_0^{(\mathrm{I};r,2r+m)} + \mathcal{L}_{\chi_1^{(r)}} f_1^{(\mathrm{I};r,r+m)} && \text{for } 0<m<r\,,\\
f_\ell^{(\mathrm{II};r,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{j!}\mathcal{L}_{\chi_1^{(r)}}^j f_{\ell+j}^{(\mathrm{I};r,s-jr)} && \text{for } \ell=0\,,\ s\ge 3r \ \text{ or } \ \ell=1\,,\ s\ge 2r\\
& && \quad\text{or } \ \ell=2\,,\ s\ge r \ \text{ or } \ \ell\ge 3\,,\ s\ge 0\,.
\end{aligned} \tag{22}$$

2.4 *Third stage of the normalization step*

In order to remove $f_2^{(\mathrm{II};r,r)}$ we proceed in two steps. First we remove the q-dependent part and then, in the next subsection, the average one.

We determine the generating function $\chi_2^{(r)}$ by solving the homological equation

$$\mathcal{L}_{\chi_2^{(r)}}\Big(\omega^{(r-1)}\cdot p + \varepsilon\sum_{j=1}^{n_2}\Omega_j^{(r-1)} z_j\bar{z}_j\Big) + f_2^{(\mathrm{II};r,r)} - \langle f_2^{(\mathrm{II};r,r)}\rangle_q = 0\ . \tag{23}$$

Considering again the Taylor-Fourier expansion

$$\begin{aligned}
f_2^{(\mathrm{II};r,r)}(p,q,z,\mathbf{i}\bar{z}) = &\sum_{|m|=1}\sum_{\substack{|k|\le rK\\ \mathcal{C}_{\mathcal{I}}(k)=0}} c_{m,0,0,k}^{(\mathrm{II};r)} p^m \exp(\mathbf{i}k\cdot q)\\
&+\sum_{|l|+|\bar{l}|=2}\sum_{\substack{|k|\le rK\\ \mathcal{C}_{\mathcal{I}}(k)=\mathcal{C}_{\mathcal{M}}(l,\bar{l})}} c_{0,l,\bar{l},k}^{(\mathrm{II};r)} z^l(\mathbf{i}\bar{z})^{\bar{l}}\exp(\mathbf{i}k\cdot q)\ ,
\end{aligned} \tag{24}$$

we get

$$
\begin{aligned}
\chi_2^{(r)}(p,q,z,\mathbf{i}\bar{z}) = & \sum_{|m|=1} \sum_{\substack{0<|k|\le rK\\ \mathcal{C}_{\mathcal{I}}(k)=0}} \frac{c_{m,0,0,k}^{(\mathrm{II};r)} p^m \exp(\mathbf{i}k\cdot q)}{\mathbf{i}k\cdot\omega^{(r-1)}} \\
& + \sum_{|l|+|\bar{l}|=2} \sum_{\substack{0<|k|\le rK\\ \mathcal{C}_{\mathcal{I}}(k)=\mathcal{C}_{\mathcal{M}}(l,\bar{l})}} \frac{c_{0,l,\bar{l},k}^{(\mathrm{II};r)} z^l (\mathbf{i}\bar{z})^{\bar{l}} \exp(\mathbf{i}k\cdot q)}{\mathbf{i}\big[k\cdot\omega^{(r-1)}+\varepsilon(l-\bar{l})\cdot\Omega^{(r-1)}\big]}\ ,
\end{aligned}
\tag{25}
$$

where the divisors cannot vanish in view of condition (10) with $|l|=0,\,2$.

The transformed Hamiltonian is calculated as

$$
H^{(\mathrm{III};r)} = \exp\left(\varepsilon^r \mathcal{L}_{\chi_2^{(r)}}\right) H^{(\mathrm{II};r)}
\tag{26}
$$

and is given the form (16), replacing the upper index I by III, with

(27)

$$
\begin{aligned}
f_\ell^{(\mathrm{III};r,r)} &= 0 && \text{for } \ell=0,1\,,\\
f_\ell^{(\mathrm{III};r,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor-1} \frac{1}{j!}\mathcal{L}^j_{\chi_2^{(r)}} f_\ell^{(\mathrm{II};r,s-jr)} && \text{for } \ell=0,1\,,\ s>r\,,\\
f_2^{(\mathrm{III};r,r)} &= \langle f_2^{(\mathrm{II};r,r)}\rangle_q\ ,\\
f_2^{(\mathrm{III};r,jr)} &= \frac{j-1}{j!}\mathcal{L}^{j-1}_{\chi_2^{(r)}} f_2^{(\mathrm{II};r,r)} + \sum_{i=0}^{j-2}\frac{1}{i!}\mathcal{L}^i_{\chi_2^{(r)}} f_2^{(\mathrm{II};r,(j-i)r)} && \text{for } j\ge 2\,,\\
f_2^{(\mathrm{III};r,jr+m)} &= \sum_{i=0}^{j-1}\frac{1}{i!}\mathcal{L}^i_{\chi_2^{(r)}} f_2^{(\mathrm{II};r,(j-i)r+m)} && \text{for } j\ge 1\,,\ 0<m<r\,,\\
f_\ell^{(\mathrm{III};r,s)} &= \sum_{j=0}^{\lfloor s/r\rfloor} \frac{1}{j!}\mathcal{L}^j_{\chi_2^{(r)}} f_\ell^{(\mathrm{II};r,s-jr)} && \text{for } \ell\ge 3\,,\ s\ge 0\,.
\end{aligned}
$$

2.4.1 Diagonalization of the quadratic normal form part in $(z,\mathbf{i}\bar{z})$

The last term to deal with is

$$
\langle f_2^{(\mathrm{II};r,r)}\rangle_q = \sum_{|m|=1} c_{m,0,0,0}^{(\mathrm{II};r)} p^m + \sum_{|l|=|\bar{l}|=1} c_{0,l,\bar{l},0}^{(\mathrm{II};r)} z^l (\mathbf{i}\bar{z})^{\bar{l}}\ .
$$

We should remove the non-diagonal terms in the latter expansion, namely the terms with $l\neq\bar{l}$. This could be done with standard algebraic methods. However, in order to construct a coherent scheme of estimates, we found it convenient to proceed with a Lie transform operator $\mathcal{T}_{\mathcal{D}_2^{(r)}}$, with a sequence of generating functions $\mathcal{D}_2^{(r)} = \big\{\varepsilon^{j(r-1)}\mathcal{D}_2^{(r;j)}\big\}_{j\ge 1}$.

The sequence must satisfy

$$\mathcal{T}_{\mathcal{D}_2^{(r)}}\left(\varepsilon Z_0^{(r)}\right)+\mathcal{T}_{\mathcal{D}_2^{(r)}}\left(\varepsilon^r g_1^{(r)}\right)=\sum_{j=0}^{+\infty}\varepsilon^{j(r-1)+1}Z_j^{(r)}\ ,\tag{28}$$

where

$$Z_0^{(r)}=\sum_{j=1}^{n_2}\Omega_j^{(r-1)}z_j\bar{z}_j\ ,\qquad g_1^{(r)}(z,\mathbf{i}\bar{z})=f_2^{(\mathrm{III};r,r)}(0,z,\mathbf{i}\bar{z})\tag{29}$$

and $Z_j^{(r)}$, for $j\ge 1$, is the polynomial

$$Z_j^{(r)}=\sum_{|l|=1}c_{0,l,l,0}^{(r;j)}z^l(\mathbf{i}\bar{z})^l\ ,\tag{30}$$

with coefficients $c_{0,l,l,0}^{(r;j)}$ to be found. The functions $\mathcal{D}_2^{(r;j)}$ are recursively defined so that

$$\mathcal{E}_j^{(r)}Z_0^{(r)}+\mathcal{E}_{j-1}^{(r)}g_1^{(r)}=Z_j^{(r)}\ .\tag{31}$$

The latter equation is solved by rearranging it as

$$\mathcal{L}_{\mathcal{D}_2^{(r;j)}}Z_0^{(r)}+\Psi_j^{(r)}=Z_j^{(r)}\ ,\tag{32}$$

with

$$\Psi_j^{(r)}=\sum_{i=1}^{j-1}\left[\frac{i}{j}\mathcal{L}_{\mathcal{D}_2^{(r;i)}}\left(Z_{j-i}^{(r)}-\mathcal{E}_{j-i-1}^{(r)}g_1^{(r)}\right)\right]+\mathcal{E}_{j-1}^{(r)}g_1^{(r)}\ .\tag{33}$$

Let us give some more details. Proceeding by induction, assume that $\Psi_j^{(r)}\in\mathcal{P}_{2,0}$ and depends only on $(z,\mathbf{i}\bar{z})$; this is true for $j=1$. Thus we can write

$$\Psi_j^{(r)}=\sum_{|l|=|\bar{l}|=1}c_{0,l,\bar{l},0}^{(r;j)}z^l(\mathbf{i}\bar{z})^{\bar{l}}\tag{34}$$

and the homological equation (32) is solved with

$$\mathcal{D}_2^{(r;j)}=\sum_{\substack{|l|=|\bar{l}|=1\\ l\neq\bar{l}}}\frac{c_{0,l,\bar{l},0}^{(r;j)}}{\mathbf{i}(l-\bar{l})\cdot\Omega^{(r-1)}}z^l(\mathbf{i}\bar{z})^{\bar{l}}\ ,\tag{35}$$

where $Z_j^{(r)}$ has the form (30). The divisors cannot vanish in view of condition (11); the cases $|l|=2$ or $|\bar{l}|=2$ cannot occur in view of d'Alembert rules for terms independent of the angles q (see (7)). Again, let us emphasize that condition $\Omega_i^{(r)}\neq 0$ for $i=1,\ldots,n_2$ is not needed here. By lemma 1 all functions so constructed depend just on $(z,\mathbf{i}\bar{z})$ and belong to $\mathcal{P}_{2,0}$. This ensures the formal consistency of the whole procedure.

Applying the Lie transform operator $\mathcal{T}_{\mathcal{D}_2^{(r)}}$ we finally get the Hamiltonian in normal form up to order r as

$$H^{(r)}=\mathcal{T}_{\varepsilon^{r-1}\mathcal{D}_2^{(r)}}H^{(\mathrm{III};r)}\ .\tag{36}$$

The transformed Hamiltonian is given the form (12), replacing $r-1$ with r, namely

$$
\begin{aligned}
H^{(r)} = \omega^{(r)}\cdot p + \varepsilon\sum_{j=1}^{n_2}\Omega_j^{(r)} z_j\bar z_j + \sum_{\ell>2}\sum_{s\ge 0}\varepsilon^s f_\ell^{(r,s)} \\
+ \sum_{s\ge r+1}\varepsilon^s f_0^{(r,s)} + \sum_{s\ge r+1}\varepsilon^s f_1^{(r,s)} + \sum_{s\ge r+1}\varepsilon^s f_2^{(r,s)}\ ,
\end{aligned}
\tag{37}
$$

possibly with a change of the frequencies $\omega^{(r)}$ and $\Omega^{(r)}$, that we briefly discuss in the next subsection.

2.4.2 *Change of frequencies and transformed Hamiltonian*

The key remark here is that the function $f_2^{(\mathrm{III};r,r)}$ still contains a part that is $\mathcal{O}(\varepsilon^r)$ and belongs to $\mathcal{P}_{2,0}$, i.e.,

$$
\sum_{|m|=1} c_{m,0,0,0}^{(\mathrm{II};r)} p^m + \sum_{i\ge 1}\varepsilon^{i(r-1)}\sum_{|l|=1} c_{0,l,l,0}^{(r;i)} z^l(\mathbf{i}\bar z)^l\ .
$$

This kind of terms cannot be eliminated and must be added to the normal form. Thus the frequencies are corrected as

$$
\omega_j^{(r)} = \omega_j^{(r-1)} + \varepsilon^r\frac{\partial f_2^{(\mathrm{III};r,r)}}{\partial p_j}\qquad \text{for } j=1,\,\ldots,\,n_1
\tag{38}
$$

and

$$
\Omega_j^{(r)} = \Omega_j^{(r-1)} + \sum_{i=1}^{+\infty}\left[\varepsilon^{i(r-1)}\frac{\partial^2 Z_i^{(r)}}{\partial z_j\partial(\mathbf{i}\bar z_j)}\right]\qquad \text{for } j=1,\,\ldots,\,n_2\ .
\tag{39}
$$

Recalling that in view of lemma 1 each class $\mathcal{P}_{\ell,sK}$ is invariant under the action of the operator $\mathcal{E}_j^{(r)} = \sum_{i=1}^{j}\frac{i}{j}\mathcal{L}_{\mathcal{D}_2^{(r;i)}}\mathcal{E}_{j-i}^{(r)}$ with $j\ge 1$, we get the explicit expressions

$$
\begin{aligned}
& f_\ell^{(r,r)} = 0 && \text{for } \ell = 0,1,2\,,\\
& f_\ell^{(r,0)} = f_\ell^{(\mathrm{III};r,0)} && \text{for } \ell\ge 3\,,\\
& f_\ell^{(r,s)} = \sum_{j\ge 0}\varepsilon^{j(r-1)}\mathcal{E}_j^{(r)} f_\ell^{(\mathrm{III};r,s)} && \text{for } 0\le\ell\le 2\,,\ s>r\\
& && \quad\text{or }\ \ell\ge 3\,,\ s\ge 1\ ,
\end{aligned}
\tag{40}
$$

Remark that the second equation means that $f_\ell^{(\mathrm{III};r,0)}$ remains unchanged through the whole normalization step, since one has $f_\ell^{(\mathrm{III};r,0)} = f_\ell^{(\mathrm{II};r,0)} = f_\ell^{(\mathrm{I};r,0)} = f_\ell^{(r-1,0)} = f_\ell^{(r-1,0)}(p;\omega^{(0)})$ for $\ell\ge 3$: recall equation (12) and formulæ (17), (22) and (27).

Let us add a few considerations. The first normalization step leave the frequencies unchanged, namely

$$
\omega^{(1)} = \omega^{(0)}\ ,\qquad \Omega^{(1)} = \Omega^{(0)}\ .
\tag{41}
$$

This plays a main role in the quantitative scheme, because the Lie transform operator $\mathcal{T}_{\mathcal{D}_2^{(1)}}$ turns out to be equal to the identity in view of the (non-restrictive) assumption $\langle f_2^{(0,1)}\rangle_q = 0$. Thus the change of the frequencies is of order ε^2, because the first correction shows up at the second step. Actually we get the chain of equalities

$$
\begin{aligned}
f_2^{(\mathrm{III};1,1)} &= \langle f_2^{(\mathrm{II};1,1)}\rangle_q = \langle f_2^{(\mathrm{I};1,1)}\rangle_q + \langle \mathcal{L}_{\chi_1^{(1)}} f_3^{(\mathrm{I};1,0)}\rangle_q = \langle f_2^{(\mathrm{I};1,1)}\rangle_q = \\
&= \langle f_2^{(0,1)}\rangle_q + \langle \mathcal{L}_{\chi_0^{(1)}} f_4^{(0,0)}\rangle_q = \langle f_2^{(0,1)}\rangle_q = 0\ .
\end{aligned}
\tag{42}
$$

This is in view of the recursive formulæ (17), (22), (27) and taking into account that $\mathcal{L}_{\chi_1^{(1)}} f_3^{(\mathrm{I};1,0)} = \mathcal{L}_{\chi_0^{(1)}} f_4^{(0,0)} = 0$, as both the generating functions $\chi_0^{(1)}$ and $\chi_1^{(1)}$ have zero angular average, while $f_4^{(0,0)} \in \mathcal{P}_{4,0}$ and $f_3^{(\mathrm{I};1,0)} \in \mathcal{P}_{3,0}$ do not depend on the angles.

We also remark that our formulation of the algorithm works for both real and complex Hamiltonians. However, if the expansion (8) contains only real functions, then all terms of type $\omega^{(r)} \cdot p$, $\sum_{j=1}^{n_2} \Omega_j^{(r)} z_j \bar{z}_j$ and $f_\ell^{(r,s)}$ generated by the algorithm are real too, as easily checked.

Finally, the Hamiltonian $H^{(r)}$ in (37) has the same form of $H^{(r-1)}$, so that the induction step can be iterated provided the conditions (10) and (11) hold true with $r+1$ in place of r.

3. Analytic settings

We introduce the complex domains $\mathcal{D}_{\varrho,R,\sigma,h} = \mathcal{G}_\varrho \times \mathbb{T}_\sigma^{n_1} \times \mathcal{B}_R \times \mathcal{W}_h$, where $\mathcal{G}_\varrho \subset \mathbb{C}^{n_1}$ and $\mathcal{B}_R \subset \mathbb{C}^{n_2} \times \mathbb{C}^{n_2}$ are open balls centered at the origin with radii ϱ and R, respectively, $\mathcal{W}$ is a subset of $\mathbb{R}^{n_1}$ while the subscripts σ and h denote the usual complex extensions† of real domains (see [10]).

Let us consider a generic analytic function $g : \mathcal{D}_{\varrho,R,\sigma,h} \to \mathbb{C}$,

$$
g(p,q,z,\mathbf{i}\bar{z};\omega) = \sum_{k\in\mathbb{Z}^{n_1}} g_k(p,z,\mathbf{i}\bar{z};\omega)\exp(\mathbf{i}k\cdot q)\ ,
\tag{43}
$$

where $g_k : \mathcal{G}_\varrho \times \mathcal{B}_R \times \mathcal{W}_h \to \mathbb{C}$. We define the weighted Fourier norm

$$
\|g\|_{\varrho,R,\sigma,h} = \sum_{k\in\mathbb{Z}^{n_1}} \left|g_k\right|_{\varrho,R,h} \exp(|k|\sigma)\ ,
$$

where

$$
|g_k|_{\varrho,R,h} = \sup_{\substack{p\in\mathcal{G}_\varrho \\ (z,\mathbf{i}\bar{z})\in\mathcal{B}_R \\ \omega\in\mathcal{W}_h}} \left|g_k(p,z,\mathbf{i}\bar{z};\omega)\right|\ .
\tag{44}
$$

† Precisely, $\mathcal{G}_\varrho = \{z \in \mathbb{C}^{n_1} : \max_{1\le j\le n_1}|z_j| < \varrho\}$, $\mathbb{T}_\sigma^{n_1} = \{q \in \mathbb{C}^{n_1} : \mathrm{Re}\, q_j \in \mathbb{T}$, $\max_{1\le j\le n_1}|\mathrm{Im}\, q_j| < \sigma\}$, $\mathcal{B}_R = \{z \in \mathbb{C}^{2n_2} : \max_{1\le j\le 2n_2}|z_j| < R\}$ and $\mathcal{W}_h = \{z \in \mathbb{C}^{n_1} : \exists\, \omega \in \mathcal{W},\ \max_{1\le j\le n_1}|z_j - \omega_j| < h\}$.

It is convenient to introduce the Lipschitz constant for the Jacobian of the function $\Omega^{(0)} : \mathcal{W}_{h_0} \to \mathbb{C}^{n_2}$, defined as

$$(45)\quad \left|\frac{\partial\Omega^{(0)}}{\partial\omega^{(0)}}\right|_{\infty;\mathcal{W}_{h_0}} = \sup_{\omega^{(0)}\in\mathcal{W}_{h_0}} \sup_{\substack{\beta\neq 0\\ \beta+\omega^{(0)}\in\mathcal{W}_{h_0}}} \frac{\max_{1\le i\le n_2}\left|\Omega_i^{(0)}(\omega^{(0)}+\beta)-\Omega_i^{(0)}(\omega^{(0)})\right|}{\max_{1\le j\le n_1}|\beta_j|} \ .$$

We notice that the dependence on the parameter ω plays no role in the following, thus we shorten the notation by ignoring it.

Lemma 2: *Let us assume the same hypotheses of theorem 1 over the family of Hamiltonians $\mathcal{H}^{(0)}$. Then, there exist positive parameters ϱ, R, σ, h_0, γ, τ, $\bar{b}$, J_0, $\bar{E}$, a compact set $\mathcal{W}\subset\mathbb{R}^{n_1}$ and a positive integer value K such that the canonical change of coordinates $(p,q,z,\mathbf{i}\bar{z})\mapsto(p,q,x,y)$ transforms $\mathcal{H}^{(0)}$ in the Hamiltonian $H^{(0)}:\mathcal{D}_{\varrho,R,\sigma}\times\mathcal{W}_{h_0}\to\mathbb{C}$ described by the expansion (8), where both $\Omega^{(0)}(\omega^{(0)})$ and all the terms of type $f_\ell^{(0,s)}$ are real analytic functions of $\omega^{(0)}\in\mathcal{W}_{h_0}$. Moreover, the following properties are satisfied:*

(a') the initial set $\mathcal{W}$ of frequencies is non-resonant *up to the finite order $2K$, namely every $\omega^{(0)}\in\mathcal{W}$ satisfies*

$$\min_{\substack{k\in\mathbb{Z}^{n_1},\,0<|k|\le 2K\\ l\in\mathbb{Z}^{n_2},\,0\le|l|\le 2}} \left|k\cdot\omega^{(0)}+\varepsilon\, l\cdot\Omega^{(0)}(\omega^{(0)})\right| > \frac{2\gamma}{K^\tau}$$

and

$$\min_{1\le i<j\le n_2}\left|\Omega_i^{(0)}(\omega^{(0)})-\Omega_j^{(0)}(\omega^{(0)})\right| > 2\bar{b}\ ;$$

(b') the Jacobian of $\Omega^{(0)}(\omega^{(0)})$ is uniformly bounded in the extended domain $\mathcal{W}_{h_0}$, namely $\left|\partial\Omega^{(0)}/\partial\omega^{(0)}\right|_{\infty;\mathcal{W}_{h_0}}\le J_0<\infty$;

(c') $f_\ell^{(0,s)}\in\mathcal{P}_{\ell,sK}$;

(d') $f_\ell^{(0,0)}=f_\ell^{(0,0)}(p;\omega^{(0)})$ for $\ell\ge 3$; moreover, $\langle f_2^{(0,1)}\rangle_q=0$;

(e') the following upper bounds hold true:

$$\left\|f_\ell^{(0,s)}\right\|_{\varrho,R,\sigma}\le\frac{\bar{E}}{2^\ell}\ .$$

We give a sketch of the proof. The Hamiltonians should be split in many parts containing a finite number of Taylor-Fourier terms. For any fixed value of index ℓ, standard arguments on the Fourier decay of the coefficients allow us to determine a suitable value of the parameters K and σ, such that for $s\ge 0$ the norms of the functions $f_\ell^{(0,s)}$ are uniformly bounded (see, e.g., the proof of lemma 5.2 in [10]). Having fixed K and using known Diophantine inequalities we determine $\gamma>0$, $\tau>n_1-1$, $h_0>0$ and a compact set $\mathcal{W}\subset\mathcal{U}$ such that property (a') is satisfied (here $\mathcal{U}$ is the initial set of frequency vectors $\omega^{(0)}$ in the hypotheses of theorem 1). Point (b') is a straightforward consequence of the analyticity of $\Omega_j^{(0)}$ on the domain $\mathcal{U}$. Property (c') follows from hypothesis (e) of theorem 1 as discussed at the beginning of subsection 2.1. Moreover, (d') is an immediate consequence of hypotheses (c)–(d) of theorem 1. Finally, by standard arguments on the Taylor expansions of homogeneous polynomials there are ϱ and R such that the inequality at point (e') of lemma 2 is satisfied. Taking into account that the usual sup-norm is bounded by the weighted Fourier one, for $\varepsilon < 1$ we have

$$\sum_{\ell\,,\,s} \varepsilon^s \| f_\ell^{(0,s)} \|_{\varrho,R,\sigma} \leq 2\bar{E}/(1-\varepsilon)\ .$$

This implies that the Hamiltonian $H^{(0)}$ is *analytic* in $\mathcal{D}_{\varrho,R,\sigma} \times \mathcal{W}_{h_0}$.

We give now some estimates for the Lie series/transforms. We shorten the notation by writing $\|\cdot\|_\alpha$ in place of the norm $\|\cdot\|_{\alpha(\varrho,R,\sigma)}$, where α is a positive number.

Lemma 3: *Let d and d' be real numbers such that $d>0$, $d'\geq 0$ and $d+d'<1$; let $\mathcal{X}$ and g be two analytic functions on $\mathcal{D}_{(1-d')(\varrho,R,\sigma)}$ having finite norms $\|\mathcal{X}\|_{1-d'}$ and $\|g\|_{1-d'}$, respectively. Then, for $j\geq 1$, we have*

$$\frac{1}{j!}\left\|\mathcal{L}_{\mathcal{X}}^j g\right\|_{1-d-d'} \leq \frac{1}{e^2}\left(\frac{2e}{\varrho\sigma}+\frac{e^2}{R^2}\right)^j \frac{1}{d^{2j}} \|\mathcal{X}\|_{1-d'}^j \|g\|_{1-d'}\ . \tag{46}$$

Actually, estimates similar to (46) are contained in some previous papers of the authors. Nevertheless, a little additional work is needed in order to adapt them to the present context.

Lemma 4: *Let d and d' be real numbers such that $d>0$, $d'\geq 0$ and $d+d'<1$; let the functions Z_0 , g and g' satisfy*

(i) $Z_0=\sum_{i=1}^{n_2}\Xi_i z_i\bar{z}_i$ with $\min_{1\leq i<j\leq n_2}\left|\Xi_i-\Xi_j\right|\geq \Xi^*>0$;

(ii) $g'=g'(z,\mathbf{i}\bar{z})$ is such that $g'\in\mathcal{P}_{2,0}$ and it is so small that

$$\varepsilon_{\rm diag}^\star = \left(\frac{2e^2}{d^2}+\frac{2^9}{(1-d')^2}\right)\frac{\|g'\|_{1-d'}}{\Xi^* R^2} \leq \frac{1}{2}\ ;$$

(iii) g is an analytic function on $\mathcal{D}_{(1-d')(\varrho,R,\sigma)}$ with finite norm $\|g\|_{1-d'}$.

Then, there exists a sequence of generating functions $\{\mathcal{X}_j\}_{j\geq 1}$ such that $\mathcal{T}_{\mathcal{X}}\,Z_0+\mathcal{T}_{\mathcal{X}}\,g'=\sum_{j=0}^{+\infty}Z_j$, where $\mathcal{T}_{\mathcal{X}}$ is the Lie transform operator introduced in (5) and the Taylor expansion of the "normal form terms" Z_j is of the same type as that of Z_0 . Moreover, for $j\geq 1$, the following inequalities hold true:

$$\|\mathcal{E}_j\, g\|_{1-d-d'} \leq \left(\varepsilon_{\rm diag}^\star\right)^j \|g\|_{1-d'}\ ,\qquad \|Z_j\|_{1-d'} \leq \left(\varepsilon_{\rm diag}^\star\right)^{j-1} \|g'\|_{1-d'}\ . \tag{47}$$

The proofs of both lemmas above are deferred to appendix A.1.

4. Analytic part of the normalization algorithm

In this section, we translate our formal algorithm into a recursive scheme of estimates on the norms of the functions involved in the normalization process.

4.1 Small divisors and selection rule

It is well known that the accumulation of the small divisors can prevent the convergence of any perturbative proof scheme, that is designed so as to ensure the existence of invariant tori for quasi-integrable systems. In the present subsection, we introduce the tools that will allow us to keep control of the accumulation of the small divisors. Here, we follow rather closely [16]; nevertheless, we think it is convenient to adapt that approach to the present context in a self-contained way, because it is one of the most delicate points of the whole proof. The key of our argument is to focus our attention on the indices corresponding to the small denominators, rather than on their actual values.

Let $I = \{j_1, \ldots, j_s\}$ and $I' = \{j'_1, \ldots, j'_s\}$ be two lists of indices with the same number s of elements. Repeated elements may appear in a list. Let us introduce the following relation of partial ordering: we say that $I \triangleleft I'$ in case there is a permutation of the indices such that the relation $j_m \le j'_m$ holds true for $m = 1, \ldots, s$. If two lists of indices contain a different number of elements, we pad the shorter one with zeros so that they can be compared.

Definition 1: *For all integers $r \ge 0$ and $s > 0$, we introduce the family of lists*

$$\mathcal{J}_{r,s} = \bigl\{ I = \{j_1, \ldots, j_{s-1}\} \ : \ 0 \le j_m \le \min\{r, \lfloor s/2 \rfloor\} \, , \ I \triangleleft I^*_s \bigr\} \ , \tag{48}$$

where

$$I^*_s = \left\{ \left\lfloor \frac{s}{s} \right\rfloor, \left\lfloor \frac{s}{s-1} \right\rfloor, \ldots, \left\lfloor \frac{s}{2} \right\rfloor \right\} \ . \tag{49}$$

In agreement with [16], sometimes we will refer to the condition $I \triangleleft I^*_s$ as the *selection rule* S. We are now ready to state two technical lemmas.

Lemma 5: *For the list of indices $I^*_s = \{j_1, \ldots, j_s\}$ the following statements hold true:*
(i) *the maximal index is $j_{\max} = \left\lfloor \frac{s}{2} \right\rfloor$;*
(ii) *for every $k \in \{1, \ldots, j_{\max}\}$ the index k appears exactly $\left\lfloor \frac{s}{k} \right\rfloor - \left\lfloor \frac{s}{k+1} \right\rfloor$ times;*
(iii) *for $0 < r \le s$ one has*

$$\bigl(\{r\} \cup I^*_r \cup I^*_s\bigr) \triangleleft I^*_{r+s} \ .$$

Lemma 6: *For the lists of indices $\mathcal{J}_{r,s}$ the following statements hold true:*
(i) $\mathcal{J}_{r,s} = \mathcal{J}_{\min\{r, \lfloor s/2 \rfloor\},s}$ *;*
(ii) $\mathcal{J}_{r-1,s} \subseteq \mathcal{J}_{r,s}$ *;*
(iii) *if $I \in \mathcal{J}_{r-1,r}$ and $I' \in \mathcal{J}_{r,s}$, then $\bigl(\{\min\{r,s\}\} \cup I \cup I'\bigr) \in \mathcal{J}_{r,r+s}$.*

The proofs of the two lemmas above are deferred to appendix A.2.

Now, we think it can be useful to describe the mechanism of accumulation of the small divisors in a rather informal way. The elementary remark is that the small divisors are created by the solution of a homological equation and accumulate through Lie derivatives. E.g., suppose we are solving (13) with $r = 1$, then in $\chi_0^{(1)}$ there will be a divisor that we estimate by a_1 according to definition (10). We describe this fact by associating to $\chi_0^{(1)}$ the list $\{1\}$. Similarly, at order r if the list $\{j_1, \ldots, j_w\}$ is associated to $f_0^{(r-1,r)}$, then $\chi_0^{(r)}$ possesses the associated list $\{j_1, \ldots, j_w, r\}$. Concerning the Poisson bracket, suppose that the functions χ and f possess the associated lists $\{j_1, \ldots, j_w\}$ and $\{j'_1, \ldots, j'_v\}$, respectively. Then $\mathcal{L}_\chi f$ possesses the list

$$\{j_1, \ldots, j_w\} \cup \{j'_1, \ldots, j'_v\} = \{j_1, \ldots, j_w, j'_1, \ldots, j'_v\} \ .$$

For a sum of several Lie derivatives, we select the greater list, according to the ordering relation $\triangleleft$ in definition 1. Thus the process of accumulation of divisors is described via the union of lists. An heuristic analysis of the mechanism of accumulation, with the aim of identifying the worst one, could be made by unfolding all the recursive definitions of the functions $f_{\ldots}^{(\ldots)}$ in (17), (22) and (27). This is an horrible task, due to the need of considering three transformations at every normalization step. However, following the process is much easier if one considers simpler cases such as the Birkhoff normal form or the Kolmogorov normal form. We refer to the heuristic description reported in [13], where it is shown that some strict selection rules apply. In rough words one can say that there is a substantial excess of low indices, so that the products of small divisors are controlled geometrically. The selection rule is stated by definition 1.

In the following table we summarize all the relevant information about the lists of indices associated to the functions in the expansion of $H^{(r)}$.

(50)

Function	conditions	list of indices
$f_0^{(r,s)}$	$0 \le r < s$	$\big(\mathcal{J}_{r,s}\big)^3$
$f_1^{(r,s)}$	$0 \le r < s$	$\big(\mathcal{J}_{r,s}\big)^3 \cup \{r\}$
$f_2^{(r,s)}$	$0 \le r < s$	$\big(\mathcal{J}_{r,s}\big)^3 \cup \big(\{r\}\big)^2$
$f_{\ell\ge3}^{(r,s)}$	$r \ge 0\,,\ s \ge 1$	$\big(\mathcal{J}_{r,s} \cup \{\min\{r,s\}\}\big)^3$
$\chi_0^{(r)}$	$r \ge 1$	$\big(\mathcal{J}_{r-1,r}\big)^3 \cup \{r\}$
$\chi_1^{(r)}$	$r \ge 1$	$\big(\mathcal{J}_{r-1,r}\big)^3 \cup \big(\{r\}\big)^2$
$\chi_2^{(r)}$	$r \ge 1$	$\big(\mathcal{J}_{r-1,r} \cup \{r\}\big)^3$

Here we made some little abuse of notation. If $\mathcal{J}$ and $\mathcal{J}'$ are two lists of indices, we use the symbol $\cup$ of union to indicate

$$\mathcal{J} \cup \mathcal{J}' = \{I \cup I' : I \in \mathcal{J}\,,\ I' \in \mathcal{J}'\} \ .$$

Moreover, we use the power representation, e.g., $\mathcal{J}^3$ for the union $\mathcal{J} \cup \mathcal{J} \cup \mathcal{J}$.

We consider the rules described in (50) as the keystones of the whole proof; they are implicitly demonstrated in appendix A.3 (dealing with the proof of the main lemma of the analytic part) through some estimates involving the sequence $\{T_{r,s}\}_{r,s\ge0}$ that will be introduced later.

As it is well known, a second source of divergence is connected with the restriction of domains required by the estimate of multiple Poisson brackets. The restrictions are controlled by a sequence d_r that should converge to some positive d small enough. Actually, the accumulation of these quantities is controlled in much the same way as that of the small divisors. We define the sequences $\{d_r\}_{r\ge 0}$ and $\{\delta_r\}_{r\ge 1}$ as

$$d_0 = 0\ , \qquad d_r = d_{r-1} + 4\delta_r\ , \qquad \delta_r = \frac{3}{8\pi^2 r^2}\ . \tag{51}$$

At the r-th step of the algorithm we find a Hamiltonian $H^{(r)}$ which is analytic in $\mathcal{D}_{(1-d_r)(\varrho,R,\sigma)}$. Thus, at every step, we restrict the domain by $4\delta_r$. The parameters are chosen so that $\lim_{r\to\infty} d_r = 1/4$, thus the sequence of domains converges to a compact set with no empty interior.

In order to give quantitative estimates of the accumulation of small divisors, we introduce the sequence of positive real numbers $T_{r,s}$ associated to the lists of indices $\mathcal{J}_{r,s}$ defined as

$$T_{0,s} = T_{s,0} = 1 \ \text{ for } s \ge 0\ , \qquad T_{r,s} = \max_{I\in\mathcal{J}_{r,s}} \prod_{j\in I\,,\, j\ge 1} \frac{1}{a_j\delta_j^2} \ \text{ for } r\ge 1\,,\ s\ge 1\ . \tag{52}$$

If $\mathcal{J}_{r,s}$ is empty, we just set $T_{r,s} = 1$.

Lemma 7: *For all $r, s \ge 1$, the sequence $T_{r,s}$ satisfies the properties:*

(i) $T_{r-1,s} \le T_{r,s}$ and $T_{r',s} = T_{s,s}$ for $r' > s$;

(ii) $T_{r-1,r}T_{r,s}/(a_m\delta_m^2) \le T_{r,r+s}$, where $m = \min\{r, s\}$.

The proof is deferred to appendix A.2.

4.2 Convergence of the algorithm under non-resonance conditions

It is convenient to introduce the constant

$$M = \max\left\{1\,,\ \bar{E}\left(\frac{2e}{\varrho\sigma} + \frac{e^2}{R^2}\right)\right\}\ . \tag{53}$$

This will allow us to produce uniform estimates in the parameters. We also define the sequence $\{\zeta_r\}_{r\ge 0}$ as

$$\zeta_0 = 0\ , \qquad \zeta_1 = 0\ , \qquad \zeta_r = \zeta_{r-1} + \frac{2^{-(r+6)}}{1-2^{-(r+6)}} \quad \text{for } r\ge 2\ . \tag{54}$$

This will be used in order to obtain a bound for the actions of the generating functions $\varepsilon^{r-1}\mathcal{D}_2^{(r)}$. We set $\zeta_1 = 0$ because $\mathcal{D}_2^{(1)} = 0$, as shown in subsection 2.4.2. Finally the number of summands in the recursive formulæ (17), (22) and (27) will be estimated by the three sequences of integers $\{\nu_{r,s}\}_{r\ge 0\,,\,s\ge 0}$, $\{\nu_{r,s}^{(\mathrm{I})}\}_{r\ge 1\,,\,s\ge 0}$ and $\{\nu_{r,s}^{(\mathrm{II})}\}_{r\ge 1\,,\,s\ge 0}$ defined as

$$
\begin{aligned}
\nu_{0,s} &= 1 && \text{for } s \ge 0\,, \\
\nu^{(\mathrm{I})}_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} \nu^{j}_{r-1,r}\nu_{r-1,s-jr} && \text{for } r \ge 1\,,\ s \ge 0\,, \\
\nu^{(\mathrm{II})}_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} (\nu^{(\mathrm{I})}_{r,r})^{j}\nu^{(\mathrm{I})}_{r,s-jr} && \text{for } r \ge 1\,,\ s \ge 0\,, \\
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} (\nu^{(\mathrm{II})}_{r,r})^{j}\nu^{(\mathrm{II})}_{r,s-jr} && \text{for } r \ge 1\,,\ s \ge 0\,.
\end{aligned}
\tag{55}
$$

Lemma 8: *Consider a Hamiltonian $H^{(0)}$ expanded as in (8) and satisfying points (c')–(e') of lemma 2. Assume that the non-resonance hypotheses (10) and (11) are satisfied with positive $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ and that*

$$
\varepsilon^{i-1}\left[\frac{M^{3i}}{b_i}\,\frac{T^3_{i,i}}{(a_i\delta_i^2)^2}\,\nu_{i,i}\exp(i\zeta_{i-1})\right] \le \frac{1}{2^{i+6}} \qquad \text{for } 2 \le i \le r\ .
\tag{56}
$$

Then, the following upper bounds on the generating functions hold true:

$$
\begin{aligned}
\left(\frac{2e}{\varrho\sigma}+\frac{e^2}{R^2}\right)\frac{1}{\delta_r^2}\,\|\chi_0^{(r)}\|_{1-d_{r-1}} &\le M^{3r-2}\,\frac{T^3_{r-1,r}}{a_r\delta_r^2}\,\nu_{r-1,r}\exp(r\zeta_{r-1})\ , \\
\left(\frac{2e}{\varrho\sigma}+\frac{e^2}{R^2}\right)\frac{1}{\delta_r^2}\,\|\chi_1^{(r)}\|_{1-d_{r-1}-\delta_r} &\le M^{3r-1}\,\frac{T^3_{r,r}}{(a_r\delta_r^2)^2}\,\nu^{(\mathrm{I})}_{r,r}\exp(r\zeta_{r-1})\ , \\
\left(\frac{2e}{\varrho\sigma}+\frac{e^2}{R^2}\right)\frac{1}{\delta_r^2}\,\|\chi_2^{(r)}\|_{1-d_{r-1}-2\delta_r} &\le M^{3r}\left(\frac{T_{r,r}}{a_r\delta_r^2}\right)^3\nu^{(\mathrm{II})}_{r,r}\exp(r\zeta_{r-1})\ , \\
\varepsilon^{j(r-1)}\|\mathcal{E}_j^{(r)}g\|_{1-d_r} &\le 2^{-j(r+6)}\|g\|_{1-d_{r-1}-3\delta_r}\ ,
\end{aligned}
\tag{57}
$$

where the latter inequality is satisfied for all $j \ge 0$ and any analytic function g with finite norm $\|g\|_{1-d_{r-1}-3\delta_r}$, being $\{\mathcal{E}_j^{(r)}\}_{j\ge 0}$ the sequence of operators defined in (5) and related to the Lie transform operator $\mathcal{T}_{\varepsilon^{r-1}\mathcal{D}_2^{(r)}}$. Furthermore, the terms appearing in the expansion of the new Hamiltonian $H^{(r)}$ in (37) are bounded by

$$
\begin{aligned}
\|f_\ell^{(r,s)}\|_{1-d_r} &\le \frac{\bar{E}M^{3s-3+\ell}}{2^\ell}\,\frac{T^3_{r,s}}{(a_r\delta_r^2)^\ell}\,\nu_{r,s}\exp(s\zeta_r) && \text{for } 0 \le \ell \le 2\,,\ s > r\,, \\
\|f_\ell^{(r,0)}\|_{1-d_r} &\le \frac{\bar{E}}{2^\ell}\nu_{r,0} && \text{for } \ell \ge 3\,, \\
\|f_\ell^{(r,s)}\|_{1-d_r} &\le \frac{\bar{E}M^{3s}}{2^\ell}\left(\frac{T_{r,s}}{a_m\delta_m^2}\right)^3\nu_{r,s}\exp(s\zeta_r) && \text{for } \ell \ge 3\,,\ s \ge 1\,, \\
& && \text{with } \ m = \min\{r,s\}\,.
\end{aligned}
\tag{58}
$$

Finally, for $r \ge 2$, the variations of the frequencies, induced by the r-th normalization *step, are bounded by*

$$\max_{\substack{1\le i\le n_1\\ 1\le j\le n_2}} \left\{ \frac{1}{\sigma}\left|\omega_i^{(r)}-\omega_i^{(r-1)}\right| ,\, \varepsilon\left|\Omega_j^{(r)}-\Omega_j^{(r-1)}\right| \right\} \le \varepsilon^r M^{3r} \left(\frac{T_{r,r}}{a_r\delta_r^2}\right)^3 \nu_{r,r}\exp(r\zeta_r)\ . \tag{59}$$

The proof requires a lot of computations that are presented in details in appendix A.3.

We recall the reader attention on the hypotheses concerning the positiveness of the sequences $a_1,\ldots,a_r$ and $b_1,\ldots,b_r$, that must be satisfied at every step. In order to prove the convergence we need a stronger non-resonance hypothesis on the sequence $a_1,\ldots,a_r$, that we borrow from [16].

Definition 2: *We say that a sequence $\{a_r\}_{r\ge1}$ satisfies the* **condition $\boldsymbol{\tau}$**, *if*

$$-\sum_{r\ge1}\frac{\log a_r}{r(r+1)} = \Gamma < \infty\ . \tag{60}$$

Lemma 9: *Let the sequence $\{a_r\}_{r\ge1}$, introduced by (10), satisfy condition $\boldsymbol{\tau}$ and the sequence $\{\delta_r\}_{r\ge1}$ be defined as in (51). Then, the sequence $\{T_{r,s}\}_{r\ge0\,,\,s\ge0}$ defined by (52) is bounded by*

$$T_{r,s}\le\frac{1}{a_s\delta_s^2}T_{r,s}\le\left(2^{15}e^{\Gamma}\right)^s\qquad\text{for } r\ge1\,,\ s\ge1\ .$$

Lemma 10: *The sequence of positive integer numbers $\{\nu_{r,s}\}_{r\ge0\,,\,s\ge0}$ defined in (55) is bounded by*

$$\nu_{r,s}\le\nu_{s,s}\le2^{8s}\qquad\text{for } r\ge0\,,\ s\ge0\ .$$

The proofs are deferred to appendix A.4.

We summarize the analytic estimates in the following

Proposition 1: *Consider an analytic Hamiltonian $H^{(0)}:\mathcal{D}_{\varrho,R,\sigma}\to\mathbb{C}$ expanded as in (8), satisfying hypotheses (c')–(e') of lemma 2.*

Assume moreover the following hypotheses:

(f') the sequences of frequency vectors $\{\omega^{(r)}\}_{r\ge0}$ and $\{\Omega^{(r)}\}_{r\ge0}$ fulfill the non-resonance conditions (10) and (11) for $r\ge1$, with $b_r\ge\bar b>0$ and $\{a_r\}$ satisfying condition $\boldsymbol{\tau}$;

(g') the parameter ε is smaller than the "analytic threshold value" $\varepsilon^{\star}_{\rm an}$, being

$$\varepsilon^{\star}_{\rm an}=\frac{1}{2^8}\left(\frac{\min\{1\,,\,\bar b\}}{\mathcal{A}}\right)^2\qquad\text{with}\qquad\mathcal{A}=\left(2^{18}M\,e^{\Gamma}\right)^3\ , \tag{61}$$

where M and Γ are defined in (53) and (60), respectively.

Then, there exists an analytic canonical transformation $\Phi^{(\infty)}_{\omega^{(0)}}:\mathcal{D}_{1/2(\varrho,R,\sigma)}\to\mathcal{D}_{3/4(\varrho,R,\sigma)}$ such that the Hamiltonian $H^{(\infty)}=H^{(0)}\circ\Phi^{(\infty)}_{\omega^{(0)}}$ is in normal form, i.e.,

$$H^{(\infty)}=\omega^{(\infty)}\cdot p+\varepsilon\sum_{j=1}^{n_2}\Omega_j^{(\infty)}z_j\bar z_j+\sum_{\ell>2}\sum_{s\ge0}\varepsilon^s f_\ell^{(\infty,s)}\ . \tag{62}$$

Furthermore, the norms of the functions $f_{\ell}^{(\infty,s)} \in \mathcal{P}_{\ell,sK}$ *are bounded by*

$$\left\| f_{\ell}^{(\infty,s)} \right\|_{3/4} \le \frac{\bar{E}}{2^{\ell}} \mathcal{A}^{s} \qquad \textit{for } \ell \ge 3\,,\ s \ge 0 \tag{63}$$

and $\{\omega^{(r)}\}_{r\ge 0}$ *and* $\{\Omega^{(r)}\}_{r\ge 0}$ *are Cauchy sequences satisfying (41) and*

$$\max_{1\le i\le n_1} \left|\omega_i^{(r)} - \omega_i^{(r-1)}\right| \le \sigma\big(\varepsilon\mathcal{A}\big)^{r}\,, \qquad \max_{1\le j\le n_2} \left|\Omega_j^{(r)} - \Omega_j^{(r-1)}\right| \le \varepsilon^{r-1}\mathcal{A}^{r}\,, \tag{64}$$

for $r \ge 2$ *so that the limit values* $\omega^{(\infty)}$ *and* $\Omega^{(\infty)}$ *do exist.*

The rest of the section contains a sketch of the proof, which depends on all the previous lemmas.

First, we apply lemma 8, remarking that condition (56) is satisfied thanks to the hypotheses of proposition 1 using also lemmas 9–10 and the elementary inequality $\exp(\zeta_s) < 2$ for $s \ge 0$ (see (54)).

Therefore, starting from the inequalities in formula (58) and using property (i) of lemma 7, by trivial calculations we check that

$$\|f_{\ell}^{(r,s)}\|_{1-d_r} \le \frac{\bar{E}}{2^{\ell}} \mathcal{A}^{s} \qquad \text{for } 0 \le \ell \le 2\,,\ s > r \ \text{ or } \ \ell \ge 3\,,\ s \ge 0\,. \tag{65}$$

Remarking that $\varepsilon\mathcal{A} < 1$ (actually $\varepsilon\mathcal{A} < 1/\mathcal{A} \ll 1$, in view of condition (g') combined with the definitions in (53) and (60)), by (65), we have that $H^{(r)}$, written as in (37), is analytic on $\mathcal{D}_{(1-d_r)(\varrho,R,\sigma)} \supset \mathcal{D}_{3/4(\varrho,R,\sigma)}$. With similar calculations, starting from (59) and in view of condition (g'), we check the inequalities in (64).

Let us now focus on the difference $H^{(r)} - H^{(r-1)}$. We use (37), (12), the recursive definitions in (40), the fact that $\mathcal{E}_0^{(r)}$ is equal to the identity and the equation $f_{\ell}^{(\mathrm{III};r,s)} = f_{\ell}^{(r-1,s)}$ for $\ell \ge 3\,,\ 1 \le s \le r-1$ (see formulæ (17), (22), (27)). So we get

$$\begin{aligned} H^{(r)} - H^{(r-1)} &= \big(\omega^{(r)} - \omega^{(r-1)}\big)\cdot p + \varepsilon \sum_{j=1}^{n_2} \big(\Omega_j^{(r)} - \Omega_j^{(r-1)}\big) z_j \bar{z}_j \\ &\quad + \sum_{\ell\ge 0}\sum_{s\ge r} \varepsilon^{s} \left(f_{\ell}^{(r,s)} - f_{\ell}^{(r-1,s)} \right) + \sum_{\ell>2}\sum_{s=1}^{r-1}\sum_{j\ge 1} \varepsilon^{s+j(r-1)} \mathcal{E}_j^{(r)} f_{\ell}^{(r-1,s)}\,. \end{aligned} \tag{66}$$

Therefore, in view of the fourth inequality in (57) and those in (64)–(65), we have

$$\left\| H^{(r)} - H^{(r-1)} \right\|_{3/4} \le \left(n_1\sigma\varrho + \frac{n_2 R^2}{\varepsilon} + \frac{4\bar{E}}{1-\varepsilon\mathcal{A}} \right) \big(\varepsilon\mathcal{A}\big)^{r} + \frac{2^{-(r+6)}}{1-2^{-(r+6)}} \frac{\bar{E}\,\varepsilon\mathcal{A}}{1-\varepsilon\mathcal{A}}\,. \tag{67}$$

Recalling that the sup-norm is bounded by the weighted Fourier one, this proves that $\{H^{(r)}\}_{r\ge 0}$ is a Cauchy sequence of analytic Hamiltonians admitting a limit $H^{(\infty)}$, since the r.h.s. of the estimate above tends to zero for $r \to \infty$. Moreover, formulæ (66)–(67) imply that also $\{\omega^{(r)}\}_{r\ge 0}$, $\{\Omega^{(r)}\}_{r\ge 0}$ and $\{f_{\ell}^{(r,s)}\}_{r\ge 0}$ for $\ell \ge 3\,,\ s \ge 0$ are Cauchy sequences, which in turn implies that $H^{(\infty)} = \lim_{r\to\infty} H^{(r)}$ has the form (62), with $f_{\ell}^{(\infty,s)} \in \mathcal{P}_{\ell,sK}$ bounded as in (63), in view of inequality (65).

It remains to prove that the canonical transformation $\Phi^{(\infty)}_{\omega^{(0)}} : \mathcal{D}_{1/2(\varrho,R,\sigma)} \to \mathcal{D}_{3/4(\varrho,R,\sigma)}$ is analytic. The proof is based on standard arguments in the Lie series theory, that we recall here, referring to subsection 4.3 of [11] for more details. Let us denote by $\varphi^{(r)}$ the canonical change of coordinates induced by the r-th step, i.e.,

$$\varphi^{(r)}(p,q,z,\mathbf{i}\bar{z}) = \exp\left(\varepsilon^r \mathcal{L}_{\chi_0^{(r)}}\right) \exp\left(\varepsilon^r \mathcal{L}_{\chi_1^{(r)}}\right) \exp\left(\varepsilon^r \mathcal{L}_{\chi_2^{(r)}}\right) \mathcal{T}_{\varepsilon^{r-1}\mathcal{D}_2^{(r)}}(p,q,z,\mathbf{i}\bar{z}) \ . \tag{68}$$

Using the fourth inequality in (57), one can easily verify that

$$\max_{1\le j\le n_2} \left\| \mathcal{T}_{\varepsilon^{r-1}\mathcal{D}_2^{(r)}} z_j - z_j \right\|_{3/4} < \delta_r R \ . \tag{69}$$

Analogous estimates can be deduced for the other Lie series appearing in (68) and for all the canonical variables, using again the inequalities in (57). Therefore, one has $\varphi^{(r)}(\mathcal{D}_{(1/2+d_{r-1})(\varrho,R,\sigma)}) \subset \mathcal{D}_{(1/2+d_r)(\varrho,R,\sigma)}$ and defining $\Phi^{(r)} = \varphi^{(1)} \circ \ldots \circ \varphi^{(r)}$ one has $\Phi^{(r)}(\mathcal{D}_{1/2(\varrho,R,\sigma)}) \subset \mathcal{D}_{3/4(\varrho,R,\sigma)}$. By repeatedly using the so-called exchange theorem for Lie series/transforms, one immediately obtains that $H^{(r)} = H^{(0)} \circ \Phi^{(r)}$. By using estimate (69) and the ones related to the other generating functions, we can prove that the canonical transformation $\Phi^{(\infty)}_{\omega^{(0)}} = \lim_{r\to\infty} \Phi^{(r)}$ is well defined in $\mathcal{D}_{1/2(\varrho,R,\sigma)}$. Finally, we get $H^{(0)} \circ \Phi^{(\infty)}_{\omega^{(0)}} = \lim_{r\to\infty} H^{(0)} \circ \Phi^{(r)} = \lim_{r\to\infty} H^{(r)} = H^{(\infty)}$.

Actually, with some additional effort, we could prove that $\Phi^{(\infty)}_{\omega^{(0)}}$ differs from the identity just for terms of order $\mathcal{O}(\varepsilon)$. As a final comment, we adopt the symbol $\Phi^{(\infty)}_{\omega^{(0)}}$ in order to emphasize the parametric dependence of that canonical transformation on the initial frequency $\omega^{(0)}$, which is relevant in the next section.

5. Measure of the resonant regions

The aim of this section is to show that there exists a set of frequencies $\omega^{(0)}$ of relatively big measure, for which our procedure converges. To this end we consider the sequence $\left\{(\omega^{(r)}\,,\,\varepsilon\Omega^{(r)})\right\}_{r\ge0}$ as function of the parameter $\omega^{(0)}$ recursively defined by (38) and (39). We start with a compact set $\mathcal{W}^{(0)} \subset \mathbb{R}^{n_1}$ complex extended as $\mathcal{W}^{(0)}_{h_0}$, where the Hamiltonian $H^{(0)}$ is well defined and the Jacobian of $\Omega^{(0)}(\omega^{(0)})$ is bounded. Then, we construct a sequence of complex extended domains $\mathcal{W}^{(0)}_{h_0} \supseteq \mathcal{W}^{(1)}_{h_1} \supseteq \mathcal{W}^{(2)}_{h_2} \supseteq \ldots$, where $\{h_r\}_{r\ge0}$ is a positive non-increasing sequence of real numbers, with the following properties: $\omega^{(r)}(\omega^{(0)})$ is an analytic function admitting an inverse $\varphi^{(r)}$ mapping the domain $\mathcal{W}^{(r)}_{h_r}$ to $\mathcal{W}^{(0)}_{h_0}$. We must prove the sequence of functions $\varphi^{(r)}$ converges to $\varphi^{(\infty)}$ mapping $\mathcal{W}^{(\infty)}_{h_\infty}$ to $\mathcal{W}^{(0)}_{h_0}$, the image having large relative measure.

Let us describe the process in some more detail. We start by setting $\mathcal{W}^{(1)} = \mathcal{W}^{(0)}$. For $r \ge 2$ and some fixed positive values of the parameters $\gamma\,,\tau \in \mathbb{R}$ and $K \in \mathbb{N}$, we define the sequence of real domains $\{\mathcal{W}^{(r)}\}_{r\ge0}$ as follows: at each step r, we remove from $\mathcal{W}^{(r-1)}$ all the resonant regions related to the new small divisors appearing in the formal algorithm (see section 2). Therefore,

$$\mathcal{W}^{(r)} = \mathcal{W}^{(r-1)} \setminus \mathcal{R}^{(r)} \ , \qquad \text{with} \quad \mathcal{R}^{(r)} = \bigcup_{\substack{rK<|k|\le(r+1)K \\ |l|\le 2}} \mathcal{R}^{(r)}_{k,l} \ , \tag{70}$$

where

$$\mathcal{R}^{(r)}_{k,l} = \left\{ \omega \in \mathcal{W}^{(r-1)} : \left| k\cdot\omega + \varepsilon l \cdot \Omega^{(r)} \circ \varphi^{(r)}(\omega) \right| < \frac{2\gamma}{((r+1)K)^{\tau}} \right\} \ . \tag{71}$$

The complex extensions $\mathcal{W}^{(r)}_{h_r}$ are technically necessary in order to work with Cauchy estimates, but it is crucial that the measure is evaluated on the real domain.

It will be also convenient to introduce the functions $\delta\omega^{(r)}$ and $\Delta\Omega^{(r)}$ defined as

$$\delta\omega^{(r)} = \omega^{(r)} \circ \varphi^{(r-1)} - \mathrm{Id} \ , \qquad \Delta\Omega^{(r)} = \Omega^{(r)} \circ \varphi^{(r-1)} - \Omega^{(r-1)} \circ \varphi^{(r-1)} \ . \tag{72}$$

By the way, let us remark that from equations in (41) we have $\delta\omega^{(1)} = 0$ and $\Delta\Omega^{(1)} = 0$.

The following proposition is an adaptation of the approach by Pöschel in [32].

Proposition 2: *Let us consider the family (8) of Hamiltonians $H^{(0)}$ parameterized by the n_1-dimensional frequency vector $\omega^{(0)}$. Assume that there exist positive parameters γ, τ, $\bar{b}$, J_0, a positive integer K and a compact set $\mathcal{W} \subset \mathbb{R}^{n_1}$ such that the function $\Omega^{(0)} : \mathcal{W}_{h_0} \to \mathbb{C}^{n_2}$ is analytic and satisfies the properties (a')–(b') of lemma 2. Define the sequence $\{h_r\}_{r\ge 0}$ of radii of the complex extensions as*

$$h_0 = \min\left\{ \frac{\gamma\eta}{4K^{\tau}} \, , \, \frac{\bar{b}}{4J_0} \right\} \qquad \text{and} \qquad h_r = \frac{h_{r-1}}{2^{\tau+2}} \quad \text{for } r \ge 1 \ , \tag{73}$$

where $\eta = \min\{1/K \, , \, \sigma\}$.
Considering the sequence of Hamiltonians $\{H^{(r)}\}_{r\ge 0}$, formally defined by the algorithm in section 2, let us assume that the functions $\omega^{(1)}$, $\Omega^{(1)}$, ... , $\omega^{(r)}$, $\Omega^{(r)}$ satisfy the following hypotheses up to a fixed normalization step $r \ge 0$:

- *(h') the function $\omega^{(s)}(\omega^{(0)})$ has analytic inverse $\varphi^{(s)}$ on $\mathcal{W}^{(s)}_{h_s}$, for $0 \le s \le r-1$, where $\varphi^{(0)} = \mathrm{Id}$ and the domains are recursively defined as extensions of those given by formulæ (70)–(71), starting from $\mathcal{W}^{(0)} = \mathcal{W}$;*
- *(i') both $\omega^{(s)} \circ \varphi^{(s-1)} : \mathcal{W}^{(s-1)}_{h_{s-1}} \to \mathbb{C}^{n_1}$ and $\Omega^{(s)} \circ \varphi^{(s-1)} : \mathcal{W}^{(s-1)}_{h_{s-1}} \to \mathbb{C}^{n_2}$ are analytic functions, for $1 \le s \le r$;*
- *(j') for $2 \le s \le r$, there exist positive parameters ε, σ and $\mathcal{A} \ge 1$ satisfying*

$$\max_{1\le j\le n_1} \sup_{\omega\in\mathcal{W}^{(s-1)}_{h_{s-1}}} \left| \delta\omega^{(s)}_j(\omega) \right| \le \sigma \big(\varepsilon\mathcal{A}\big)^s \ , \quad \max_{1\le j\le n_2} \sup_{\omega\in\mathcal{W}^{(s-1)}_{h_{s-1}}} \left| \Delta\Omega^{(s)}_j(\omega) \right| \le \varepsilon^{s-1}\mathcal{A}^s \ , \tag{74}$$

  where $\delta\omega^{(s)}$ and $\Delta\Omega^{(s)}$ are defined as in (72), with $\delta\omega^{(1)} = 0$ and $\Delta\Omega^{(1)} = 0$;
- *(k') the parameter ε is smaller than the "geometric threshold value"*

$$\varepsilon^{*}_{\mathrm{ge}} = \min\left\{ \frac{1}{(2J_0+1)\eta} \, , \, \frac{1}{2^{\tau+3}\mathcal{A}} \min\left\{ 1 \, , \, \frac{h_0}{8\mathcal{A}} \, , \, \frac{\bar{b}}{8\mathcal{A}} \right\} \right\} \ . \tag{75}$$

Then, the function $\omega^{(r)}(\omega^{(0)})$ *admits an analytic inverse* $\varphi^{(r)} : \mathcal{W}^{(r)}_{h_r} \to \mathcal{W}^{(0)}_{h_0}$ *on its domain of definition and satisfies the inclusion relation* $\varphi^{(r)}\big(\mathcal{W}^{(r)}_{h_r}\big) \subset \varphi^{(r-1)}\big(\mathcal{W}^{(r-1)}_{h_{r-1}}\big)$ *. Moreover, the following non-resonance inequalities hold true:*

$$
\begin{aligned}
&\min_{\substack{k\in\mathbb{Z}^{n_1},\, 0<|k|\le(r+1)K \\ l\in\mathbb{Z}^{n_2},\, 0\le|l|\le 2}} \ \inf_{\omega\in\mathcal{W}^{(r)}_{h_r}} \left| k\cdot\omega + \varepsilon l \cdot \Omega^{(r)}\big(\varphi^{(r)}(\omega)\big)\right| \ge \frac{\gamma}{\big((r+1)K\big)^{\tau}} \ ,\\
&\min_{1\le i<j\le n_2} \ \inf_{\omega\in\mathcal{W}^{(r)}_{h_r}} \left|\Omega^{(r)}_i\big(\varphi^{(r)}(\omega)\big) - \Omega^{(r)}_j\big(\varphi^{(r)}(\omega)\big)\right| \ge \bar{b} \ .
\end{aligned}
\tag{76}
$$

Finally, the Lipschitz constants related to the Jacobians of the functions $\varphi^{(r)}$ *and* $\Omega^{(r)} \circ \varphi^{(r)}$ *are uniformly bounded as*

$$
\left|\frac{\partial\big(\varphi^{(r)} - \mathrm{Id}\big)}{\partial\omega}\right|_{\infty;\mathcal{W}^{(r)}_{h_r}} \le \varepsilon\sigma \ , \qquad \left|\frac{\partial\big(\Omega^{(r)}\circ\varphi^{(r)}\big)}{\partial\omega}\right|_{\infty;\mathcal{W}^{(r)}_{h_r}} \le 2J_0 + 1 \ .
\tag{77}
$$

The proof is deferred to appendix A.5.

In order to prove the proposition, we consider a set of tori with "Diophantine" frequencies.

Definition 3: *For any fixed* $\omega^{(0)}$ *, we say that the sequence of frequency vectors* $\big\{\big(\omega^{(r)}(\omega^{(0)})\, ,\, \varepsilon\Omega^{(r)}(\omega^{(0)})\big)\big\}_{r\ge0}$ *is Diophantine, if there are positive constants* γ *,* τ *and* $\bar{b}$ *such that (10) and (11) are satisfied, for all* $r \ge 1$*, with* $a_r = \gamma/(rK)^{\tau}$ *and* $b_r \ge \bar{b}$ *.*

The latter condition is more manageable than the weaker condition $\boldsymbol{\tau}$ in (60) considered in section 4.

We denote by $\mathcal{K}^{(r)}_l$ the closed convex hull of the gradient set

$$
\mathcal{G}^{(r)}_l = \Big\{\partial_\omega\big[\varepsilon l\cdot\Omega^{(r)}\big(\varphi^{(r)}(\omega)\big)\big] \ : \ \omega\in\mathcal{W}^{(r)}\Big\} \ .
\tag{78}
$$

For ε small enough, the closed convex hull contains no integer vector except the null one. The smallness condition is

$$
\varepsilon < \frac{1}{4(2J_0+1)} \ .
\tag{79}
$$

Indeed, using the Euclidean distance we have $\mathrm{dist}(k, \mathcal{K}^{(r)}_l) \ge 1/2$ for $k \in \mathbb{Z}^{n_1}\setminus\{0\}$.

The estimate of the volume of resonant zones that are removed at each step is based on lemma 8.1 of Pöschel's paper [32], that we report here (the Lebesgue measure is denoted by $\mathrm{m}(\cdot)$).

Lemma 11: *If* $\mathrm{dist}(k, \mathcal{K}^{(r)}_l) = s > 0$ *then*

$$
\mathrm{m}(\mathcal{R}^{(r)}_{k,l}) \le \frac{4\gamma}{((r+1)K)^{\tau}}\frac{D^{n_1-1}}{s} \ ,
$$

where D *is the diameter of* $\mathcal{W}^{(0)}$ *with respect to the sup-norm.*

The volume of the resonant regions must be compared with respect to the initial set $\mathcal{W}^{(0)}$. To this end, we estimate the measure of $\varphi^{(r)}(\mathcal{R}_{k,l}^{(r)})$ in the original coordinates $\omega^{(0)}$, recalling that $\varphi^{(r)}$ is the inverse function of $\omega^{(r)}(\omega^{(0)})$. Using lemma 11 and assumption (79), for $k \in \mathbb{Z}^{n_1} \setminus \{0\}$ we have

$$\mathrm{m}\Big(\varphi^{(r)}\big(\mathcal{R}_{k,l}^{(r)}\big)\Big) \le \frac{8\gamma D^{n_1-1}}{((r+1)K)^{\tau}} \sup_{\omega\in\mathcal{W}^{(r)}} \det\left(\frac{\partial \varphi^{(r)}}{\partial \omega}\right) . \tag{80}$$

Starting from the first inequality in (77) and using the well known Gershgorin circle theorem, we control the actual expansion of the resonant zones due to the stretching of the frequencies, by verifying that

$$\sup_{\omega\in\mathcal{W}^{(r)}} \det\left(\frac{\partial \varphi^{(r)}}{\partial \omega}\right) \le 2 \qquad \text{when} \quad \varepsilon \le \frac{\log 2}{\sigma n_1^2} . \tag{81}$$

Using the inequalities (80)–(81), we easily get a final estimate of the total volume of the resonant regions included in $\mathcal{W}^{(0)}$:

$$\begin{aligned} \sum_{r=2}^{\infty} \sum_{\substack{rK<|k|\le(r+1)K \\ |l|\le 2}} \mathrm{m}\Big(\varphi^{(r)}\big(\mathcal{R}_{k,l}^{(r)}\big)\Big) &\le c_{n_2} \sum_{r=2}^{\infty} \sum_{(r-1)K<|k|\le rK} \frac{16\gamma D^{n_1-1}}{((r+1)K)^{\tau}} \\ &\le \gamma\, \frac{2^{n_1+4} c_{n_2} D^{n_1-1}}{K^{\tau-n_1}} \sum_{r=3}^{\infty} \frac{1}{r^{\tau-n_1+1}} , \end{aligned} \tag{82}$$

where $c_{n_2} = (2n_2+2)(2n_2+1)/2$ is the maximum number of polynomial terms having degree ≤ 2 in the transverse variables $(z, \mathrm{i}\bar{z})$. The last series is convergent if $\tau > n_1$ and the sum is of order $\mathcal{O}(\gamma)$.

6. Proof of theorem 1

The proof is a straightforward combination of propositions 1 and 2, which summarize the analytic study of the convergence of our algorithm and the geometric part, respectively. We sketch the argument.

According to lemma 2 the family of Hamiltonians $\mathcal{H}^{(0)}$, parameterized by the frequency vectors $\omega^{(0)}$ and defined on a real domain, can be extended to a complex domain $\mathcal{D}_{\varrho,R,\sigma} \times \mathcal{W}_{h_0}$ with suitable parameters; moreover, their expansions can be written as $H^{(0)}$ in (8). Possibly modifying the values of parameters γ and τ, we can choose γ and $\tau > n_1$ such that the estimate of the resonant volume in the last row of (82) is smaller than $\mathrm{m}(\mathcal{W}_{h_0})$ and property (a') of lemma 2 is still satisfied. Let us consider values of the small parameter ε such that $\varepsilon < \varepsilon^{\star}$, with

$$\varepsilon^{\star} = \min\left\{\varepsilon_{\rm an}^{\star}\,,\, \varepsilon_{\rm ge}^{\star}\,,\, \frac{1}{4(2J_0+1)}\,,\, \frac{\log 2}{\sigma n_1^2}\right\} , \tag{83}$$

where $\varepsilon_{\rm an}^{\star}$ and $\varepsilon_{\rm ge}^{\star}$ are defined in (61) and (75), respectively.

Recall that the r-th normalization step of the formal algorithm described in section 2 can be performed if the non-resonance conditions (10)–(11) are satisfied. Assuming the threshold value $\varepsilon^\star$ as in (83) lemma 2 and proposition 2 ensure that the first step can be performed for every frequency vector $\omega^{(0)}\in\mathcal{W}^{(0)}_{h_0}$.

We now proceed by induction. Let us suppose that $r-1$ steps have been performed and proposition 2 applies. In view of the non-resonance condition (76) the r-th normalization step can be performed. We now check that proposition 2 applies again. By construction, both $\omega^{(r)}(\omega^{(0)})$ and $\Omega^{(r)}(\omega^{(0)})$ are analytic functions on $\varphi^{(r-1)}\big(\mathcal{W}^{(r-1)}_{h_{r-1}}\big)$. In view of $\varepsilon<\varepsilon^\star$, hypothesis (g') of proposition 1 is satisfied, so lemma 8 applies and the estimate (64) on the shift of the frequencies holds true. Thus proposition 2 can be applied at the r-th step which completes the induction.

We conclude that the non-resonance conditions (76) hold true for $r\ge 0$ and that the sequence of frequency vectors $\big\{\big(\omega^{(r)}(\omega^{(0)})\,,\,\varepsilon\Omega^{(r)}(\omega^{(0)})\big)\big\}_{r\ge 0}$ is Diophantine for $\omega^{(0)}\in\lim_{r\to\infty}\varphi^{(r)}\big(\mathcal{W}^{(r)}_{h_r}\big)=\bigcap_{r=0}^{\infty}\varphi^{(r)}\big(\mathcal{W}^{(r)}_{h_r}\big)$, where we used the inclusion relation $\varphi^{(r)}\big(\mathcal{W}^{(r)}_{h_r}\big)\subset\varphi^{(r-1)}\big(\mathcal{W}^{(r-1)}_{h_{r-1}}\big)$ between open sets. Finally, also hypothesis (f') of proposition 1 is satisfied and so, for $\omega^{(0)}\in\bigcap_{r=0}^{\infty}\varphi^{(r)}\big(\mathcal{W}^{(r)}_{h_r}\big)$, there exists an analytic canonical transformation $\Phi^{(\infty)}_{\omega^{(0)}}$ which gives the initial Hamiltonian $H^{(0)}$ the normal form (62). Since $\bigcap_{r=0}^{\infty}\varphi^{(r)}\big(\mathcal{W}^{(r)}_{h_r}\big)$ is a countable intersection of open sets, it is measurable and

$$\mathrm{m}\bigg(\bigcap_{r=0}^{\infty}\varphi^{(r)}\big(\mathcal{W}^{(r)}_{h_r}\big)\bigg)\ge\mathrm{m}\big(\mathcal{W}^{(0)}\big)-\sum_{r=2}^{\infty}\sum_{\substack{rK<|k|\le(r+1)K\\ |l|\le 2}}\mathrm{m}\Big(\varphi^{(r)}\big(\mathcal{R}^{(r)}_{k,l}\big)\Big)>0\ ,$$

where we have taken into account the fact that the complex extension radius $h_r\to 0$ for $r\to\infty$, the estimate (82) and the initial choice of the parameters γ and τ at the beginning of the present section. This concludes the argument proving theorem 1.

We now add a short remark concerning the comparison with the estimates in previous works. The threshold value $\varepsilon^\star$ on the small parameter ε is explicitly defined in (83). Although the definition involves many parameters, we might produce an asymptotic estimate of the volume of the resonant regions for $\varepsilon\to 0$. In view of inequality (82) it is $\mathcal{O}(\gamma)$. Moreover, one can easily show† that $\varepsilon^\star\le\varepsilon^\star_{\rm an}=\mathcal{O}(\gamma^6)$. Thus the complement of the set of the invariant elliptic tori, i.e., $\mathcal{W}^{(0)}\setminus\bigcap_{r=0}^{\infty}\varphi^{(r)}\big(\mathcal{W}^{(r)}_{h_r}\big)$, has a measure estimated by $\mathcal{O}(\varepsilon^{1/6})$. This is definitely worse than the results obtained in [3] and [4], where this same quantity has been proven to be smaller than a bound $\mathcal{O}(\varepsilon^{b_1})$, with $b_1<1/2$. However, let us emphasize that our main interest is to establish the convergence of a constructive algorithm suitable for computer assisted applications. As a matter of facts, by explicitly performing a number of perturbation steps, both the applicability threshold and the estimate of the measure can be significantly improved, possibly giving realistic

† First, let us remark that $\varepsilon^\star_{\rm an}=\mathcal{O}(e^{-6\Gamma})$ in view of the definitions in (61). By definitions 2 and 3 we get that $\Gamma=\sum_{r\ge1}[-\log\gamma+\tau\log(rK)]/[r(r+1)]$. Therefore, $e^{-\Gamma}=\mathcal{O}(\gamma)$, as one can easily verify that $\sum_{r\ge1}1/[r(r+1)]=1$. For this purpose, it is enough to check by induction that $\sum_{r=1}^{s}1/[r(r+1)]=s/(s+1)$ for $s\ge1$.

estimates for physical systems. On the other hand it is well known that purely analytic estimates are usually unrealistically small. For this reason we did not pay attention to producing optimal estimates.

A. Technicalities

The appendix is devoted to technical details and proofs which have been moved here in order to avoid the overloading of the text.

A.1 Estimates for multiple Poisson brackets

We shorten the notation by replacing $|\cdot|_{\alpha\varrho,\alpha R}$ with $|\cdot|_\alpha$ and $\|\cdot\|_{\alpha\varrho,\alpha R,\alpha\sigma}$ with $\|\cdot\|_\alpha$.

Some Cauchy estimates on the derivatives in the restricted domains will be useful during the following proof. We recall that for any function g satisfying the hypotheses of lemma 3 Cauchy inequality reads

$$\left|\frac{\partial g}{\partial p_j}\right|_{1-d-d'} \le \frac{|g|_{1-d'}}{d\varrho}\ ,\qquad \left|\frac{\partial g}{\partial z_j}\right|_{1-d-d'} \le \frac{|g|_{1-d'}}{dR}\ . \tag{84}$$

Of course, the latter inequality holds true also by replacing z_j with $\bar{z}_j$.

Lemma 12: *Let d, $d' \in \mathbb{R}_+$ such that $d+d'<1$ and g, g' be two analytic functions with bounded norms $\|g\|_{1-d-d'}$ and $\|g'\|_{1-d'}$. Then, for all $\delta \in \mathbb{R}_+$ such that $d+d'+\delta<1$ one has*

$$\|\{g,g'\}\|_{1-d-d'-\delta} \le \left(\frac{2}{e\varrho\sigma}+\frac{1}{R^2}\right)\frac{1}{(d+\delta)\delta}\,\|g\|_{1-d-d'}\,\|g'\|_{1-d'}\ . \tag{85}$$

Proof. We separately consider the parts of the Poisson bracket involving the (p,q) variables and the $(z,\mathbf{i}\bar{z})$ ones. For the first part we have

$$\begin{aligned}
&\left\|\sum_{j=1}^{n_1}\left(\frac{\partial g}{\partial q_j}\frac{\partial g'}{\partial p_j}-\frac{\partial g}{\partial p_j}\frac{\partial g'}{\partial q_j}\right)\right\|_{1-d-d'-\delta} \le \\
&\qquad \sum_{k\in\mathbb{Z}^{n_1}}\sum_{k'\in\mathbb{Z}^{n_1}}\left[\left(\frac{|k|\,|g_k|_{1-d-d'}\,|g'_{k'}|_{1-d'}}{(d+\delta)\varrho}\right.\right. \\
&\qquad\qquad \left.\left.+\frac{|g_k|_{1-d-d'}\,|k'|\,|g'_{k'}|_{1-d'}}{\delta\varrho}\right)e^{(|k|+|k'|)(1-d-d'-\delta)\sigma}\right] \\
&\qquad \le \frac{2}{e\varrho\sigma}\frac{1}{(d+\delta)\delta}\,\|g\|_{1-d-d'}\,\|g'\|_{1-d'}\ ,
\end{aligned} \tag{86}$$

being $g'_{k'}=g'_{k'}(p,z,\mathbf{i}\bar{z})$ as in the expansion (43). Here the Cauchy estimate (84) and the elementary inequality $ae^{-ab}\le 1/(eb)$, for positive a and b have been used.

Let us now focus on the second part of the Poisson bracket. For every point $(p,z,\mathbf{i}\bar{z}) \in \mathcal{G}_{(1-d-d'-\delta)\varrho}\times\mathcal{B}_{(1-d-d'-\delta)R}$ and for all pairs of vectors k, $k' \in \mathbb{Z}^{n_1}$, we introduce an auxiliary function

$$G_{(p,z,\mathbf{i}\bar{z});k,k'}(t) = g_k\Big(p,\, z - t\frac{\partial g'_{k'}}{\partial(\mathbf{i}\bar{z})},\, \mathbf{i}\bar{z} + t\frac{\partial g'_{k'}}{\partial z}\Big)\ . \tag{87}$$

Since g_k is analytic on $\mathcal{G}_{(1-d-d')\varrho} \times \mathcal{B}_{(1-d-d')R}$, then $G_{(p,z,\mathbf{i}\bar{z});k,k'}$ is analytic for $|t| \le \bar{t}$, with

$$\bar{t} = \frac{\delta R}{\displaystyle\max_{1\le j\le n_2} \left\{ \left|\frac{\partial g'_{k'}}{\partial z_j}\right|_{1-d-d'-\delta}\,,\, \left|\frac{\partial g'_{k'}}{\partial(\mathbf{i}\bar{z}_j)}\right|_{1-d-d'-\delta} \right\}}\ . \tag{88}$$

Thus, by the Cauchy's estimate we get

$$\left|\,\{g_k, g'_{k'}\}\,\right|_{1-d-d'-\delta} \le \left| \frac{\mathrm{d}}{\mathrm{d}t} G_{(p,z,\mathbf{i}\bar{z});k,k'}(t)\ \Big|_{t=0} \right|_{1-d-d'-\delta} \le \frac{|g_k|_{1-d-d'}}{\bar{t}}\ . \tag{89}$$

By the definition (44) of the norm, we get

$$\left\| \sum_{j=1}^{n_2} \left(\frac{\partial g}{\partial(\mathbf{i}\bar{z}_j)} \frac{\partial g'}{\partial z_j} - \frac{\partial g}{\partial z_j} \frac{\partial g'}{\partial(\mathbf{i}\bar{z}_j)} \right) \right\|_{1-d-d'-\delta} \le \frac{1}{R^2} \frac{\|g\|_{1-d-d'}\, \|g'\|_{1-d'}}{(d+\delta)\delta} \tag{90}$$

The wanted inequality (85) follows by adding up (86) and (90). *Q.E.D.*

Proof of lemma 3. For $j \ge 1$ let $\delta = d/j$. By repeated application of lemma 12, we get the recursive chain of inequalities

$$\begin{aligned} \left\|\mathcal{L}^j_{\mathcal{X}} g\right\|_{1-d-d'} &\le \left(\frac{2}{e\varrho\sigma} + \frac{1}{R^2}\right) \frac{1}{j\delta^2} \|\mathcal{X}\|_{1-d'} \left\|\mathcal{L}^{j-1}_{\mathcal{X}} g\right\|_{1-d'-(j-1)\delta} \\ &\le \ldots \\ &\le \frac{j!}{e^2} \left(\frac{2e}{\varrho\sigma} + \frac{e^2}{R^2}\right)^j \frac{1}{(d^2)^j} \|\mathcal{X}\|^j_{1-d'} \|g\|_{1-d'}\ . \end{aligned} \tag{91}$$

In the last row, we used the trivial inequality $j^j \le j! e^{j-1}$, holding true for $j \ge 1$. *Q.E.D.*

Proof of lemma 4. We refer to the formal procedure in subsection 2.4.1. We simplify the notation in equations (28)–(35), by replacing the symbols ε , $\mathcal{D}^{(r;j)}_2$, $Z^{(r)}_j$, $g^{(r)}_1$, $\Omega^{(r-1)}_i$, $\mathcal{E}^{(r)}_j$, $\Psi^{(r)}_j$ with 1 , $\mathcal{X}_j$, Z_j , g' , Ξ_i , $\mathcal{E}_j$, Ψ_j , respectively. Recall that by construction $\mathcal{X}_j$, Z_j , and Ψ_j, for $j \ge 1$, belong to $\mathcal{P}_{2,0} \cap \widehat{\mathcal{P}}_{0,2,0}$, so that they depend only on the conjugate canonical coordinates $(z, \mathbf{i}\bar{z})$. A generic function $g'' \in \widehat{\mathcal{P}}_{0,2,0}$ is a quadratic polynomial and so its norm is well defined on any domain $\mathcal{D}_{\varrho,R,\sigma}$ and moreover $\|g''\|_{d''} = (d'')^2 \|g''\|_1$ for $d'' \in \mathbb{R}_+$. Using this property we rewrite the estimate of the Poisson bracket for functions in $\widehat{\mathcal{P}}_{0,2,0}$ as

$$\left\|\mathcal{L}_{\mathcal{X}_j} g''\right\|_{1-d'} \le 16\, \frac{\|\mathcal{X}_j\|_{1-d'}\, \|g''\|_{1-d'}}{(1-d')^2 R^2} \qquad \text{for } j \ge 1\ . \tag{92}$$

To this end just rewrite the estimate (90) putting $d = 0$ and $\delta = (1 - d')/2$. We stress that the upper bound in the latter inequality does not require any restriction of the domain. The solution of the homological equation (32) and (35) are estimated as

$$\|\mathcal{X}_j\|_{1-d'} \le \frac{\|\Psi_j\|_{1-d'}}{\Xi^*} \ , \qquad \|Z_j\|_{1-d'} \le \|\Psi_j\|_{1-d'} \qquad \text{for } j \ge 1 \ , \tag{93}$$

where we also used hypothesis (i). Putting inequalities (92)–(93) in (33) and in view of the definition (5) of $\mathcal{E}_j$, for $j \ge 1$, we get the recursive estimates

$$\begin{aligned} \|\Psi_j\|_{1-d'} &\le \|\mathcal{E}_{j-1}\, g'\|_{1-d'} \\ &\quad + \sum_{i=1}^{j-1} \left[\frac{16\, i\, \|\Psi_i\|_{1-d'}}{j(1-d')^2 \Xi^* R^2} \left(\|\Psi_{j-i}\|_{1-d'} + \|\mathcal{E}_{j-i-1}\, g'\|_{1-d'} \right) \right] \ , \\ \|\mathcal{E}_j\, g'\|_{1-d'} &\le \sum_{i=1}^{j} \left[\frac{16\, i\, \|\Psi_i\|_{1-d'}}{j(1-d')^2 \Xi^* R^2} \|\mathcal{E}_{j-i}\, g'\|_{1-d'} \right] \ , \end{aligned} \tag{94}$$

Starting with $\|\mathcal{E}_0\, g'\|_{1-d'} = \|g'\|_{1-d'}$ and proceeding by induction, we get

$$\max\left\{ \|\mathcal{E}_{j-1}\, g'\|_{1-d'} \ , \ \frac{1}{2}\, \|\Psi_j\|_{1-d'} \right\} \le \frac{\lambda_j}{j} \left(\frac{2^7\, \|g'\|_{1-d'}}{(1-d')^2 \Xi^* R^2} \right)^{j-1} \|g'\|_{1-d'} \ , \tag{95}$$

where $\{\lambda_j\}_{j\ge1}$ is the well-known Catalan sequence, i.e.,

$$\lambda_1 = 1 \ , \qquad \lambda_j = \sum_{i=1}^{j-1} \lambda_i \lambda_{j-i} \ . \tag{96}$$

Using the estimate $\lambda_j \le 4^{j-1}$ and inequalities (93) and (95), for $j \ge 1$ we obtain

$$\max\left\{ \Xi^*\, \|\mathcal{X}_j\|_{1-d'} \ , \ \|Z_j\|_{1-d'} \right\} \le \frac{2\, \|g'\|_{1-d'}}{j} \left(\frac{2^9\, \|g'\|_{1-d'}}{(1-d')^2 \Xi^* R^2} \right)^{j-1} \ . \tag{97}$$

The latter estimate, together with the smallness condition on $\varepsilon^{\star}_{\rm diag}$ in hypothesis (ii) allow us to prove that the Lie transform operator $T_{\mathcal{X}}$ properly defines a linear canonical transformation. The proof is a straightforward adaptation of proposition 4.3 in Giorgilli (2003), and this concludes the proof of the lemma. *Q.E.D.*

A.2 On the lists of indices

We report here the proofs of lemmas 5–7.

Proof of lemma 5. The claim (i) is a trivial consequence of the definition.
(ii) For each fixed value of $s > 0$ and $1 \le k \le \lfloor s/2 \rfloor$, we have to determine the cardinality of the set $\mathcal{M}_{k,s} = \{m \in \mathbb{N} : \ 2 \le m \le s \, , \ \lfloor s/m \rfloor = k\}$. For this purpose, we use the obvious inequalities

$$\left\lfloor \frac{s}{\lfloor s/k \rfloor} \right\rfloor \ge k \quad \text{and} \quad \left\lfloor \frac{s}{\lfloor s/k \rfloor + 1} \right\rfloor < k \ .$$

After having rewritten the same relations with $k+1$ in place of k, one immediately realizes that an index $m \in \mathcal{M}_{k,s}$ *if and only if* $m \le \lfloor s/k \rfloor$ *and* $m \ge \lfloor s/(k+1) \rfloor + 1$, therefore $\#\mathcal{M}_{k,s} = \left\lfloor \frac{s}{k} \right\rfloor - \left\lfloor \frac{s}{k+1} \right\rfloor$.
(iii) Since $r \le s$, the definition in (49) implies that neither $\{r\} \cup I_r^* \cup I_s^*$ nor I_{r+s}^* can include any index exceeding $\lfloor (r+s)/2 \rfloor$. Thus, let us define some finite sequences of non-negative integers as follows:

$$
\begin{aligned}
R_k &= \#\bigl\{j \in I_r^* \ : \ j \le k\bigr\} \ , & S_k &= \#\bigl\{j \in I_s^* \ : \ j \le k\bigr\} \ , \\
M_k &= \#\bigl\{j \in \{r\} \cup I_r^* \cup I_s^* \ : \ j \le k\bigr\} \ , & N_k &= \#\bigl\{j \in I_{r+s}^* \ : \ j \le k\bigr\} \ ,
\end{aligned}
$$

where the integer index k ranges in $\bigl[1,\, \lfloor (r+s)/2 \rfloor\bigr]$. When $k < r$, the property (ii) of the present lemma allows us to write

$$
R_k = r - \left\lfloor \frac{r}{k+1} \right\rfloor \ , \quad S_k = s - \left\lfloor \frac{s}{k+1} \right\rfloor \ , \quad N_k = r + s - \left\lfloor \frac{r+s}{k+1} \right\rfloor \ ;
$$

using the elementary estimate $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x + y \rfloor$, from the equations above it follows that $M_k \ge N_k$ for $1 \le k < r$. In the remaining cases, i.e., when $r \le k \le \lfloor (r+s)/2 \rfloor$, we have that

$$
R_k = r - 1 \ , \quad S_k = s - \left\lfloor \frac{s}{k+1} \right\rfloor \ , \quad N_k = r + s - \left\lfloor \frac{r+s}{k+1} \right\rfloor \ ;
$$

therefore, $M_k = 1 + R_k + S_k \ge N_k$. Since we have just shown that $M_k \ge N_k$ for $1 \le k \le \lfloor (r+s)/2 \rfloor$, it is now an easy matter to complete the proof. Let us first imagine to have reordered both the lists of indices $\{r\} \cup I_r^* \cup I_s^*$ and I_{r+s}^* in increasing order; moreover, let us recall that $\#\bigl(\{r\} \cup I_r^* \cup I_s^*\bigr) = \# I_{r+s}^* = r + s - 1$, in view of the definition in (49). Thus, since $M_1 \ge N_1$, every element equal to 1 in $\{r\} \cup I_r^* \cup I_s^*$ has a corresponding index in I_{r+s}^* the value of which is at least 1. Analogously, since $M_2 \ge N_2$, every index 2 in $\{r\} \cup I_r^* \cup I_s^*$ has a corresponding index in I_{r+s}^* which is at least 2, and so on up to $k = \lfloor (r+s)/2 \rfloor$. This allows us to conclude that $\{r\} \cup I_r^* \cup I_s^* \triangleleft I_{r+s}^*$.
Q.E.D.

Proof of lemma 6. The points (i) and (ii) of the statement immediately follow from the definition of $\mathcal{J}_{r,s}$.
Concerning (iii), we first remark that $\#\bigl(\{\min\{r,s\}\} \cup I \cup I'\bigr) = 1 + \#(I) + \#(I') = r+s-1$. Moreover, after having recalled the definition in (48), it is easy to verify that $0 \le j \le \min\{r, \lfloor (r+s)/2 \rfloor\}$, for $j \in \bigl(\{\min\{r,s\}\} \cup I \cup I'\bigr)$. In order to complete the proof, we check that the selection rule S is satisfied. We first remark that $\bigl(\{\min\{r,s\}\} \cup I \cup I'\bigr) \triangleleft \bigl(\{\min\{r,s\}\} \cup I_r^* \cup I_s^*\bigr)$, because $I \in \mathcal{J}_{r-1,r}$ and $I' \in \mathcal{J}_{r,s}$. Therefore, property (iii) of lemma 5 allows us to conclude that $\bigl(\{\min\{r,s\}\} \cup I \cup I'\bigr) \triangleleft I_{r+s}^*$. *Q.E.D.*

Proof of lemma 7. The point (i) of the present lemma immediately follows from property (ii) of lemma 6. In fact, we can write

$$
T_{r-1,s} = \max_{I \in \mathcal{J}_{r-1,s}} \prod_{j \in I\,,\, j \ge 1} \frac{1}{a_j \delta_j^2} \le \max_{I \in \mathcal{J}_{r,s}} \prod_{j \in I\,,\, j \ge 1} \frac{1}{a_j \delta_j^2} \ ,
$$

because the maximum is evaluated over a bigger list of indices. Moreover, the equation $T_{r',s} = T_{s,s}$ holds true when $r' > s$, as a trivial consequence of the definition in (52)

and property (i) of lemma 6.
Concerning the point (ii), we can evaluate $T_{r-1,r}T_{r,s}/(a_m\delta_m^2)$, where $m=\min\{r,s\}$, as follows:

$$
\begin{aligned}
\frac{1}{a_m\delta_m^2}T_{r-1,r}T_{r,s} &= \frac{1}{a_m\delta_m^2}\left(\max_{I\in\mathcal{J}_{r-1,r}}\prod_{j\in I\,,\,j\ge1}\frac{1}{a_j\delta_j^2}\right)\left(\max_{I'\in\mathcal{J}_{r,s}}\prod_{j'\in I'\,,\,j'\ge1}\frac{1}{a_{j'}\delta_{j'}^2}\right)\\
&= \max_{I\in\mathcal{J}_{r-1,r}\,,\,I'\in\mathcal{J}_{r,s}}\prod_{j\in(\{m\}\cup I\cup I')\,,\,j\ge1}\frac{1}{a_j\delta_j^2}\\
&\le \max_{J\in\mathcal{J}_{r,r+s}}\prod_{j\in J\,,\,j\ge1}\frac{1}{a_j\delta_j^2}=T_{r,r+s}\ ,
\end{aligned}
$$

where the inequality above holds true in view of property (iii) of lemma 6. *Q.E.D.*

A.3 On the main estimates of the analytic part

Proof of lemma 8. We prove by induction the estimates (58), from which the estimates (57) readily follow.

For $r=1$, the upper bounds at point (e') of lemma 2 can be written as

$$
\|f_\ell^{(0,s)}\|_{1-d_0}\le\frac{\bar{E}M^{3s-3}}{2^\ell}\,T_{0,s}^3\,\nu_{0,s}\exp(s\zeta_0)\qquad\text{for } \ell\ge0\,,\ s\ge1\ , \tag{98}
$$

where we used the definitions in formulæ (51)–(55). We stress that no divisor $a_m\delta_m^2$ appears in the estimate. Checking this inequality and performing the step to $r-1$ to r requires essentially the same calculations, so we do both things together.

For $r-1$ we write the estimates (58) as

(99)

$$
\begin{aligned}
&\|f_\ell^{(r-1,s)}\|_{1-d_{r-1}}\le\frac{\bar{E}M^{3s-3+\ell}}{2^\ell}\,\frac{T_{r-1,s}^3}{\left(a_{r-1}\delta_{r-1}^2\right)^\ell}\,\nu_{r-1,s}\exp(s\zeta_{r-1}) && \text{for } 0\le\ell\le2\,,\ s\ge r\,,\\
&\|f_\ell^{(r-1,0)}\|_{1-d_{r-1}}\le\frac{\bar{E}}{2^\ell}\nu_{r-1,0} && \text{for } \ell\ge3\,,\\
&\|f_\ell^{(r-1,s)}\|_{1-d_{r-1}}\le\frac{\bar{E}M^{3s}}{2^\ell}\left(\frac{T_{r-1,s}}{a_m\delta_m^2}\right)^3\nu_{r-1,s}\exp(s\zeta_{r-1}) && \text{for } \ell\ge3\,,\ s\ge1\,,\\
& && m=\min\{r-1,s\}\,.
\end{aligned}
$$

We check the estimate for r. Let us deal separately with the easy case $s=0$. The estimate in the second row of (58) follows from the upper bounds at point (e') of lemma 2, because $f_\ell^{(r-1,0)}=f_\ell^{(\mathrm{I};r,0)}=f_\ell^{(\mathrm{II};r,0)}=f_\ell^{(\mathrm{III};r,0)}=f_\ell^{(r,0)}$, $d_r\ge0$ and $\nu_{r-1,0}=\nu_{r,0}=1$ for $r\ge1$, $\ell\ge3$, as one can easily verify by looking at the recursive formulæ (17), (22), (27), (40), (51) and (55).

We follow the different stages of the r-th normalization step, starting with the first one. By (13)–(15) and using inequalities (10) and (99), we obtain

$$
\|\chi_0^{(r)}\|_{1-d_{r-1}}\le\frac{1}{a_r}\|f_0^{(r-1,r)}\|_{1-d_{r-1}}\le M^{3r-3}\,\frac{T_{r-1,r}^3}{a_r}\,\nu_{r-1,r}\exp(r\zeta_{r-1})\ .
$$

The upper bound for $\chi_0^{(r)}$ in (57) immediately follows from (53) and the inequality above. For $r=1$, we must replace the estimate of $f_0^{(r-1,r)}$ in (99) with that of $f_0^{(0,1)}$ in (98).

The functions appearing in the expansion (16) of the Hamiltonian $H^{(\mathrm{I};r)}$ are estimated by

$$\begin{aligned} &\|f_\ell^{(\mathrm{I};r,s)}\|_{1-d_{r-1}-\delta_r} \le \frac{\bar{E}M^{3s-3+\ell}}{2^\ell}\,\frac{T_{r,s}^3}{(a_r\delta_r^2)^\ell}\,\nu_{r,s}^{(\mathrm{I})}\exp(s\zeta_{r-1}) \qquad \text{for } 0\le\ell\le 2\,,\ s\ge r\,,\\ &\|f_\ell^{(\mathrm{I};r,s)}\|_{1-d_{r-1}-\delta_r} \le \frac{\bar{E}M^{3s}}{2^\ell}\left(\frac{T_{r,s}}{a_m\delta_m^2}\right)^3\nu_{r,s}^{(\mathrm{I})}\exp(s\zeta_{r-1}) \qquad \text{for } \ell\ge 3\,,\ s\ge 1\,,\ m=\min\{r,s\}\,. \end{aligned} \tag{100}$$

Here we omitted the inequality $\|f_\ell^{(\mathrm{I};r,0)}\|_{1-d_{r-1}-\delta_r}\le \bar{E}/2^\ell$ for $\ell\ge 3$, that has been already proved verifying the estimates in the second row of (58). In order to justify the inequalities (100), we focus on the recursive definitions in (17). For $\ell=0$ and $s=r$ there is nothing to do. When $\ell=0$ and $r<s=r+m<2r$, starting from the corresponding estimate in (99), we can write

$$\begin{aligned} \|f_0^{(\mathrm{I};r,r+m)}\|_{1-d_{r-1}-\delta_r} &\le \bar{E}M^{3(r+m)-3}\,T_{r-1,r+m}^3\,\nu_{r-1,r+m}\exp\big((r+m)\zeta_{r-1}\big)\\ &\le \bar{E}M^{3(r+m)-3}\,T_{r,r+m}^3\,\nu_{r,r+m}^{(\mathrm{I})}\exp\big((r+m)\zeta_{r-1}\big)\ , \end{aligned} \tag{101}$$

where we used property (i) of lemma 7 and the obvious inequality $\nu_{r-1,r+m}\le\nu_{r,r+m}^{(\mathrm{I})}$.

Most of the work to verify the estimates (100) concerns the third definition in (17). There, it is convenient to separately consider the cases where s is a multiple of the normalization step r. Moreover, it is useful to introduce the sequence of non-negative integer numbers $\{w_\ell\}_{\ell\ge 0}$ defined as

$$w_\ell = 3-\ell \quad \text{for } 0\le\ell\le 2\ , \qquad w_\ell = 0 \quad \text{for } \ell\ge 3\ . \tag{102}$$

Thus, for $s=mr$ with $m\ge 2$ when $\ell=0$ or $m\ge 1$ when $\ell\ge 1$, one has

$$\begin{aligned} \|f_\ell^{(\mathrm{I};r,mr)}\|_{1-d_{r-1}-\delta_r} \le{}& \frac{\bar{E}M^{3mr-w_\ell}}{2^\ell}\exp(mr\zeta_{r-1})\Bigg\{M^{w_\ell-2m}\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^m \nu_{r-1,r}^m\nu_{r-1,0}\\ &+\sum_{j=0}^{m-1}\left[M^{w_\ell-2j-w_{\ell+2j}}\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^j\frac{T_{r-1,(m-j)r}^3}{(a_r\delta_r^2)^{3-w_{\ell+2j}}}\,\nu_{r-1,r}^j\nu_{r-1,(m-j)r}\right]\Bigg\}\\ \le{}& \frac{\bar{E}M^{3mr-w_\ell}}{2^\ell}\nu_{r,mr}^{(\mathrm{I})}\exp(mr\zeta_{r-1})\\ &\cdot\max\left\{\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^m,\ \max_{0\le j\le m-1}\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^j\frac{T_{r-1,(m-j)r}^3}{(a_r\delta_r^2)^{3-w_{\ell+2j}}}\right\}\\ \le{}& \frac{\bar{E}M^{3mr-w_\ell}}{2^\ell}\,\frac{T_{r,mr}^3}{(a_r\delta_r^2)^{3-w_\ell}}\,\nu_{r,mr}^{(\mathrm{I})}\exp(mr\zeta_{r-1})\ , \end{aligned} \tag{103}$$

where we used lemma 4, the estimate for $\chi_0^{(1)}$ in formula (57), the inductive inequalities in (99), the fact that $\{a_r\delta_r^2\}_{r\ge1}$ is a non-increasing sequence and $M\ge1$ (recall the definitions in (10), (51) and (53)), some elementary properties[†] of the sequence $\{w_\ell\}_{\ell\ge0}$, the definition of $\{\nu_{r,s}^{(\mathrm{I})}\}_{r\ge1\,,\,s\ge0}$ in (55) and the inequality

$$\max\left\{\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^m\,,\,\max_{0\le j\le m-1}\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^j\frac{T_{r-1,(m-j)r}^3}{(a_r\delta_r^2)^{3-w_{\ell+2j}}}\right\}\le\frac{T_{r,mr}^3}{(a_r\delta_r^2)^{3-w_\ell}}\tag{104}$$

holding true for $r\ge1$, for $\ell=0$, $m\ge2$ or $\ell\ge1$, $m\ge1$. We remark that the inequality (103) holds true also for $r=1$; just replace the first inductive estimate (99) with that in (98) and use $a_1\delta_1^2<1$. Thus, in order to complete the justification of (103), we should check the inequality (104). It is convenient to distinguish some sub-cases.

(i) For $m=1$ and so $\ell\ge1$, inequality (104) is rather obvious, because $T_{r-1,r}\le T_{r,r}$ (recall property (i) of lemma 7), $a_r\delta_r^2<1$ (see the definitions in (10) and (51)) and $3-w_\ell\ge1$ for $\ell\ge1$.

(ii) For $m\ge2$ and so $\ell\ge0$, it is convenient to verify that each term in the l.h.s. of (104) is bounded by the r.h.s., as we do in the following points (ii.a) and (ii.b).

(ii.a) Concerning the first term in (104), we can write

$$\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^m\le\left[\left(\frac{T_{r-1,r}}{a_r\delta_r^2}\right)^{m-1}T_{r,r}\right]^3\le T_{r,mr}^3\le\frac{T_{r,mr}^3}{(a_r\delta_r^2)^{3-w_\ell}}\ ,$$

where we used lemma 7 and the elementary inequalities $3m-3\ge m$ for $m\ge2$, $a_r\delta_r^2<1$ and $3-w_\ell\ge0$ for $\ell\ge0$.

(ii.b) For $j=0,\,\ldots,\,m-1$, we estimate the remaining terms of the l.h.s. of (104) as

$$\begin{aligned}\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^j\frac{T_{r-1,(m-j)r}^3}{(a_r\delta_r^2)^{3-w_{\ell+2j}}}&\le\left[\left(\frac{T_{r-1,r}}{a_r\delta_r^2}\right)^jT_{r,(m-j)r}\right]^3\frac{1}{(a_r\delta_r^2)^{3-2j-w_{\ell+2j}}}\\&\le\frac{T_{r,mr}^3}{(a_r\delta_r^2)^{3-w_\ell}}\ ,\end{aligned}$$

using again lemma 7, $a_r\delta_r^2<1$ and $w_\ell\le2j+w_{\ell+2j}$ for $j\ge0$.

This completes the justification of (104), and then also of (103).

It remains to verify the estimates (100) when s is not a multiple of the normalization step r. The case with $0<s<r$ is trivial and can be treated in a way similar to (101). For $s>r$ let us put $m=\lfloor s/r\rfloor$ and $s=mr+i$; thus, we focus on the third definition in (17) with $0<i<r$, for $m\ge2$, $\ell=0$ or $m\ge1$, $\ell\ge1$, for which we can write the following chain of inequalities:

[†] Actually, we used two elementary inequalities: $w_\ell\le2m$ for $\ell=0$ and $m\ge2$ or $\ell\ge1$ and $m\ge1$; $w_\ell\le2j+w_{\ell+2j}$ for $j\ge0$. Both immediately follow from the definition in (102).

(105)
$$
\begin{aligned}
\|f_{\ell}^{(\mathrm{I};r,s)}\|_{1-d_{r-1}-\delta_r} \le \frac{\bar{E}M^{3s-w_\ell}}{2^\ell}\nu_{r,s}^{(\mathrm{I})}\exp(s\zeta_{r-1})\max\Biggl\{ &\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^m \frac{T_{r-1,i}^3}{\bigl(a_i\delta_i^2\bigr)^{3-w_{\ell+2m}}}\,,\\
&\max_{0\le j\le m-1}\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^j \frac{T_{r-1,s-jr}^3}{\bigl(a_r\delta_r^2\bigr)^{3-w_{\ell+2j}}}\Biggr\}\\
\le \frac{\bar{E}M^{3s-w_\ell}}{2^\ell}\,\frac{T_{r,s}^3}{\bigl(a_r\delta_r^2\bigr)^{3-w_\ell}}\,\nu_{r,s}^{(\mathrm{I})}\exp(s\zeta_{r-1})\ ,&
\end{aligned}
$$

where we proceeded in a similar way as for (103), replacing (104) by

$$
\begin{aligned}
\max\Biggl\{ &\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^m \frac{T_{r-1,i}^3}{\bigl(a_i\delta_i^2\bigr)^{3-w_{\ell+2m}}}\,,\\
&\max_{0\le j\le m-1}\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^j \frac{T_{r-1,s-jr}^3}{\bigl(a_r\delta_r^2\bigr)^{3-w_{\ell+2j}}}\Biggr\} \le \frac{T_{r,s}^3}{\bigl(a_r\delta_r^2\bigr)^{3-w_\ell}}\ .
\end{aligned}
\tag{106}
$$

The latter holds true for $\ell=0$, $m\ge 2$ or $\ell\ge 1$, $m\ge 1$. Therefore, in order to complete the justification of (105), we should verify the inequality (106). To this purpose, it is convenient to work out the verification term-by-term. First, one checks that

$$
\begin{aligned}
\left(\frac{T_{r-1,r}^3}{a_r\delta_r^2}\right)^m \frac{T_{r-1,i}^3}{\bigl(a_i\delta_i^2\bigr)^{3-w_{\ell+2m}}} &\le \left[T_{r-1,r}\left(\frac{T_{r-1,r}}{a_r\delta_r^2}\right)^{m-1}\frac{T_{r,i}}{a_i\delta_i^2}\right]^3 \frac{1}{\bigl(a_r\delta_r^2\bigr)^{3-2m}}\\
&\le \frac{T_{r,s}^3}{\bigl(a_r\delta_r^2\bigr)^{3-w_\ell}}\,\frac{1}{\bigl(a_r\delta_r^2\bigr)^{w_\ell-2m}} \le \frac{T_{r,s}^3}{\bigl(a_r\delta_r^2\bigr)^{3-w_\ell}}\ ,
\end{aligned}
\tag{107}
$$

using lemma 7, $a_r\delta_r^2<1$ and $w_\ell\le 2m$, both for $\ell=0$, $m\ge 2$ and $\ell\ge 1$, $m\ge 1$. For $j=0,\ldots,m-1$, the remaining terms of the l.h.s. of (106) are estimated in a way similar to (107). This concludes the justification of inequalities (106), so that also (105) and (100) are verified.

We come now to the second stage of the r-th normalization step. Recalling formulæ (18)–(20), the upper bound on the generating function $\chi_1^{(r)}$ in (57) is easily justified by using inequalities (10) and (100) and the definition in (53). We note that the upper bounds in (100) include the case $r=1$ which starts the induction process. Concerning the functions $f_\ell^{(\mathrm{II};r,s)}$ appearing in the expansion of $H^{(\mathrm{II};r)}$, we can prove estimates analogous to those in (100), just replacing the upper index I with II. This can be verified, by starting from the recursive definitions in (22) and by patiently dealing with many different cases, as we have done for (100).

Concerning the first part of the third stage, the upper bound on the generating function $\chi_2^{(r)}$ in (57) is proved in a way similar to that for $\chi_0^{(r)}$. By patiently estimating all terms appearing in the recursive definitions in (27), we provide the upper bounds (108)

$$\|f_\ell^{(\mathrm{III};r,s)}\|_{1-d_{r-1}-3\delta_r} \le \frac{\bar{E}M^{3s-3+\ell}}{2^\ell}\,\frac{T_{r,s}^3}{(a_r\delta_r^2)^\ell}\,\nu_{r,s}\exp(s\zeta_{r-1}) \quad \text{for } 0\le\ell\le 2\,,\ s\ge r\,,$$

$$\|f_\ell^{(\mathrm{III};r,s)}\|_{1-d_{r-1}-3\delta_r} \le \frac{\bar{E}M^{3s}}{2^\ell}\left(\frac{T_{r,s}}{a_m\delta_m^2}\right)^3\nu_{r,s}\exp(s\zeta_{r-1}) \quad \text{for } \ell\ge 3\,,\ s\ge 1\,, \ \text{with } m=\min\{r,s\}\,.$$

We omit all the tedious calculations necessary to fully justify (108) because they are essentially a repetition of the arguments for the first stage.

We finally come to the diagonalization of the quadratic part of the normal form of order $\mathcal{O}(\varepsilon^r)$. For $r=1$, since $f_2^{(\mathrm{III};1,1)}=0$ (see (42)), then equations (28)–(35) imply $\mathcal{D}_2^{(1;j)}=0$ for $j\ge 1$; therefore, $f_\ell^{(\mathrm{III};1,s)}=f_\ell^{(1,s)}$ for $0\le\ell\le 2$, $s>1$ or $\ell\ge 3$, $s\ge 0$, in view of the definitions in (40). Hence, the upper bounds in (58) immediately follow from those in (108) and from the fact that $\zeta_0=\zeta_1=0$ (see (54)). For the generic case $r\ge 2$, comparing formulæ (11) and (28)–(31) with the hypotheses of lemma 4, one realizes that the smallness condition

$$\varepsilon_{\mathrm{diag}}^{\star}=\varepsilon^{r-1}\left(\frac{2e^2}{\delta_r^2}+\frac{2^9}{(1-d_{r-1}-3\delta_r)^2}\right)\frac{\|f_2^{(\mathrm{III};r,r)}\|_{1-d_{r-1}-\delta_r}}{b_r\,R^2}\le\frac{1}{2}\ ,$$

must be satisfied. Actually, the stronger inequality $\varepsilon_{\mathrm{diag}}^{\star}\le 2^{-(r+6)}$ holds true, as it is checked by using (56), (108) and the definitions in (10), (51), (53). The inequality bounding the effect of the operator $\mathcal{E}_j^{(r)}$ in formula (57) is nothing but the first estimate in (47) with $\varepsilon_{\mathrm{diag}}^{\star}=2^{-(r+6)}$. Applying lemma 4 to the third equation in (40), we obtain

$$\|f_\ell^{(r,s)}\|_{1-d_r}\le\|f_\ell^{(\mathrm{III};r,s)}\|_{1-d_{r-1}-\delta_r}\,\exp\left(\frac{2^{-(r+6)}}{1-2^{-(r+6)}}\right)\ ,$$

for $0\le\ell\le 2$, $s>r\ge 2$ or $\ell\ge 3$, $s\ge 1$. Using the estimates in (108), the inequality above and the definition in (54), the justification of (58) is completed. Recall that we have already considered the case $s=0$ at the beginning of the proof.

In order to complete the proof of the lemma, it remains to evaluate the change of the frequencies induced by the r-th normalization step. Starting from equation (38), using (51), (108) and the first Cauchy inequality in (84), we obtain

$$\max_{1\le j\le n_1}|\omega_j^{(r)}-\omega_j^{(r-1)}|\le\varepsilon^r\frac{\bar{E}}{3\,\varrho}M^{3r-1}\frac{T_{r,s}^3}{(a_r\delta_r^2)^2}\nu_{r,r}\exp(r\zeta_{r-1})\ . \tag{109}$$

Analogously, starting from (39), using (51), (54), (108), the second estimate (again with $\varepsilon_{\mathrm{diag}}^{\star}=2^{-(r+6)}$) in (47) and twice the second Cauchy inequality in (84), we get

$$\max_{1\le j\le n_2}|\Omega_j^{(r)}-\Omega_j^{(r-1)}|\le\varepsilon^{r-1}\frac{8\,\bar{E}}{9\,R^2}M^{3r-1}\frac{T_{r,s}^3}{(a_r\delta_r^2)^2}\nu_{r,r}\exp(r\zeta_r)\ . \tag{110}$$

By using inequality $a_r\delta_r^2<1$ and the definition in (53), one can easily verify that both the estimates (109) and (110) are gathered in (59). *Q.E.D.*

A.4 Estimates of some special numerical sequences

We report here the proofs of lemmas 9–10.

Proof of lemma 9. Since $a_s\delta_s^2 < 1$ (see (10) and (51)), it is enough to prove the second part of the inequality stated in the lemma, i.e., $T_{r,s}/(a_s\delta_s^2) \le A^s e^{s\Gamma}$ for $r \ge 1$, $s \ge 1$. Starting from the definition in (52), using properties (i) and (ii) of lemma 6, the selection rule S and the decreasing character of the sequence $\{a_s\delta_s^2\}_{s\ge1}$, we get

$$\frac{T_{r,s}}{a_s\delta_s^2} = \frac{1}{a_s\delta_s^2}\max_{I\in\mathcal{J}_{s,s}}\prod_{j\in I\,,\,j\ge1}\frac{1}{a_j\delta_j^2} \le \prod_{j\in\{s\}\cup I_s^*\,,\,j\ge1}\frac{1}{a_j\delta_j^2}\ .$$

Starting from the estimate above, we have

$$\begin{aligned}
\log\frac{T_{r,s}}{a_s\delta_s^2} &\le -\log(a_s\delta_s^2) - \sum_{k=1}^{\lfloor s/2\rfloor}\left(\left\lfloor\frac{s}{k}\right\rfloor - \left\lfloor\frac{s}{k+1}\right\rfloor\right)\log(a_k\delta_k^2)\\
&\le -\sum_{k=1}^{s}\left(\left\lfloor\frac{s}{k}\right\rfloor - \left\lfloor\frac{s}{k+1}\right\rfloor\right)(\log a_k + 2\log\delta_k) \qquad (111)\\
&\le -s\sum_{k\ge1}\frac{\log a_k + 2\log\delta_k}{k(k+1)} = s\left(\Gamma - \sum_{k\ge1}\frac{2\log\delta_k}{k(k+1)}\right)\ ,
\end{aligned}$$

where we used properties (i) and (ii) of lemma 5, the fact that the sequence $\{a_s\delta_s^2\}_{s\ge1}$ is decreasing and the condition $\boldsymbol{\tau}$ in (60). Thus, using (51), one gets

$$\begin{aligned}
-\sum_{k\ge1}\frac{2\log\delta_k}{k(k+1)} &= 2\sum_{k\ge1}\frac{\log\frac{8\pi^2}{3} + 2\log k}{k(k+1)}\\
&< 2\log\frac{8\pi^2}{3} + 4\left(\frac{\log 2}{6} + \int_2^\infty\frac{\log x\,\mathrm{d}x}{x^2}\right)\\
&< 15\log 2\ ,
\end{aligned}$$

where the relation $\sum_{k\ge1} 1/[k(k+1)] = 1$ is used. Putting the estimate above into (111), we conclude the proof. *Q.E.D.*

Proof of lemma 10. First, it is convenient to replace the definition in (55) for the sequence $\{\nu_{r,s}\}_{r\ge0\,,\,s\ge0}$ with a closed formula, avoiding to introduce $\{\nu^{(\mathrm{I})}_{r,s}\}_{r\ge1\,,\,s\ge0}$ and $\{\nu^{(\mathrm{II})}_{r,s}\}_{r\ge1\,,\,s\ge0}$. Thus, let us remove the symbol $\nu^{(\mathrm{II})}$, by writing

$$\begin{aligned}
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r\rfloor}\left(\nu^{(\mathrm{I})}_{r,r} + \nu^{(\mathrm{I})}_{r,r}\nu^{(\mathrm{I})}_{r,0}\right)^j\sum_{i=0}^{\lfloor s/r\rfloor-j}(\nu^{(\mathrm{I})}_{r,r})^i\nu^{(\mathrm{I})}_{r,s-(i+j)r} = \sum_{j=0}^{\lfloor s/r\rfloor}(2\nu^{(\mathrm{I})}_{r,r})^j\sum_{i=j}^{\lfloor s/r\rfloor}(\nu^{(\mathrm{I})}_{r,r})^{i-j}\nu^{(\mathrm{I})}_{r,s-ir}\\
&= \sum_{i=0}^{\lfloor s/r\rfloor}(\nu^{(\mathrm{I})}_{r,r})^i\nu^{(\mathrm{I})}_{r,s-ir}\sum_{j=0}^{i}2^j = \sum_{i=0}^{\lfloor s/r\rfloor}\left(2^{i+1}-1\right)(\nu^{(\mathrm{I})}_{r,r})^i\nu^{(\mathrm{I})}_{r,s-ir}\ .
\end{aligned}$$

Analogously, we eliminate the occurrence of $\nu^{(\mathrm{I})}$, by writing

$$
\begin{aligned}
\nu_{r,s} &= \sum_{j=0}^{\lfloor s/r \rfloor} 2^j \left(2^{j+1}-1\right) \nu_{r-1,r}^{j} \sum_{i=j}^{\lfloor s/r \rfloor} \nu_{r-1,r}^{i-j} \nu_{r-1,s-ir} \\
&= \sum_{i=0}^{\lfloor s/r \rfloor} \nu_{r-1,r}^{i} \nu_{r-1,s-jr} \sum_{j=0}^{i} \left(2^{2j+1}-2^j\right) = \sum_{i=0}^{\lfloor s/r \rfloor} \theta_i \nu_{r-1,r}^{i} \nu_{r-1,s-ir}\ ,
\end{aligned}
\tag{112}
$$

where we introduced the shorthand notation

$$
\theta_i = \sum_{j=0}^{i} \left(2^{2j+1}-2^j\right) = \frac{2}{3}\left(2^{2(i+1)}-1\right) - 2^{i+1} + 1\ .
$$

From the definition above we immediately verify that

$$
\theta_0 = 1\ , \qquad \theta_1 = 7\ , \qquad \frac{1}{3} 2^{2(j+1)} \le \theta_j \le \frac{2}{3} 2^{2(j+1)} \quad \text{for } j \ge 1\ .
\tag{113}
$$

Using such basic properties of the sequence $\{\theta_j\}_{j\ge 0}$, one gets

$$
\theta_{j+1} \le 8\theta_j \qquad \text{for } j \ge 0\ .
\tag{114}
$$

Let us recall that combining (55) with the recursive equation (112), we can provide a new and more compact definition of the sequence $\{\nu_{r,s}\}_{r\ge 0\,,\, s\ge 0}$, namely

$$
\nu_{0,s} = 1 \quad \text{for } s \ge 0\ , \qquad \nu_{r,s} = \sum_{j=0}^{\lfloor s/r \rfloor} \theta_j \nu_{r-1,r}^{j} \nu_{r-1,s-jr} \quad \text{for } r \ge 1\,,\ s \ge 0\ .
\tag{115}
$$

As an immediate consequence, we remark that

$$
\nu_{0,s} \le \nu_{1,s} \le \ldots \le \nu_{s,s} = \nu_{s+1,s} = \ldots\ .
\tag{116}
$$

Moreover, since $\nu_{r,r} = \theta_0 \nu_{r-1,r} + \theta_1 \nu_{r-1,r}$, $\theta_0 = 1$ and $\theta_1 = 7$ we have

$$
\nu_{r,r} = 8\nu_{r-1,r} \qquad \text{for } r \ge 1\ .
\tag{117}
$$

The following chains of inequalities allow us to justify other useful properties of the sequence $\{\nu_{r,s}\}_{r\ge 0\,,\, s\ge 0}$. Indeed, starting from (115), for $r \ge 2\,,\ s > r$, we can write

$$
\begin{aligned}
\nu_{r,s} &= \nu_{r-1,s} + \nu_{r-1,r} \sum_{j=0}^{\lfloor s/r \rfloor - 1} \theta_{j+1} \nu_{r-1,r}^{j} \nu_{r-1,s-r-jr} \\
&\le \nu_{r-1,s} + 8\nu_{r-1,r} \sum_{j=0}^{\lfloor s/r \rfloor - 1} \theta_j \nu_{r-1,r}^{j} \nu_{r-1,s-r-jr} \\
&\le \nu_{r-1,s} + 8\nu_{r-1,r}\nu_{r-1,s-r} \le \nu_{r-1,s} + \nu_{r,r}\nu_{s-r,s-r}\ ,
\end{aligned}
\tag{118}
$$

where we used inequality (114) and equation (117). In a similar way, for $r = 1$ (and, again, $s > r$) we provide a more accurate estimate, namely

$$(119)\qquad \begin{aligned} \nu_{1,s} &= \nu_{0,s} + \nu_{0,1}\sum_{j=0}^{s-1}\theta_{j+1}\nu_{0,1}^{j}\nu_{0,s-1-j} \\ &\le (1+\theta_1)\nu_{0,s-1} + 8\sum_{j=1}^{s-1}\theta_j\nu_{0,1}^{j}\nu_{0,s-1-j} \le 8\nu_{0,s-1} \le \nu_{s-1,s-1}\ , \end{aligned}$$

where some particular values of the sequences involved have been inserted (namely, $\theta_1 = 7$ and $\nu_{0,s} = 1$ for $s \ge 0$). We aim to control all the sequence $\{\nu_{r,s}\}_{r\ge 0\,,\,s\ge 0}$; thus, looking at formula (116) one immediately realizes that it is enough to provide an upper bound on the diagonal elements, for which we can write

$$(120)\qquad \begin{aligned} \nu_{r,r} &= 8\nu_{r-1,r} \le 8\nu_{r-2,r} + 8\nu_{r-1,r-1}\nu_{1,1} \le \ldots \\ &\le 8\nu_{1,r} + 8\left(\nu_{2,2}\nu_{r-2,r-2} + \ldots + \nu_{r-1,r-1}\nu_{1,1}\right) \le 8\sum_{j=1}^{r-1}\nu_{j,j}\nu_{r-j,r-j}\ , \end{aligned}$$

for $r \ge 2$, where we orderly used all the properties described by formulæ (117)–(119).

Using the fact that $\nu_{1,1} = 8$ (see (115) and (117)) and the recursive inequality for the diagonal elements in (120), by induction one can easily verify that

$$\nu_{r,r} \le \frac{64^r}{8}\lambda_r \qquad \text{for } r \ge 1\ ,$$

where $\{\lambda_r\}_{r\ge 1}$ is the Catalan sequence, whose definition is recalled in (96). By combining the information provided by formula (116), the inequality above and the well known upper bound $\lambda_r \le 4^{r-1}$, one can fully justify the statement. *Q.E.D.*

A.5 On the geometry of the resonant regions

Proof of proposition 2. The proof proceeds by induction. However, in order to highlight the key points, we describe in detail the first two steps. By hypotheses, $\Omega^{(0)}(\omega)$ is an analytic function on the complex extended domain $\mathcal{W}_{h_0}^{(0)}$ and its Jacobian is uniformly bounded in $\mathcal{W}_{h_0}^{(0)}$, namely $\left|\partial\Omega^{(0)}/\partial\omega\right|_{\infty;\mathcal{W}_{h_0}^{(0)}} \le J_0$, where $\left|\,\cdot\,\right|_{\infty;\mathcal{W}_{h_0}^{(0)}}$ is defined in (45). Thus, starting from the inequality at point (a') of lemma 2, if

$$(121)\qquad h_0 \le \min\left\{\frac{1}{K+2J_0\varepsilon}\frac{\gamma}{2K^\tau}\,,\ \frac{\bar b}{4J_0}\right\}\ ,$$

then the non-resonance conditions (10) and (11) are satisfied in the complexified domain $\mathcal{W}_{h_0}^{(0)}$ for $0 < |k| \le K$, $|l| \le 2$ and $0 \le i < j \le n_2$. More precisely,

$$(122)\qquad \left|k\cdot\omega + \varepsilon l\,\cdot\Omega^{(0)}(\omega)\right| \ge \frac{(2-1/2)\gamma}{K^\tau} \quad\text{and}\quad \left|\Omega_i^{(0)}(\omega) - \Omega_j^{(0)}(\omega)\right| \ge \left(2-\frac{1}{2}\right)\bar b\ .$$

For $r=0$, the inequalities in (76) immediately follow from the previous ones, recalling that $\varphi^{(0)} = \mathrm{Id}$. Moreover, the frequencies are not modified by the first normalization step (see (41)), therefore we set

$$\mathcal{W}^{(1)} = \mathcal{W}^{(0)} \quad \text{and} \quad J_1 = J_0 \ .$$

Let us now require the new radius of the complex extension be so small that

$$h_1 \le \min\left\{ h_0 \, , \, \frac{1}{\max\{K \, , \, 1/\sigma\} + \varepsilon J_1} \cdot \frac{\gamma}{4(2K)^\tau} \right\} \ . \tag{123}$$

Using condition above and the first non-resonant condition in (a') of lemma 2, the first inequality in (122) is replaced by

$$\left| k\cdot\omega + \varepsilon l \ \cdot \Omega^{(1)}(\omega) \right| \ge \frac{(2-1/2)\gamma}{(2K)^\tau} \ , \tag{124}$$

for $\omega \in \mathcal{W}^{(1)}_{h_1}$, $0 < |k| \le 2K$ and $|l| \le 2$. This concludes the proof in the case $r = 1$.

The first actual change of the frequencies might occur at the end of the second perturbation step and the transformed fast frequencies read

$$\omega^{(2)}(\omega^{(0)}) = \omega^{(2)}(\omega^{(1)}) = \omega^{(1)} + \delta\omega^{(2)}(\omega^{(1)}) = \left(\mathrm{Id} + \delta\omega^{(2)} \right) (\omega^{(1)})$$

where $\max_{1\le j\le n_1} \sup_{\omega\in\mathcal{W}^{(1)}_{h_1}} \left|\delta\omega^{(2)}_j(\omega)\right| \le \sigma\big(\varepsilon\mathcal{A}\big)^2$ in view of the assumption in (74). If

$$\mu_1 = \frac{4\sigma(\varepsilon\mathcal{A})^2}{h_1} < 1 \ , \tag{125}$$

in view of lemma D.1 in Pöschel (1989), the function $(\mathrm{Id} + \delta\omega^{(2)})$ admits an analytic inverse $\varphi^{(2)} : \mathcal{W}^{(1)}_{h_1/4} \to \mathcal{W}^{(1)}_{h_1/2}$ and, in the domain $\mathcal{W}^{(1)}_{h_1/4}$, the following estimates hold true:

$$\max_{1\le j\le n_2} \sup_{\omega\in\mathcal{W}^{(1)}_{h_1/4}} \left| \varphi^{(2)}_j(\omega) - \omega_j \right| \le \sigma(\varepsilon\mathcal{A})^2 \ , \qquad \left| \frac{\partial\left(\varphi^{(2)} - \mathrm{Id}\right)}{\partial\omega} \right|_{\infty;\mathcal{W}^{(1)}_{h_1/4}} \le \mu_1 \ , \tag{126}$$

where the norm $\left|\,\cdot\,\right|_{\infty;\mathcal{W}^{(1)}_{h_1/4}}$ on the Jacobian of the function $\varphi^{(2)} - \mathrm{Id} : \mathcal{W}_{h_1/4} \to \mathbb{C}^{n_1}$ is defined in a way analogous to (45). The growth of the Lipschitz constants for the sequence of functions $\left\{\varphi^{(r)} - \mathrm{Id}\right\}_{r\ge0}$ is controlled by setting

$$\bar{J}_0 = \bar{J}_1 = 0 \ , \qquad \bar{J}_2 = e^{\mu_1} - 1 \ , \tag{127}$$

so that, in particular, we have $\left|\partial\left(\varphi^{(2)} - \mathrm{Id}\right)/\partial\omega\right|_{\infty;\mathcal{W}^{(1)}_{h_1/4}} \le \bar{J}_2$.

Recall now that the step $r = 2$ includes also the preparation of the next step $r = 3$, namely cutting out the resonant regions

$$\mathcal{R}^{(2)}_{k,l} = \left\{ \omega\in\mathcal{W}^{(2)} : \left| k\cdot\omega + \varepsilon l \ \cdot \Omega^{(2)} \circ \varphi^{(2)}(\omega) \right| \le 2\gamma/(3K)^\tau \right\} \ .$$

Thus we need an upper bound on both the sup-norm of $\Omega^{(2)}\circ\varphi^{(2)}$ and the Lipschitz constant of its Jacobian. The new transverse frequencies can be written as

$$\varepsilon\Omega^{(2)}(\omega^{(1)}) = \varepsilon\Omega^{(1)}(\omega^{(1)}) + \varepsilon\Delta\Omega^{(2)}(\omega^{(1)}) \ ,$$

where we have $\max_{1\le j\le n_1}\sup_{\omega\in\mathcal{W}^{(1)}_{h_1}}\bigl|\varepsilon\Delta\Omega^{(2)}_j(\omega)\bigr| \le (\varepsilon\mathcal{A})^2$, in view of the hypothesis in (74). Moreover, in the domain $\mathcal{W}^{(1)}_{h_1/4}$ the new transverse frequencies are functions of the transformed fast frequencies, namely

$$\Omega^{(2)}\bigl(\varphi^{(2)}(\omega^{(2)})\bigr) = \Omega^{(1)}\bigl(\varphi^{(2)}(\omega^{(2)})\bigr) + \Delta\Omega^{(2)}\bigl(\varphi^{(2)}(\omega^{(2)})\bigr) \ . \tag{128}$$

Using this formula we can bound the Jacobian of the function $\Omega^{(2)}\circ\varphi^{(2)}$. To this end, we need the preliminary estimate

$$\begin{aligned}
\left|\frac{\partial\bigl(\Omega^{(2)}\circ\varphi^{(2)}-\Omega^{(1)}\bigr)}{\partial\omega}\right|_{\infty;\mathcal{W}^{(1)}_{h_1/4}}
&\le \left|\frac{\partial\Omega^{(1)}}{\partial\omega}\right|_{\infty;\mathcal{W}^{(1)}_{h_1/2}}\left|\frac{\partial\bigl(\varphi^{(2)}-\mathrm{Id}\bigr)}{\partial\omega}\right|_{\infty;\mathcal{W}^{(1)}_{h_1/4}}\\
&\quad+\left|\frac{\partial\Delta\Omega^{(2)}}{\partial\omega}\right|_{\infty;\mathcal{W}^{(1)}_{h_1/2}}\left|\frac{\partial\varphi^{(2)}}{\partial\omega}\right|_{\infty;\mathcal{W}^{(1)}_{h_1/4}}\\
&\le J_1\mu_1 + \frac{2(\varepsilon\mathcal{A})^2}{\varepsilon h_1}(1+\mu_1) \le J_1\mu_1 + \frac{\mu_1}{2\varepsilon\sigma}(1+\mu_1) \ ,
\end{aligned}$$

where we used formulæ (125)–(126), the estimate $\bigl|\partial\Omega^{(1)}/\partial\omega\bigr|_{\infty;\mathcal{W}^{(1)}_{h_1/2}} \le J_1$ and Cauchy inequality to estimate $\bigl|\partial\Delta\Omega^{(2)}/\partial\omega\bigr|_{\infty;\mathcal{W}^{(1)}_{h_1/2}}$. Thus, we can ensure that the Jacobian of the transformed transverse frequencies is bounded as

$$\left|\frac{\partial\bigl(\Omega^{(2)}\circ\varphi^{(2)}\bigr)}{\partial\omega}\right|_{\infty;\mathcal{W}^{(1)}_{h_1/4}} \le \left(J_1+\frac{\mu_1}{2\varepsilon\sigma}\right)(1+\mu_1) =: J_2 \ . \tag{129}$$

Using again formula (128), we can bound the deterioration of the non-resonance conditions involving the transverse frequencies. In fact, for $l\in\mathbb{Z}^{n_2}$, $|l|\le 2$, we have

$$\begin{aligned}
&\sup_{\omega\in\mathcal{W}^{(1)}_{h_1/4}}\left|\varepsilon l\cdot\Bigl[\Omega^{(2)}\bigl(\varphi^{(2)}(\omega)\bigr)-\Omega^{(1)}(\omega)\Bigr]\right|\\
&\quad\le 2\varepsilon\max_j\sup_{\omega\in\mathcal{W}^{(1)}_{h_1/4}}\left|\Omega^{(1)}_j\bigl(\varphi^{(2)}(\omega)\bigr)-\Omega^{(1)}_j(\omega)\right| + 2\max_j\sup_{\omega\in\mathcal{W}^{(1)}_{h_1/4}}\left|\varepsilon\Delta\Omega^{(2)}_j\bigl(\varphi^{(2)}(\omega)\bigr)\right|\\
&\quad\le 2\left|\varepsilon\frac{\partial\Omega^{(1)}}{\partial\omega}\right|_{\infty;\mathcal{W}^{(1)}_{h_1/2}}\max_j\sup_{\omega\in\mathcal{W}^{(1)}_{h_1/4}}\left|\varphi^{(2)}_j(\omega)-\omega_j\right| + 2\max_j\sup_{\omega\in\mathcal{W}^{(1)}_{h_1/2}}\left|\varepsilon\Delta\Omega^{(2)}_j(\omega)\right|\\
&\quad\le 2\varepsilon J_1\sigma(\varepsilon\mathcal{A})^2 + 2(\varepsilon\mathcal{A})^2 = 2(1+\varepsilon J_1\sigma)(\varepsilon\mathcal{A})^2 = \mu_1\left(\frac{1}{\sigma}+\varepsilon J_1\right)\frac{h_1}{2} \ .
\end{aligned}\tag{130}$$

Thus, for $\omega \in \mathcal{W}^{(1)}_{h_1/4}$, $0 < |k| \le 2K$ and $|l| \le 2$, we obtain the non-resonance estimate

(131)
$$\begin{aligned}\left|k\cdot\omega + \varepsilon l\,\cdot\Omega^{(2)}\big(\varphi^{(2)}(\omega)\big)\right| &\ge \frac{(2-1/2)\gamma}{(2K)^\tau} - \sup_{\omega\in\mathcal{W}^{(1)}_{h_1/4}}\left|\varepsilon l\cdot\left[\Omega^{(2)}\big(\varphi^{(2)}(\omega)\big) - \Omega^{(1)}(\omega)\right]\right| \\ &\ge \frac{(2-1/2)\gamma}{(2K)^\tau} - \mu_1\left(\frac{1}{\sigma} + \varepsilon J_1\right)\frac{h_1}{2} \ge \frac{(2-1/2-1/4)\gamma}{(2K)^\tau}\ ,\end{aligned}$$

where we started from inequality (124) (holding true on all the complex domain $\mathcal{W}^{(1)}_{h_1/4}$), we used (130), (125) and the definition of h_1 in (123). Moreover, requiring also

$$\varepsilon\mathcal{A} \le \frac{1}{(1+\varepsilon J_1\sigma)\mathcal{A}}\,\frac{\bar b}{8}\ , \tag{132}$$

one can easily obtain the lower bound

$$\left|\Omega^{(2)}_i\big(\varphi^{(2)}(\omega)\big) - \Omega^{(2)}_j\big(\varphi^{(2)}(\omega)\big)\right| \ge \left(2 - \frac{1}{2} - \frac{1}{4}\right)\bar b \tag{133}$$

uniformly with respect to $\omega \in \mathcal{W}^{(1)}_{h_1/4}$, when $i \ne j$. It is now time to consider the new subset of resonant regions $\mathcal{R}^{(2)}_{k,l}$ for $2K < |k| \le 3K$, $|l| \le 2$. First, we remove them from the domain, by defining $\mathcal{W}^{(2)}$ according to (70)–(71). Having required that the new radius of the complex extension is so small that

$$h_2 \le \min\left\{\frac{h_1}{4}\,,\ \frac{1}{\max\{3K/2\,,\,1/\sigma\} + \varepsilon J_2}\,\frac{\gamma}{8(3K)^\tau}\right\}\ , \tag{134}$$

for $2K < |k| \le 3K$, $|l| \le 2$ we have

(135)
$$\begin{aligned}\inf_{\omega\in\mathcal{W}^{(2)}_{h_2}}\left|k\cdot\omega + \varepsilon l\,\cdot\Omega^{(2)}\big(\varphi^{(2)}(\omega)\big)\right| &\ge \inf_{\omega\in\mathcal{W}^{(2)}}\left|k\cdot\omega + \varepsilon l\,\cdot\Omega^{(2)}\big(\varphi^{(2)}(\omega)\big)\right| \\ &\quad - 3Kh_2 - 2\varepsilon J_2 h_2 \ge \frac{(2-1/2-1/4)\gamma}{(3K)^\tau}\ .\end{aligned}$$

Here we used the definitions (70)–(71) for the real set $\mathcal{W}^{(2)}$ and subtracted the contribution due to the complex extension; moreover, we also used formulæ (134), (129) (recall that $\mathcal{W}^{(2)}_{h_2} \subseteq \mathcal{W}^{(1)}_{h_1/4}$, in view of (70) and (134)). The inequalities in (76) are justified in view of (131), (133) and (135). This concludes the proof of the statement for $r = 2$.

Iterating the procedure for a generic step $r > 2$ is now straightforward, provided the sequence of restrictions of the frequency domain is suitably selected. Let us give some details. For $r > 2$, we restart from the relation

$$\omega^{(r)}(\omega^{(0)}) = \omega^{(r)}\circ\varphi^{(r-1)}\circ\omega^{(r-1)}(\omega^{(0)}) = \Big(\mathrm{Id} + \delta\omega^{(r)}\Big)\left(\omega^{(r-1)}(\omega^{(0)})\right)\ ,$$

where $\max_{1\le j\le n_1}\sup_{\omega\in\mathcal{W}^{(r-1)}_{h_{r-1}}}\left\{\left|\delta\omega^{(r)}_j(\omega)\right|\right\} \le \sigma\big(\varepsilon\mathcal{A}\big)^r$ in view of assumption (74). Thus, using again lemma D.1 in Pöschel (1989), we obtain $\omega^{(r-1)}(\omega^{(0)})$ from $\omega^{(r)}(\omega^{(0)})$ via the function $\phi^{(r)}:\mathcal{W}^{(r-1)}_{h_{r-1}/4}\to\mathcal{W}^{(r-1)}_{h_{r-1}/2}$. The function $\varphi^{(r)}$, namely the inverse of $\omega^{(r)}(\omega^{(0)})$, is obtained by composition, i.e., $\varphi^{(r)}=\varphi^{(r-1)}\circ\phi^{(r)}=\phi^{(2)}\circ\ldots\circ\phi^{(r)}$ and, by construction, we have $\varphi^{(r)}\big(\mathcal{W}^{(r-1)}_{h_{r-1}/4}\big)\subset\varphi^{(r-1)}\big(\mathcal{W}^{(r-1)}_{h_{r-1}}\big)$. Replacing $\varphi^{(2)}$ with $\phi^{(r)}$, formulæ (125)–(135) can be suitably adapted to the case $r>2$, so as to prove the inequalities corresponding to (131), (133) and (135). For this purpose, for $r\geq 2$, it is convenient to impose the following conditions:

$$\text{(i)}\qquad \mu_{r-1}=\frac{4\sigma(\varepsilon\mathcal{A})^r}{h_{r-1}}\leq\frac{\min\big\{1\,,\,\varepsilon\sigma\big\}}{2^r}\ ,$$

$$\text{(ii)}\qquad \bar{J}_r=\bar{J}_{r-1}(1+\mu_{r-1})+\mu_{r-1}\leq e^{\tilde{\mu}_1}-1\leq\varepsilon\sigma\ ,$$

$$\text{(iii)}\qquad J_r=\left(J_{r-1}+\frac{\mu_{r-1}}{2\varepsilon\sigma}\right)(1+\mu_{r-1})\leq\left(J_0+\frac{\tilde{\mu}_1}{2\varepsilon\sigma}\right)e^{\tilde{\mu}_1}\leq 2J_0+1\ ,$$

$$\text{(iv)}\qquad (\varepsilon\mathcal{A})^{r-1}\leq\frac{\bar{b}}{2^{r+1}\mathcal{A}(1+\varepsilon J_{r-1}\sigma)}\ ,$$

$$\text{(v)}\qquad h_r\leq\min\left\{\frac{h_{r-1}}{4}\,,\,\frac{1}{\max\left\{\frac{(r+1)K}{2}\,,\,\frac{1}{\sigma}\right\}+\varepsilon J_r}\,\frac{\gamma}{2^{r+1}\big((r+1)K\big)^{\tau}}\right\}\ ,$$

where $\tilde{\mu}_r=\sum_{s=r}^{\infty}\mu_s$. Let us remark that the estimates (ii)–(iii) are straightforward consequences of condition (i). The sequence $\{h_r\}_{r\geq 0}$ in (73) has been chosen so as to satisfy all the smallness conditions (121), (123) and (v), which are required along this proof. This is seen because $\varepsilon J_r\sigma\leq\varepsilon(2J_0+1)\sigma<1$ in view of point (iii), hypothesis $\varepsilon<\varepsilon^*_{\rm ge}$ and $\varepsilon^*_{\rm ge}\leq 1/[(2J_0+1)\sigma]$ due to (75).

The smallness conditions (125), (132), (i) and (iv) on ε are satisfied in view of definition (75) since $\varepsilon<\varepsilon^*_{\rm ge}$. This concludes the proof. *Q.E.D.*

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